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Research Paper

Bending of Beams with Consideration of a Seventh-Order Shear Deformation Theory

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The subject of the paper is a simply-supported prismatic beam with bisymmetrical cross-sections under non-uniformly distributed load. The shapes of the cross-sections and the non-uniformly distributed load are described analytically. The individual seventh-order shear deformation theory-hypothesis of the planar beam cross-sections is assumed. Based on the principle of stationary potential energy two differential equations of equilibrium are obtained. The system of the equations is analytically solved, and the shear and deflection coefficients of the beam are derived. Moreover, the shear stress patterns for selected cross-sections are determined and compared with stresses determined by Zhuravsky's formula. The results of example calculations are presented in tables and figures.

Key words: nonlinear hypothesis; shear deformation theory; shear stresses.

1. Introduction

The problem of the shear stresses arising in the elastic beams while bending was analyzed in the mid of the 19th century by D.I. Zhuravsky. In 1921, S.P. Timoshenko formulated the linear beam theory with consideration of the shear effect. Gere and Timoshenko [1] described in details the distribution of the shear stresses in selected cross- sections based on the Zhuravsky formula and the influence of the shear effect on beams deflections. Rychter [2] presented the beam-bending theories with consideration of the shear-deformation of the cross-section. The Bernoulli-Euler theory, being a special case of the above theories, gave the accuracy order characteristic for the shear-deformation theories. Wang *et al.* [3] reviewed the theories developed in the 20th century aimed at explaining the behavior of the beams and plates with consideration of the shear effect. They proposed the shear deformation theories based on the classical Euler-Bernoulli/Kirchhoff approach, giving more accurate solutions. HUTCHINSON [4] described the Timoshenko beam theory with consideration of the shear effect. The paper includes new formulae for the Timoshenko shear coefficients in the cases of various beam cross-sections. REDDY [5] presented a new approach to the classical and shear deformation beam and plate theories and derived the equations of equilibrium based on the nonlocal beam theories. This allowed to obtain the finite element results and to determine the effect of the geometric nonlinearity on bending response. SHI and VOYIADJIS [6] derived a new beam theory with the sixth-order differential equilibrium equations for the analysis of shear deformable beams. The authors indicated practical applications of this theory. The example solutions agreed well with the elasticity solutions, reflecting well the boundary layer behaviour near the beam ends and load application points. BECK and DA SILVA JR [7] compared the Euler-Bernoulli and Timoshenko beam theories. The two formulations were solved with the use of a Monte Carlo-Galerkin scheme. The authors pointed out that the behavior of the two beam models is significantly different with regard to the beam reliability or risk analysis. KIM [8] developed a shear deformable beam element designed for analysing thin-walled composite I-beams. The element enabled consideration of transverse shear deformation with the use of the first-order shear deformation beam theory. Numerical solutions have been presented with a view to be compared with the results obtained based on the new beam elements. MAGNUCKA-BLANDZI [9] focused on the dynamic stability and static stress state of simply-supported sandwich beams with a metal foam core. Three different hypotheses of the fields of displacement for the planar cross- section of the beam have been applied to solve the shear effect problem. MAGNUCKA-BLANDZI *et al.* [10] presented mathematical modeling of shearing effect in the case of sandwich beams with corrugated cores. Crosswise and lengthwise core corrugation was considered. The transverse shearing effect affected the deflections and critical loads of these sandwich beams. SCHNEIDER and KIENZLER [11] demonstrated that the general problem of three-dimensional elasticity theory can be decoupled into four independent one-dimensional subproblems corresponding to a rod-, a shaft- and two orthogonal beam cases. Additionally, it was shown that these four subproblems can be derived from the stiffness tensor of anisotropic materials. SENJANOVIĆ *et al.* [12] presented a formulation alternative to the Timoshenko beam theory, dealing with total deflection and bending deflection of the beam. The new formulation of the beam theory takes into account the coupling between flexural and in-plane shear vibrations, using the Hamilton principle. The proposed theory better describes the beam behavior than former first-order shear deformation beam theories. ENDO [13] used the Hamilton principle to derive the governing equations of typical thin-walled beams and plates. The paper took into account bending with consideration of shearing deflections of these structures. The alternative theory allows modelling of the moderately thick structures. Kienzler and Schneider [14] developed a second-order plate theory without any kinematical assumptions or shear-correction factors. The three-dimensional boundary conditions and local equilibrium equations are met due to the a posteriori determined coefficients of the resulting displacement distribution. Adámek [15] discussed possible applications of the classical Timoshenko beam theory to the problems of three-layered elastic beams with consideration of the shear effect. The equivalent single-layer first-order shear deformation theory of the beam was applied, with the use of the Timoshenko shear coefficient. It was demonstrated that the proposed modification of the theory gives very accurate results. Magnucki *et al.* [16] dealt with a beam with symmetrically varying mechanical properties in the depth direction. Two differential equations of motion have been obtained based on the Hamilton principle. Distributions of the normal and shear stresses arising in the cross-section of the beam have been determined. Magnucki and Lewiński [17] presented an analytical model of I-beam with consideration of the shear effect. The governing differential equations for the I-beam have been obtained based on the principle of stationary total potential energy. The shear effect of the beam was graphically demonstrated for the beam under three-point bending. Magnucki [18] described the bending problem of simply- supported sandwich beams and I-beams of the symmetrical structure. Two models of deformation of planar cross-sections of these beams have been proposed. The equations of equilibrium have been formulated based on the principle of stationary total potential energy. The system of equations was solved for exemplary beams with consideration of the shear effect. Magnucki *et al.* [19] studied the bending of simply-supported beams under non-uniformly distributed loads with consideration of a shear effect. An expression for maximum deflection of the beams was formulated. Detailed calculations of the maximum deflection were carried out analytically and numerically (FEM).

