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A Characterization of the Rivlin-Ericksen Viscoelastic Fluid in the Presence of a Magnetic Field and Rotation

Ajaib S. BANYAL

Department of Mathematics, Govt. College Nadaun, Hamirpur
(HP) India 177033
e-mail: ajaibbanyal@rediffmail.com

A layer of Rivlin-Ericksen viscoelastic fluid heated from below is considered in the presence of an uniform vertical magnetic field and rotation. Following the linearized stability theory and normal mode analysis, this paper mathematically establishes the condition for characterizing oscillatory motion, which may be neutral or unstable, for rigid boundaries at the top and bottom of the fluid. It is established that all non-decaying slow motions starting from rest, in the configurations, are necessarily non-oscillatory in the regime

$$\frac{T_A F}{\pi^2} + \frac{T_A}{\pi^4} + \frac{Q p_2}{\pi^2} \leq 1,$$

where T_A is the Taylor number, Q is the Chandrasekhar number, p_2 is the magnetic Prandtl number, and F is the viscoelasticity parameter. This result is important, since it holds for all wave numbers for rigid boundaries of infinite horizontal extension at the top and bottom of the fluid, and the exact solutions of the problem investigated in closed form are not obtainable.

Key words: thermal convection, Rivlin-Ericksen fluid, rotation, magnetic field PES, Rayleigh number, Chandrasekhar number, Taylor number.

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Notations

a – dimensionless wave number,
F – viscoelasticity parameter,
g – acceleration due to gravity [m/s^2],
k – wave number [1/m],
k_x, k_y – wave numbers in x- and y-directions [1/m],
n – growth rate [1/s],
Q – Chandrasekhar number,
T_A – Taylor number,
R – Rayleigh number,
$\mathbf{\Omega}(0, 0, \Omega)$ – rotation vector having components $(0, 0, \Omega)$,
$\mathbf{H}(h_x, h_y, h_z)$ – magnetic field having components (h_x, h_y, h_z),

T – temperature [K],
$\mathbf{q}(u, v, w)$ – components of velocity after perturbation,
p_1 – thermal Prandtl number,
p_2 – magnetic Prandtl number,
α – coefficient of thermal expansion [1/K],
β – uniform temperature gradient [K/m],
Θ – perturbation in temperature [K],
κ – thermal diffusivity [m^2/s],
ν – kinematic viscosity [m^2/s],
ν' – kinematic viscoelasticity [m^2/s],
∇, ∂, D – Del operator, Curly operator and Derivative with respect to z $(= d/dz)$.

1. Introduction

The stability of a dynamical system is close to real life in the sense that the realization of a dynamical system depends upon its stability. Right from the conceptualization of turbulence, the instability of fluid flow is regarded as being at its root. The thermal instability of a fluid layer with maintained adverse temperature gradient by heating the underside, plays an important role in Geophysics, interiors of the Earth, Oceanography, and Atmospheric Physics; and has been investigated by several authors under different conditions (e.g., Bénard [1], Rayleigh [2], and Jeffreys [3]). A detailed account of the theoretical and experimental study of the onset of Bénard Convection in Newtonian fluids, under various assumptions of hydrodynamics and hydromagnetics, has been given by Chandrasekhar [4]. The use of Boussinesq's approximation has been made throughout, which states that density changes are disregarded in all terms in the equations of motion except the external force term. Bhatia and Steiner [5] have considered the affect of uniform rotation on the thermal instability of a viscoelastic (Maxwell) fluid and found that rotation has a destabilizing influence in contrast to the stabilizing affect on Newtonian fluids. The thermal instability of a Maxwell fluid in hydromagnetics has been studied by Bhatia and Steiner [6]. They found that the magnetic field stabilizes a viscoelastic (Maxwell) fluid, just as it stabilizes a Newtonian fluid. Sharma [7] studied the thermal instability of a layer of viscoelastic (Oldroydian) fluid acted upon by a uniform rotation and found that rotation has a destabilizing as well as a stabilizing effect under certain conditions, in contrast to that of a Maxwell fluid where it has a destabilizing effect. In another study, Sharma [8] has considered the stability of a layer of an electrically conducting Oldroyd fluid [9] in the presence of magnetic field and found that the magnetic field has a stabilizing influence.