The main goal of the presented work is to describe the shear effect in the bending beam based on the proposed individual seventh-order shear deformation theory of a planar cross- section and determination of the shear stress patterns for selected bisymmetrical cross-sections. This work is a generalization of the problem of the bending beam presented by Magnucki *et al.* [19].

2. The analytical model of the beam

The subject of the study is a simply supported prismatic beam of length L with a bisymmetrical cross-section of depth h (Fig. 1).

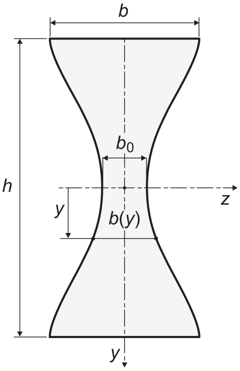

FIG. 1. Scheme of the bisymmetrical cross-section of the beam.

The width of the cross-section is

$$b(y) = b f_b(\eta), \tag{2.1}$$

where

$$f_b(\eta) = \beta_0 + (1 - \beta_0)(6\eta^2 - 32\eta^6)^{k_c}, \tag{2.2}$$

and $\beta_0 = b_0/b$ – parameter, k_c – exponent (positive real number), $\eta = y/h$ – dimensionless coordinate $(-1/2 \leq \eta \leq 12)$. The values of the parameter β_0 and exponent k_c shape the cross-section profile.

The area and the second moment (moment of inertia) of the cross-section are as follows:

$$A = bh\widetilde{A}, \qquad J_z = bh^3\widetilde{J}_z, \tag{2.3}$$

where the dimensionless area and second moment are

$$\widetilde{A} = \int\limits_{-1/2}^{1/2} f_b(\eta)\,\mathrm{d}\eta, \qquad \widetilde{J}_z = \int\limits_{-1/2}^{1/2} \eta^2 f_b(\eta)\,\mathrm{d}\eta. \tag{2.4}$$

The beam is under the non-uniformly distributed load of intensity $q(x)$ (Fig. 2).