There are many viscoelastic fluids that cannot be characterized by Maxwell's constitutive relations, nor by Oldroyd's [9] constitutive relations. Two such classes of fluids are Rivlin-Ericksen's and Walter's (model B') fluids. Rivlin-Ericksen [10] has proposed a theoretical model for one such class of elastic-viscous fluids. Sharma and Kumar [11] have studied the affect of rotation on thermal instability in a Rivlin-Ericksen elastico-viscous fluid and found that rotation has a stabilizing effect and introduces oscillatory modes in the system. Kumar *et al.* [12] considered the affect of rotation and magnetic field on a Rivlin-Ericksen viscoelastic fluid and found that rotation has a stabilizing effect, whereas the magnetic field has both stabilizing and destabilizing effects. A layer of such fluid heated from below or under the action of either a magnetic field or rotation, or both, may find applications in Geophysics, interior of the Earth, Oceanography, and Atmospheric Physics.

Pellow and Southwell [13] proved the validity of the 'Principle of Exchange of Stability' (PES) for the classical Rayleigh-Bénard convection problem. Banerjee *et al.* [14] gave a new scheme for combining the governing equations of thermohaline convection, which was shown to lead to bounds for the complex growth rate of arbitrary oscillatory perturbations, neutral or unstable, for all combinations of dynamically rigid or free boundaries. Banerjee and Banerjee [15] established a criterion for the characterization of non-oscillatory motions in hydrodynamics, which was further extended by Gupta *et al.* [16]. However, no such result exists for non-Newtonian fluid configurations in general and for Rivlin-Ericksen viscoelastic fluid configurations in particular. Banyal [17] have characterized the non-oscillatory motions in coupled-stress fluids.

Keeping in mind the importance of Rivlin-Ericksen viscoelastic fluids, this paper is an attempt to study a Rivlin-Ericksen viscoelastic fluid heated from below in the presence of a uniform vertical magnetic field and rotation. It is established that the onset of instability in a Rivlin-Ericksen viscoelastic fluid in the present configuration cannot manifest itself as oscillatory motion of growing amplitude if the Taylor number T_A, the Chandrasekhar number Q, the magnetic Prandtl number p_2, and the viscoelasticity parameter F, together satisfy the inequality $\frac{T_A F}{\pi^2}+\frac{T_A}{\pi^4}+\frac{Qp_2}{\pi^2}\le 1$. These results hold for all wave numbers with rigid boundaries of infinite horizontal extension at the top and bottom of the fluid.

2. Formulation of the problem and perturbation equations

Consider an infinite, horizontal, incompressible, electrically conducting, Rivlin-Ericksen viscoelastic fluid layer of thickness d, heated from below such that the temperature and density on the bottom surface $z = 0$ are T_0 and ρ_0, and on the upper surface $z = d$ are T_d and ρ_d respectively, and that a uniform adverse temperature gradient $\beta\left(=\left|\frac{dT}{dz}\right|\right)$ is maintained. The fluid is acted upon by a uniform vertical rotation $\mathbf{\Omega}(0, 0, \Omega)$ and a uniform vertical magnetic field $\mathbf{H}(0, 0, H)$.

The equations of motion, continuity, heat conduction, and Maxwell's equations governing the flow of Rivlin-Ericksen viscoelastic fluid in the presence of magnetic field and rotation (RIVLIN and ERICKSEN [10]; CHANDRASEKHAR [4], and KUMAR *et al.* [12]) are:

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}.\nabla)\,\mathbf{q} = -\nabla\left(\frac{p}{\rho_0} - \frac{1}{2}\left|\mathbf{\Omega}\times\mathbf{r}\right|^2\right) + \mathbf{g}\left(1 + \frac{\delta\rho}{\rho_0}\right) + \left(\nu + \nu'\frac{\partial}{\partial t}\right)\nabla^2\mathbf{q} + \frac{\mu_e}{4\pi\rho_0}(\nabla\times\mathbf{H})\times\mathbf{H} + 2\left(\mathbf{q}\times\mathbf{\Omega}\right), \tag{2.1}$$

$$\nabla.\mathbf{q} = 0, \tag{2.2}$$

$$\frac{\partial T}{\partial t} + (\mathbf{q}.\nabla)T = \kappa\nabla^2 T, \tag{2.3}$$

$$\nabla.\mathbf{H} = 0, \tag{2.4}$$

$$\frac{\partial \mathbf{H}}{\partial t} = (\mathbf{H}.\nabla)\mathbf{q} + \eta\nabla^2\mathbf{H}. \tag{2.5}$$

The equation of state for the fluid is

$$\rho = \rho_0\left[1 - \alpha(T - T_0)\right], \tag{2.6}$$

where ρ, p, T, ν, ν', and $\mathbf{q}(u, v, w)$ denote the density, pressure, temperature, kinematic viscosity, kinematic viscoelasticity, and velocity of the fluid respectively. Furthermore, $\mathbf{r}(x, y, z)$ and the suffix zero refers to values at the reference level $z = 0$. Here, $\mathbf{g}(0, 0, -g)$ is the acceleration due to gravity and α is the coefficient of thermal expansion. In writing Eq. (2.1), we made use of the Boussinesq approximation, which states that variations of the density are ignored in all terms in the equation of motion, except the external force term. The magnetic permeability μ_e, thermal diffusivity κ, and electrical resistivity η, are all assumed to be constant.

The initial state is one in which the velocity, density, pressure, and temperature at any point in the fluid are, respectively, given by

$$\mathbf{q} = (0, 0, 0), \qquad \rho = \rho(z), \qquad p = p(z), \qquad T = T(z). \tag{2.7}$$

Let us assume small perturbations around the basic solution and let $\delta\rho$, δp, θ, $\mathbf{q}(u, v, w)$, and $\mathbf{h} = (h_x, h_y, h_z)$ denote respectively perturbations in the density ρ, pressure p, temperature T, velocity $\mathbf{q}(0, 0, 0)$, and magnetic field

$\mathbf{H} = (0, 0, H)$. The change in density $\delta\rho$, caused mainly by the perturbation θ in temperature, is given by

$$\rho + \delta\rho = \rho_0 \left[1 - \alpha(T + \theta - T_0)\right] = \rho - \alpha\rho_0\theta, \qquad \text{i.e.} \quad \delta\rho = -\alpha\rho_0\theta. \tag{2.8}$$

Then the linearized perturbation equations are:

$$\frac{\partial \mathbf{q}}{\partial t} = -\frac{1}{\rho_0}\nabla\delta p - \mathbf{g}\alpha\theta + \left(\nu + \nu'\frac{\partial}{\partial t}\right)\nabla^2\mathbf{q} + \frac{\mu_e}{4\pi\rho_0}\left(\nabla \times \mathbf{h}\right) \times \mathbf{H} + 2\left(\mathbf{q} \times \mathbf{\Omega}\right), \tag{2.9}$$

$$\nabla.\mathbf{q} = 0, \tag{2.10}$$

$$\frac{\partial \theta}{\partial t} = \beta w + \kappa\nabla^2\theta, \tag{2.11}$$

$$\nabla.\mathbf{h} = 0, \tag{2.12}$$

$$\frac{\partial \mathbf{h}}{\partial t} = \left(\mathbf{H}.\nabla\right)\mathbf{q} + \eta\nabla^2\mathbf{h}. \tag{2.13}$$

Within the framework of Boussinesq's approximation, Eqs. (2.9)–(2.13), become

$$\frac{\partial}{\partial t}\nabla^2 w = \left(\nu + \nu'\frac{\partial}{\partial t}\right)\nabla^4 w + \frac{\mu_e H}{4\pi\rho_0}\nabla^2\left(\frac{\partial h_z}{\partial z}\right) + g\alpha\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) - 2\Omega\frac{\partial \varsigma}{\partial z}, \tag{2.14}$$

$$\frac{\partial \varsigma}{\partial t} = \left(\nu + \nu'\frac{\partial}{\partial t}\right)\nabla^2\varsigma + 2\Omega\frac{\partial w}{\partial z} - \frac{\mu_e H}{4\pi\rho_0}\frac{\partial \xi}{\partial z}, \tag{2.15}$$

$$\frac{\partial \theta}{\partial t} = \beta w + \kappa\nabla^2\theta, \tag{2.16}$$

$$\frac{\partial h_z}{\partial t} = H\frac{\partial w}{\partial z} + \eta\nabla^2 h_z, \tag{2.17}$$

$$\frac{\partial \xi}{\partial t} = H\frac{\partial \varsigma}{\partial z} + \eta\nabla^2\xi, \tag{2.18}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$; and $\varsigma = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ and $\xi = \dfrac{\partial h_y}{\partial x} - \dfrac{\partial h_x}{\partial y}$ denote the z-component of vorticity and current density respectively.