The intensity of the load is as follows:

$$q(\xi) = C_q\left[(1-\xi)\xi\right]^n \frac{F}{L}, \tag{2.5}$$

where n – exponent-natural number, $\xi = x/L$ – dimensionless coordinate $(0 \leq \xi \leq 1)$, F – force-total load,

$$C_q = \left\{\int\limits_0^1 \left[(1-\xi)\xi\right]^n \mathrm{d}\xi\right\}^{-1} \quad \text{– coefficient.}$$

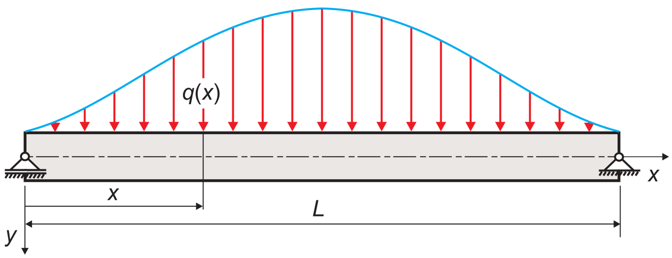

Fig. 2. Scheme of the simply-supported beam under non-uniformly distributed load.

The natural number n is decisive for the load type. Therefore, the expression (2.5) describes the loads from uniformly distributed ($n = 0$) to concentrated force (three-point bending, $n \to \infty$).

Example values of the coefficient C_q are specified in Table 1.

Table 1. Values of the coefficient C_q.

n	0	1	2	3	4	5	6
C_q	1	6	30	140	630	2772	12012

The shear force and bending moment are as follows:

$$T(\xi) = \left\{ \frac{1}{2} - C_q \int_0^{\xi} [(1-\xi_1)\xi_1]^n \, \mathrm{d}\xi_1 \right\} F, \tag{2.6}$$

$$M_b(\xi) = \left\{ \frac{1}{2}\xi - C_q \int_0^{\xi} (\xi - \xi_1) [(1-\xi_1)\xi_1]^n \, \mathrm{d}\xi_1 \right\} FL. \tag{2.7}$$

The example diagrams of the dimensionless intensity of the load (2.5), dimensionless shear force (2.6) and dimensionless bending moment (2.7) for $n = 0$, 2 and 500 are shown in Fig. 3.

The deformation of a planar cross-section of the beam is shown in Fig. 4.

The longitudinal displacement in accordance with the scheme (Fig. 4) is as follows:

$$u(x,y) = -h\left[\eta \frac{\mathrm{d}v}{\mathrm{d}x} - f_d(\eta)\,\psi(x)\right], \tag{2.8}$$

where $v(x)$ is deflection, $f_d(\eta)$ – dimensionless function of deformation, $\psi(x) = u_1(x)/h$ – dimensionless function of shear effect.