3. Normal mode of analysis

Analyzing the disturbances in normal modes, we assume that the perturbation quantities are of the form

$$[w, \theta, h_z, \varsigma, \xi] = [W(z), \Theta(z), K(z), Z(z), X(z)] \exp(ik_x x + ik_y y + nt), \tag{3.1}$$

where k_x, k_y are the wave numbers along the x- and y-directions respectively, $k = (k_x^2 + k_y^2)^{1/2}$, is the resultant wave number and n is the growth rate, which in general is a complex constant.

Using (3.1), Eqs. (2.14)–(2.18), in non-dimensional form, transform to:

$$(D^2 - a^2)\left[(1+F\sigma)(D^2 - a^2) - \sigma\right] W = Ra^2\Theta + T_A DZ - Q(D^2 - a^2)DK, \tag{3.2}$$

$$\left[(1+F\sigma)(D^2 - a^2) - \sigma\right] Z = -DW - QDX, \tag{3.3}$$

$$(D^2 - a^2 - p_1\sigma)\Theta = -W, \tag{3.4}$$

$$(D^2 - a^2 - p_2\sigma)K = -DW, \tag{3.5}$$

$$(D^2 - a^2 - p_2\sigma)X = -DZ, \tag{3.6}$$

where we have introduced the new coordinates $(x', y', z') = (x/d, y/d, z/d)$ in units of length d, and $D = d/dz'$. For convenience, the primes are dropped hereafter. We have substituted, $a = kd$, $\sigma = \dfrac{nd^2}{\nu}$, and $p_1 = \dfrac{\nu}{\kappa}$ is the thermal Prandtl number, $p_2 = \dfrac{\nu}{\eta}$ is the magnetic Prandtl number, $F = \dfrac{\nu'}{d^2}$ is the Rilvin-Ericksen kinematic viscoelasticity parameter, $R = \dfrac{g\alpha\beta d^4}{\kappa\nu}$ is the thermal Rayleigh number, $Q = \dfrac{\mu_e H^2 d^2}{4\pi\rho_0\nu\eta}$ is the Chandrasekhar number, and $T_A = \dfrac{4\Omega^2 d^4}{\nu^2}$ is the Taylor number. Also, we have substituted $W = W_\oplus$, $\Theta = \dfrac{\beta d^2}{\kappa}\Theta_\oplus$, $Z = \dfrac{2\Omega d}{\nu} Z_\oplus$, $K = \dfrac{Hd}{\eta}K_\oplus$, $X = \left(\dfrac{Hd}{\eta}\right)\left(\dfrac{2\Omega d}{\nu}\right)X_\oplus$, and $D_\oplus = dD$, and dropped $(\oplus)$ for convenience.

We now consider the case where both the boundaries are rigid and perfectly conducting and are maintained at constant temperature. Then, the perturbations in the temperature are zero on the boundaries. The appropriate boundary conditions, with respect to which Eqs. (3.2)–(3.6) must possess a solution, are:

$$\begin{gathered} W = DW = 0, \quad \Theta = 0, \quad Z = 0, \quad K = 0, \quad DX = 0, \\ \text{at} \quad z = 0 \quad \text{and} \quad z = 1. \end{gathered} \tag{3.7}$$

The Eqs. (3.2)–(3.6), along with appropriate boundary conditions (3.7), constitute an eigenvalue problem for σ and we want to characterize σ_i, when $\sigma_r \geq 0$.

We first note that, since W and Z satisfy $W(0) = 0 = W(1)$, then $K(0) = 0 = K(1)$ and $Z(0) = 0 = Z(1)$ in addition to satisfying the governing equations; and hence we have from the Rayleigh-Ritz inequality (Schultz [18]):

$$
\begin{aligned}
&\int_0^1 |DW|^2 dz \geq \pi^2 \int_0^1 |W|^2 \, dz,\\
&\int_0^1 |DK|^2 dz \geq \pi^2 \int_0^1 |K|^2 \, dz,\\
&\int_0^1 |DZ|^2 dz \geq \pi^2 \int_0^1 |Z|^2 \, dz.
\end{aligned}
\tag{3.8}
$$

Furthermore, for $W(0) = 0 = W(1)$, $K(0) = 0 = K(1)$, and $Z(0) = 0 = Z(1)$, Banerjee *et al.* [19] have shown that

$$
\begin{aligned}
&\int_0^1 \left|D^2 W\right|^2 dz \geq \pi^2 \int_0^1 |DW|^2 \, dz,\\
&\int_0^1 \left|D^2 K\right|^2 dz \geq \pi^2 \int_0^1 |DK|^2 \, dz,\\
&\int_0^1 \left|D^2 Z\right|^2 dz \geq \pi^2 \int_0^1 |DZ|^2 \, dz.
\end{aligned}
\tag{3.9}
$$