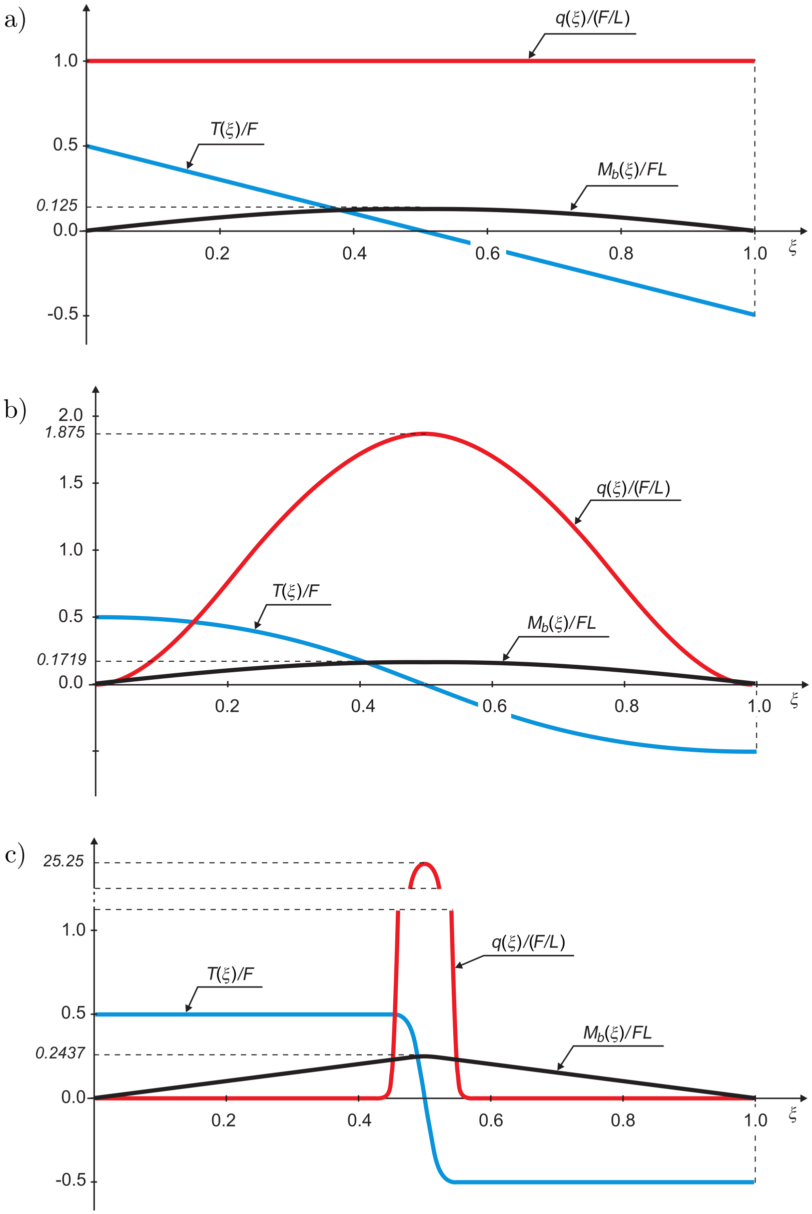

FIG. 3. The example diagrams of the intensity of the load (2.5), shear force (2.6) and bending moment (2.7): a) for $n = 0$ – the uniformly distributed load, b) for $n = 2$ – the non-uniformly distributed load, c) for $n = 500$ – the load approximating the three-point bending.

By taking into account the shear deformation theories presented in several papers quoted in the References, the individual function of deformation of a planar cross-section of the beam is developed. The graph of the function is perpendic-

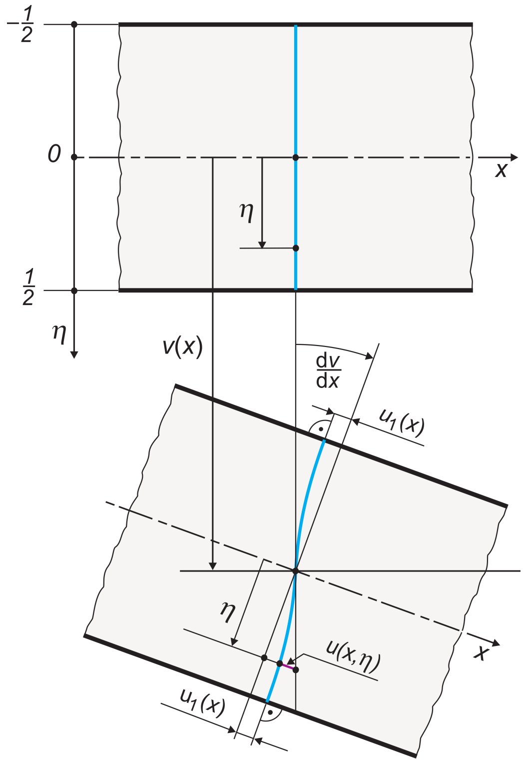

FIG. 4. Scheme of the deformation of a planar cross-section – the nonlinear hypothesis.

ular to the upper and lower surfaces of the beam (Fig. 4). This dimensionless function of deformation takes the following form:

$$f_d\,(\eta) = \sum_{j=1}^{4} \beta_{2j-1}\left(3\eta - 4\eta^3\right)^{2j-1}, \tag{2.9}$$

where β_{2j-1} – coefficients, $j = 1, 2, 3, 4$ – numbers.