4. Mathematical analysis

We prove the following lemma:

Lemma 1: *For any arbitrary oscillatory perturbation, neutral or unstable*

$$
\int_0^1 \left\{ |DK|^2 + a^2 |K|^2 \right\} dz \leq \frac{1}{\pi^2} \int_0^1 |DW|^2 \, dz.
$$

Proof. Multiplying Eq. (3.5) by K^* (the complex conjugate of K) and integrating by parts each term of the resulting equation on the left-hand side an appropriate number of times, and making use of boundary conditions on K, namely $K(0) = 0 = K(1)$, it follows that:

$$\int_0^1 \left\{ |DK|^2 + a^2 |K|^2 \right\} dz + \sigma_r p_2 \int_0^1 |K|^2 \, dz \tag{4.1}$$
$$= \text{Real part of } \left\{ \int_0^1 K^* DW \, dz \right\} \leq \left| \int_0^1 K^* DW \, dz \right| \leq \int_0^1 |K^* DW| \, dz$$
$$\leq \int_0^1 |K^*| \, |DW| \, dz \leq \int_0^1 |K| \, |DW| \, dz$$
$$\leq \left\{ \int_0^1 |K|^2 \, dz \right\}^{1/2} \left\{ \int_0^1 |DW|^2 \, dz \right\}^{1/2}$$

(utilizing the Cauchy-Schwartz inequality).

This gives that:

$$\int_0^1 |DK|^2 \, dz \leq \left\{ \int_0^1 |K|^2 \, dz \right\}^{1/2} \left\{ \int_0^1 |DW|^2 \, dz \right\}^{1/2}. \tag{4.2}$$

The inequality (4.1), on utilizing the inequalities (3.8) and (4.2), gives:

$$\left\{ \int_0^1 |K|^2 \, dz \right\}^{1/2} \leq \frac{1}{\pi^2} \left\{ \int_0^1 |DW|^2 \, dz \right\}^{1/2}. \tag{4.3}$$

Since $\sigma_r \geq 0$ and $p_2 > 0$, and hence inequality (4.1), on utilizing (4.3) gives:

$$\int_0^1 \left(|DK|^2 + a^2 |K|^2 \right) dz \leq \frac{1}{\pi^2} \int_0^1 |DW|^2 \, dz. \tag{4.4}$$

This completes the proof of Lemma 1.

Lemma 2: *For any arbitrary oscillatory perturbation, neutral or unstable*

$$\int_0^1 |Z|^2 \, dz \leq \frac{1}{\pi^4} \int_0^1 |DW|^2 \, dz,$$

$$\int_0^1 \left(|DZ|^2 + a^2 |Z|^2 \right) dz \leq \frac{1}{\pi^2} \int_0^1 |DW|^2 \, dz.$$

P r o o f. Multiplying Eq. (3.3) by Z^* (the complex conjugate of Z) and integrating by parts each term of the resulting equation on the left-hand side an appropriate number of times, on utilizing Eq. (3.6) and the appropriate boundary conditions (3.7), it follows that:

$$
\begin{aligned}
(4.5)\qquad (1+F\sigma_r)\int_0^1 \left\{|DZ|^2 + a^2|Z|^2\right\} dz + \sigma_r \int_0^1 |Z|^2\, dz \\
+ Q\int_0^1 \left\{|DX|^2 + a^2|X|^2\right\} dz + Qp_2\sigma_r \int_0^1 |X|^2\, dz \\
= \text{ Real part of } \left\{\int_0^1 DW^* Z\, dz\right\} \le \left|\int_0^1 DW^* Z\, dz\right| \\
\le \int_0^1 |DW^* Z|\, dz \le \int_0^1 |DW^*|\,|Z|\, dz \\
= \int_0^1 |DW||Z|\, dz \le \left\{\int_0^1 |Z|^2 dz\right\}^{1/2} \left\{\int_0^1 |DW|^2 dz\right\}^{1/2}
\end{aligned}
$$

(utilizing the Cauchy-Schwartz inequality).