Three boundary conditions for $\eta = \pm 1/2$ impose the constraints on this function:

1) $f_d\left(\pm\frac{1}{2}\right) = 1$, from which $\beta_7 = 1 - (\beta_1 + \beta_3 + \beta_5)$, therefore, the function (2.9) for $j = 1, 2, 3$ is as follows:

$$f_d(\eta) = \left\{\sum_{j=1}^{3} \beta_{2j-1}\left[1 - \left(3\eta - 4\eta^3\right)^{2(4-j)}\right] \cdot \left(3\eta - 4\eta^3\right)^{2(j-1)} + \left(3\eta - 4\eta^3\right)^6\right\}\left(3\eta - 4\eta^3\right), \tag{2.10}$$

2) $\left.\frac{\mathrm{d} f_d}{\mathrm{d}\eta}\right|_{\pm 1/2} = 0$, the first derivative of the function (2.10) satisfies this condition,

$$\frac{\mathrm{d} f_d}{\mathrm{d}\eta} = 3\left\{\sum_{j=1}^{3} \beta_{2j-1}\left[j - 7\left(3\eta - 4\eta^3\right)^{2(4-j)}\right] \cdot \left(3\eta - 4\eta^3\right)^{2(j-1)} + 7(3\eta - 4\eta^3)^6\right\}(1 - 4\eta^2), \tag{2.11}$$

3) $0 \le \left.\frac{\mathrm{d}^2 f_d}{\mathrm{d}\eta^2}\right|_{-1/2}$ and $\left.\frac{\mathrm{d}^2 f_d}{\mathrm{d}\eta^2}\right|_{1/2} \le 0$ – condition for the convex function (2.10), from which

$$\beta_5 \le \frac{7}{2} - (3\beta_1 + 2\beta_3)\,. \tag{2.12}$$

The strains:

$$\begin{aligned} \varepsilon_x(x,y) &= \frac{\partial u}{\partial x} = -h\left[\eta\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - f_d\left(\eta\right)\frac{\mathrm{d}\psi}{\mathrm{d}x}\right], \\ \gamma_{xy}(x,y) &= \frac{\partial u}{\partial y} + \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d} f_d}{\mathrm{d}\eta}\psi(x). \end{aligned} \tag{2.13}$$

The stresses – Hooke's law

$$\begin{aligned} \sigma_x(x,y) &= -Eh\left[\eta\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - f_d(\eta)\frac{\mathrm{d}\psi}{\mathrm{d}x}\right], \\ \tau_{xy}(x,y) &= G\frac{\mathrm{d} f_d}{\mathrm{d}\eta}\psi(x), \end{aligned} \tag{2.14}$$

where E, $G = \frac{E}{2(1+\nu)}$, ν – moduli of elasticity and Poisson's ratio.

Taking into account the expressions (2.1) and (2.14) the bending moment is as follows:

$$M_b(x) = \int\limits_A y\sigma_x(x,y)\,\mathrm{d}A = -Ebh^3\left(\widetilde{J}_z\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - C_{v\psi}\frac{\mathrm{d}\psi}{\mathrm{d}x}\right), \tag{2.15}$$

where $C_{v\psi} = \int\limits_{-1/2}^{1/2} \eta f_b(\eta) f_d(\eta)\,\mathrm{d}\eta$ – dimensionless coefficients.

The elastic strain energy of the beam

$$U_\varepsilon = \frac{1}{2} E b h^3 \int_0^L \int_{-1/2}^{1/2} \left\{ \left[\eta \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - f_d(\eta) \frac{\mathrm{d}\psi}{\mathrm{d}x} \right]^2 + \frac{1}{2(1+\nu)} \left(\frac{\mathrm{d} f_d}{\mathrm{d}\eta} \right)^2 \frac{\psi^2(x)}{h^2} \right\} f_b(\eta) \, \mathrm{d}\eta \, \mathrm{d}x. \tag{2.16}$$

The work of the load

$$W = \int_0^L q(x) v(x) \, \mathrm{d}x. \tag{2.17}$$

The system of two differential equations of equilibrium, based on the principle of stationary total potential energy $\delta(U_\varepsilon - W) = 0$, is in the following form:

$$\begin{aligned} \widetilde{J}_z \frac{\mathrm{d}^4 v}{\mathrm{d}x^4} - C_{v\psi} \frac{\mathrm{d}^3 \psi}{\mathrm{d}x^3} &= \frac{q(x)}{Ebh^3}, \\ C_{v\psi} \frac{\mathrm{d}^3 v}{\mathrm{d}x^3} - C_{\psi\psi} \frac{\mathrm{d}^2 \psi}{\mathrm{d}x^2} + \frac{C_\psi}{2(1+\nu)} \frac{\psi(x)}{h^2} &= 0, \end{aligned} \tag{2.18}$$

where

$$C_{\psi\psi} = \int_{-1/2}^{1/2} f_b(\eta) f_d^2(\eta) \, \mathrm{d}\eta,$$

$$C_\psi = \int_{-1/2}^{1/2} f_b(\eta) \left(\frac{\mathrm{d} f_d}{\mathrm{d}\eta} \right)^2 \mathrm{d}\eta \text{ – dimensionless coefficients.}$$

The first equation of the system is equivalent to the bending moment (2.15) since the moment (2.15) differentiated twice gives the equation $(2.18)_1$. Therefore, for the analytical study of the beam bending with consideration of a shear effect, the two following governing equations are applied:

$$\begin{aligned} \widetilde{J}_z \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - C_{v\psi} \frac{\mathrm{d}\psi}{\mathrm{d}x} &= -\frac{M_b(x)}{Ebh^3}, \\ C_{v\psi} \frac{\mathrm{d}^3 v}{\mathrm{d}x^3} - C_{\psi\psi} \frac{\mathrm{d}^2 \psi}{\mathrm{d}x^2} + \frac{C_\psi}{2(1+\nu)} \frac{\psi(x)}{h^2} &= 0. \end{aligned} \tag{2.19}$$

3. Analytical solution

This system is reduced to one differential equation in the following form:

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} - (\alpha\lambda)^2\,\psi(\xi) = -\frac{C_{v\psi}}{\widetilde{J}_z C_{\psi\psi} - C_{v\psi}^2}\lambda^2 \widetilde{T}(\xi)\,\frac{F}{Ebh}, \tag{3.1}$$

where dimensionless shear force with consideration of the expression (2.6) is

$$\widetilde{T}(\xi) = \frac{1}{2} - C_q \int\limits_0^{\xi} \left[(1-\xi_1)\,\xi_1\right]^n \,\mathrm{d}\xi_1, \tag{3.2}$$

and $\alpha = \sqrt{\frac{1}{2(1+\nu)}\frac{\widetilde{J}_z C_{\psi}}{\widetilde{J}_z C_{\psi\psi} - C_{v\psi}^2}}$ – dimensionless coefficient, $\lambda = \frac{L}{h}$ – relative length.

The solution of the equation (3.1) is a sum of the homogeneous and particular solutions in the form

$$\psi(\xi) = \left[C_1 \sinh(\alpha\lambda\xi) + C_2\cosh(\alpha\lambda\xi) + \widetilde{\psi}_p(\xi)\right]\frac{F}{Ebh}, \tag{3.3}$$

where C_1, C_2 – arbitrary constants, $\widetilde{\psi}_p(\xi)$ – particular solution.

The function of shear effect (3.3) at the middle of the beam length ($\xi = 1/2$) is zero, and also $\psi_p(1/2) = 0$, from which $C_2 = -C_1\tanh(\alpha\lambda/2)$, therefore

$$\psi(\xi) = \left[-C_1\frac{\sinh\left[\alpha\lambda\,(1/2-\xi)\right]}{\cosh(\alpha\lambda/2)} + \widetilde{\psi}_p(\xi)\right]\frac{F}{Ebh}. \tag{3.4}$$

This function for selected values of the natural numbers ($n = 0, 1, 2$) of the dimensionless force (3.2), with consideration of the boundary condition $\left.\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right|_0 = 0$, is as follows:

- $\boldsymbol{n} = \mathbf{0}$, then $\widetilde{T}(\xi) = \frac{1}{2} - \xi$ (uniformly distributed load), therefore

$$\psi(\xi) = 2\,(1+\nu)\left\{-\frac{\sinh\left[\alpha\lambda\,(1/2-\xi)\right]}{\alpha\lambda\cosh(\alpha\lambda/2)} + \frac{1}{2} - \xi\right\}\frac{C_{v\psi}}{\widetilde{J}_z C_{\psi}}\frac{F}{Ebh}, \tag{3.5}$$