This gives that

$$
(4.6)\qquad \int_0^1 |DZ|^2\, dz \le \left\{\int_0^1 |Z|^2\, dz\right\}^{1/2} \left\{\int_0^1 |DW|^2\, dz\right\}^{1/2}.
$$

Inequality (4.5) on utilizing inequalities (3.8) and (4.6), gives

$$
(4.7)\qquad \left\{\int_0^1 |Z|^2\, dz\right\}^{1/2} \le \frac{1}{\pi^2}\left\{\int_0^1 |DW|^2\, dz\right\}^{1/2}.
$$

Since $\sigma_r \ge 0$ and $p_2 > 0$, hence inequality (4.5) on utilizing (4.7), give

$$
(4.8)\qquad
\begin{aligned}
\int_0^1 |Z|^2\, dz &\le \frac{1}{\pi^4}\int_0^1 |DW|^2\, dz, \\
\int_0^1 \left(|DZ|^2 + a^2|Z|^2\right) dz &\le \frac{1}{\pi^2}\int_0^1 |DW|^2\, dz.
\end{aligned}
$$

This completes the proof of Lemma 2.

We now prove the following theorems:

THEOREM 1: *If $R > 0$, $F > 0$, $Q > 0$, $T_A > 0$, $p_1 > 0$, $p_2 > 0$, $\sigma_r \geq 0$, and $\sigma_i \neq 0$, then the necessary condition for the existence of a non-trivial solution (W, Θ, K, Z, X) of Eqs. (3.2)–(3.6), together with boundary conditions (3.7), is that*

$$\frac{T_A F}{\pi^2} + \frac{T_A}{\pi^4} + \frac{Q p_2}{\pi^2} > 1.$$

P r o o f. Multiplying Eq. (3.2) by W^* (the complex conjugate of W) throughout and integrating the resulting equation over the vertical range of z, we get

$$(4.9)\quad (1 + F\sigma) \int_0^1 W^*(D^2 - a^2)^2 W \, dz - \sigma \int_0^1 W^*(D^2 - a^2) W \, dz$$
$$= Ra^2 \int_0^1 W^* \Theta \, dz + T_A \int_0^1 W^* DZ \, dz - Q \int_0^1 W^* D(D^2 - a^2) K \, dz.$$

Taking the complex conjugate of both sides of Eq. (3.4), we get:

$$(4.10)\qquad \left(D^2 - a^2 - p_1 \sigma^*\right) \Theta^* = -W^*.$$

Therefore, using (4.10), we get:

$$(4.11)\qquad \int_0^1 W^* \Theta \, dz = - \int_0^1 \Theta (D^2 - a^2 - p_1 \sigma^*) \Theta^* \, dz.$$

Taking the complex conjugate of both sides of Eq. (3.3), we get:

$$(4.12)\qquad (1 + F\sigma^*)(D^2 - a^2) Z^* - \sigma^* Z^* = -DW^* - QDX^*.$$

Therefore, using (4.12), we get:

$$(4.13)\quad \int_0^1 W^* DZ \, dz = - \int_0^1 DW^* Z \, dz = (1 + F\sigma^*) \int_0^1 Z^*(D^2 - a^2) Z \, dz$$
$$- \sigma^* \int_0^1 Z^* Z \, dz + Q \int_0^1 Z DX^* \, dz.$$

Now, integrating by parts, the third term on left-hand side and using Eq. (3.6) and the appropriate boundary condition (3.7), we get:

$$\int_0^1 W^* DZ\,dz = (1+F\sigma^*)\int_0^1 Z^*(D^2-a^2)Z\,dz - \sigma^*\int_0^1 Z^*Z\,dz + Q\int_0^1 X(D^2-a^2-p_2\sigma)X^*\,dz. \tag{4.14}$$

Taking the complex conjugate of both sides of Eq. (3.5), we get:

$$\left[D^2-a^2-p_2\sigma^*\right]K^* = -DW^*. \tag{4.15}$$

Therefore, Eq. (4.15) with the appropriate boundary condition (3.7), we get:

$$\int_0^1 W^* D(D^2-a^2)K\,dz = -\int_0^1 DW^*(D^2-a^2)K\,dz = \int_0^1 K(D^2-a^2)(D^2-a^2-p_2\sigma^*)K^*\,dz. \tag{4.16}$$

Substituting (4.11), (4.14), and (4.16) into the right-hand side of Eq. (4.9), we get:

$$\begin{aligned}(1+F\sigma)\int_0^1 W^*(D^2-a^2)^2W\,dz - \sigma\int_0^1 W^*(D^2-a^2)W\,dz \\ = -Ra^2\int_0^1 \Theta(D^2-a^2-p_1\sigma^*)\Theta^*\,dz + T_A(1+F\sigma^*)\int_0^1 Z(D^2-a^2)Z^*\,dz \\ - T_A\sigma^*\int_0^1 Z^*Z\,dz + T_AQ\int_0^1 X(D^2-a^2-p_2\sigma^*)X^*\,dz \\ - Q\int_0^1 K^*(D^2-a^2)^2K\,dz - Qp_2\sigma^*\int_0^1 K^*(D^2-a^2)K\,dz.\end{aligned} \tag{4.17}$$