- $\boldsymbol{n} = \mathbf{1}$, then $\widetilde{T}(\xi) = \frac{1}{2} - 3\xi^2 + 2\xi^3$, therefore

$$\psi(\xi) = 2\,(1+\nu)\left\{\alpha_1^{(1)}\frac{\sinh\left[\alpha\lambda\,(1/2-\xi)\right]}{\alpha\lambda\cosh(\alpha\lambda/2)} + f_{\psi}^{(1)}(\xi)\right\}\frac{C_{v\psi}}{\widetilde{J}_z C_{\psi}}\frac{F}{Ebh}, \tag{3.6}$$

where

$$f_{\psi}^{(1)}(\xi) = \alpha_0^{(1)} + \alpha_1^{(1)}\xi - 3\xi^2 + 2\xi^3, \qquad \alpha_0^{(1)} = \frac{1}{2}\left[1-\alpha_1^{(1)}\right], \qquad \alpha_1^{(1)} = \frac{12}{(\alpha\lambda)^2},$$

- $\boldsymbol{n = 2}$, then $\widetilde{T}(\xi) = \frac{1}{2} - 10\xi^3 + 15\xi^4 - 6\xi^5$, therefore

$$\psi(\xi) = 2\,(1+\nu)\left\{-\alpha_1^{(2)} \frac{\sinh\left[\alpha\lambda\,(1/2-\xi)\right]}{(\alpha\lambda)^2 \cosh\left(\alpha\lambda/2\right)} + f_\psi^{(2)}(\xi)\right\} \frac{C_{v\psi}}{\widetilde{J}_z C_\psi} \frac{F}{Ebh}, \tag{3.7}$$

where

$$f_\psi^{(2)}(\xi) = \alpha_0^{(2)} - \alpha_1^{(2)}\xi + \alpha_2^{(2)}\xi^2 - \alpha_3^{(2)}\xi^3 + 15\xi^4 - 6\xi^5,$$

$$\alpha_0^{(2)} = \frac{1}{2}\left[1 + \frac{720}{(\alpha\lambda)^4}\right], \qquad \alpha_1^{(2)} = \frac{6}{(\alpha\lambda)^2}\alpha_3^{(2)},$$

$$\alpha_2^{(2)} = \frac{180}{(\alpha\lambda)^2}, \qquad \alpha_3^{(2)} = 10\left[1 + \frac{12}{(\alpha\lambda)^2}\right].$$

By taking into account the expressions (3.5), (3.6), and (3.7), with consideration of the condition $50 \le \alpha\lambda$ for the engineering structures of slender beams, the shear function may be assumed in the following form

$$\psi(\xi) = 2\,(1+\nu)\,\widetilde{T}(\xi)\,\frac{C_{v\psi}}{\widetilde{J}_z C_\psi} \frac{F}{Ebh}. \tag{3.8}$$

The first equation of the system (2.19) in the dimensionless coordinate ξ is as follows:

$$\widetilde{J}_z \frac{\mathrm{d}^2\widetilde{v}}{\mathrm{d}\xi^2} = \left[C_{v\psi}\frac{\mathrm{d}\widetilde{\psi}}{\mathrm{d}\xi} - \widetilde{M}_b(\xi)\lambda^2\right] \frac{F}{Ebh}, \tag{3.9}$$

where $\widetilde{v}(\xi) = \frac{v(\xi)}{L}$ – relative deflection, and $\widetilde{\psi}(\xi) = 2\,(1+\nu)\,\widetilde{T}(\xi)\frac{C_{v\psi}}{\widetilde{J}_z C_\psi}$.