Integrating the terms on both sides of Eq. (4.17) an appropriate number of times and making use of the appropriate boundary conditions (3.7), we get:

$$(1+F\sigma)\int_0^1 \left\{\left|D^2W\right|^2+2a^2\left|DW\right|^2+a^4\left|W\right|^2\right\}dz \tag{4.18}$$
$$+\sigma\int_0^1\left(\left|DW\right|^2+a^2\left|W\right|^2\right)dz = Ra^2\int_0^1\left(\left|D\Theta\right|^2+a^2\left|\Theta\right|^2\right)dz$$
$$+Ra^2p_1\sigma^*\int_0^1\left|\Theta\right|^2dz - T_A(1+F\sigma^*)\int_0^1\left\{\left|DZ\right|^2+a^2\left|Z\right|^2\right\}dz$$
$$-T_A\sigma^*\int_0^1\left|Z\right|^2dz - T_AQ\int_0^1\left(\left|DX\right|^2+a^2\left|X\right|^2\right)dz$$
$$-T_AQp_2\sigma\int_0^1\left|X\right|^2dz - Q\int_0^1\left(\left|D^2K\right|^2+2a^2\left|DK\right|^2+a^4\left|K\right|^2\right)dz$$
$$-Qp_2\sigma^*\int_0^1\left(\left|DK\right|^2+a^2\left|K\right|^2\right)dz.$$

Now, equating the imaginary parts on both sides of Eq. (4.18), and cancelling $\sigma_i(\neq 0)$ throughout from the imaginary part, we get:

$$F\int_0^1\left\{\left|D^2W\right|^2+2a^2\left|DW\right|^2+a^4\left|W\right|^2\right\}dz+\int_0^1\left\{\left|DW\right|^2+a^2\left|W\right|^2\right\}dz \tag{4.19}$$
$$=-Ra^2p_1\int_0^1\left|\Theta\right|^2dz+T_AF\int_0^1\left\{\left|DZ\right|^2+a^2\left|Z\right|^2\right\}dz$$
$$+T_A\int_0^1\left|Z\right|^2dz - T_AQp_2\int_0^1\left|X\right|^2dz+Qp_2\int_0^1\left\{\left|DK\right|^2+a^2\left|K\right|^2\right\}dz.$$

Now, for $R>0$, $p_2>0$, $p_1>0$, $Q>0$, and $T_A>0$, and utilizing the inequalities (3.8), (3.9), (4.4), and (4.8), Eq. (4.19) gives,

$$\left[1-\left(\frac{T_AF}{\pi^2}+\frac{T_A}{\pi^4}+\frac{Qp_2}{\pi^2}\right)\right]\int_0^1\left|DW\right|^2dz+I_1<0, \tag{4.20}$$

where

$$(4.21)\quad I_1 = F\int_0^1 \left\{\left|D^2W\right|^2 + 2a^2\left|DW\right|^2 + a^4\left|W\right|^2\right\} dz + a^2\int_0^1 |W|^2\,dz + Ra^2p_1\int_0^1 |\Theta|^2\,dz + T_AQp_2\int_0^1 |X|^2\,dz,$$

is positive definite, and therefore, we must have that

$$(4.22)\qquad \frac{T_AF}{\pi^2} + \frac{T_A}{\pi^4} + \frac{Qp_2}{\pi^2} > 1.$$

Hence, if

$$(4.23)\qquad \sigma_r \geq 0 \quad\text{and}\quad \sigma_i \neq 0, \quad\text{then}\quad \frac{T_AF}{\pi^2} + \frac{T_A}{\pi^4} + \frac{Qp_2}{\pi^2} > 1.$$

That completes the proof of Theorem 1.

Presented otherwise, from the point of view of the existence of instability as a stationary convection, the above theorem can be put in the form as follows:

THEOREM 2: *The sufficient condition for the onset of instability as a non-oscillatory motions of non-growing amplitude in a Rivlin-Ericksen fluid heated from below, in the presence of uniform vertical magnetic field and rotation, is that,* $\frac{T_AF}{\pi^2} + \frac{T_A}{\pi^4} + \frac{Qp_2}{\pi^2} \leq 1$, *where* T_A *is the Taylor number,* Q *is the Chandrasekhar number,* p_2 *is the magnetic Prandtl number, and* F *is the viscoelasticity parameter when both the boundaries are rigid.*

OR:

The onset of instability in a Rivlin-Ericksen fluid heated from below in the presence of an uniform vertical magnetic field and rotation, cannot manifest itself as oscillatory motion of growing amplitude if the Taylor number T_A*, the Chandrasekhar number* Q*,* p_2 *the magnetic Prandtl number, and the viscoelasticity parameter* F*, satisfy the inequality* $\frac{T_AF}{\pi^2} + \frac{T_A}{\pi^4} + \frac{Qp_2}{\pi^2} \leq 1$ *when both the bounding surfaces are rigid.*

The sufficient condition for the validity of the PES can be expressed in the form:

THEOREM 3: *If* $(W, \Theta, K, Z, X, \sigma)$, $\sigma = \sigma_r + i\sigma_i$, $\sigma_r \geq 0$ *is a solution of Eqs. (2.15)–(3.1), with* $R > 0$ *and*

$$\frac{T_AF}{\pi^2} + \frac{T_A}{\pi^4} + \frac{Qp_2}{\pi^2} \leq 1,$$

then $\sigma_i = 0$. *In particular, a sufficient condition for the validity of the 'exchange principle', i.e.* $\sigma_r = 0 \Rightarrow \sigma_i = 0$, *is that* $\frac{T_AF}{\pi^2} + \frac{T_A}{\pi^4} + \frac{Qp_2}{\pi^2} \leq 1$.

In the context of the existence of instability in 'oscillatory modes' and of 'over-stability' in the present configuration, we can state the above theorem as follows:

THEOREM 4: *The necessary condition for the existence of instability in 'oscillatory modes' and that of 'overstability' in a Rivlin-Ericksen fluid heated from below in the presence of uniform vertical magnetic field and rotation, is that the Taylor number T_A, the Chandrasekhar number Q, p_2 the magnetic Prandtl number, and the viscoelasticity parameter F must satisfy the inequality* $\frac{T_A F}{\pi^2} + \frac{T_A}{\pi^4} + \frac{Qp_2}{\pi^2} > 1$, *when both the bounding surfaces are rigid.*

SPECIAL CASES: It follows from Theorem 1 that an arbitrary neutral or unstable mode is non-oscillatory in character and PES is valid for:

1. Thermal convection in a Rivlin-Ericksen fluid heated from below, i.e. when $Q = 0 = T_A$ (KUMAR *et al.* [12]).
2. Magneto-thermal convection in a Rivlin-Ericksen fluid heated from below ($T_A = 0$), if $\left(\frac{Qp_2}{\pi^2}\right) \leq 1$ (GUPTA *et al.* [16]).
3. Rotatory-thermal convection in a Rivlin-Ericksen fluid heated from below ($Q = 0$), if $\frac{T_A F}{\pi^2} + \frac{T_A}{\pi^4} \leq 1$.
4. When $F = 0$ we retrieve the result for a Newtonian fluid by GUPTA *et al.* [16] in the presence of a uniform vertical magnetic field and rotation; i.e., $\frac{T_A}{\pi^4} + \frac{Qp_2}{\pi^2} \leq 1$.

5. CONCLUSIONS

This theorem mathematically establishes that the onset of instability in a Rivlin-Ericksen fluid in the presence of uniform vertical magnetic field and rotation cannot manifest itself as oscillatory motion of growing amplitude if the Taylor number T_A, the Chandrasekhar number Q, p_2 the magnetic Prandtl number, and the viscoelasticity parameter F, satisfy the inequality $\frac{T_A F}{\pi^2} + \frac{T_A}{\pi^4} + \frac{Qp_2}{\pi^2} \leq 1$ when both the bounding surfaces are rigid.

The essential content of the theorem from the point of view of linear stability theory, is that for the configuration of coupled-stress fluid of infinite horizontal extension heated from below having rigid boundaries at the top and bottom of the fluid and in the presence of an uniform vertical magnetic field and rotation parallel to the force field of gravity, an arbitrary neutral or unstable modes of the system are definitely non-oscillatory in character if $\frac{T_A F}{\pi^2} + \frac{T_A}{\pi^4} + \frac{Qp_2}{\pi^2} \leq 1$, and in particular, if the PES is valid.

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