Integrating the equation (3.9) one obtains

$$\widetilde{J}_z \frac{\mathrm{d}\widetilde{v}}{\mathrm{d}\xi} = \left[C_3 + C_{v\psi}\widetilde{\psi}(\xi) - \lambda^2 \int \widetilde{M}_b(\xi)\,\mathrm{d}\xi\right] \frac{F}{Ebh}, \tag{3.10}$$

where the integration constant C_3, based on the condition $\mathrm{d}\widetilde{v}/\,\mathrm{d}\xi|_{1/2} = 0$ for $\xi = 1/2$, is $C_3 = \lambda^2 \int\limits_0^{1/2} \widetilde{M}_b(\xi)\,\mathrm{d}\xi$.

Consequently, after the second integration

$$\widetilde{J}_z\widetilde{v}(\xi) = \left\{C_4 + C_3\xi + 2\,(1+\nu)\,\widetilde{M}_b(\xi)\,\frac{C_{v\psi}^2}{\widetilde{J}_z C_\psi} - \lambda^2 \iint \widetilde{M}_b(\xi)\,\mathrm{d}\xi^2\right\} \frac{F}{Ebh}, \tag{3.11}$$

where the integration constant C_4, based on the condition $\widetilde{v}(0) = 0$, is zero ($C_4 = 0$).

Thus, the maximum relative deflection is

$$\widetilde{v}_{\max} = \widetilde{v}\left(\frac{1}{2}\right) = \widetilde{v}^{o}_{\max}\frac{F}{Ebh}, \tag{3.12}$$

where

$$\widetilde{v}^{o}_{\max} = \left(1 + \frac{C_{vs}}{\lambda^2}\right) C_v \frac{\lambda^2}{\widetilde{J}_z},$$

$$C_v = \frac{1}{2}\int_0^{1/2} \widetilde{M}_b(\xi)\,\mathrm{d}\xi - \iint_0^{1/2} \widetilde{M}_b(\xi)\,\mathrm{d}\xi^2 \text{ – deflection coefficient,}$$

$$C_{vs} = 2(1+\nu)\frac{C_m}{C_v}\widetilde{M}_b\left(\frac{1}{2}\right) \text{ – shear coefficient,}$$

$$C_m = \max_{\beta_{2j-1}}\left(\frac{C^2_{v\psi}}{\widetilde{J}_z C_\psi}\right), \quad \text{for } j = 1, 2, 3.$$

The shear stresses (2.14), with consideration of the shear function (3.8), are as follows:

$$\tau^{(An)}_{\xi\eta}(\xi,\eta) = \widetilde{\tau}^{(An)}_{\xi\eta}(\xi,\eta)\frac{F}{bh}, \tag{3.13}$$

where dimensionless shear stresses are

$$\widetilde{\tau}^{(An)}_{\xi\eta}(\xi,\eta) = \widetilde{T}(\xi)\frac{\mathrm{d}f_d}{\mathrm{d}\eta}\frac{C_{v\psi}}{\widetilde{J}_z C_\psi}. \tag{3.14}$$

The problem of the shear stresses for elastic beams was analyzed in the mid of the 19th century by D.I. Zhuravsky and described in the following formula [1]:

$$\tau^{(Zh)}_{\xi\eta}(\xi,\eta) = \frac{T(\xi)S^*_z(\eta)}{b(\eta)J_z}, \tag{3.15}$$

where $T(\xi)$ – shear force, $S^*_z(\eta)$ – first moment of the part of the cross-section, $b(\eta)$ – width of the cross-section (Fig. 1), J_z – second moment of the cross-section.

Therefore, the Zhuravsky formula of dimensionless shear stress is as follows:

$$\widetilde{\tau}^{(Zh)}_{\xi\eta}(\xi,\eta) = \frac{\widetilde{T}(\xi)\widetilde{S}^*_z(\eta)}{f_b(\eta)\widetilde{J}_z}, \tag{3.16}$$

where

$$\widetilde{S}_z^*(\eta) = -\int\limits_{-1/2}^{\eta} \eta_1 f_b(\eta_1)\,\mathrm{d}\eta_1, \qquad -\frac{1}{2} \le \eta_1 \le \eta.$$

4. Example calculations

Three example cross-sections of the prismatic beam (Fig. 5) are assumed for analytical studies.

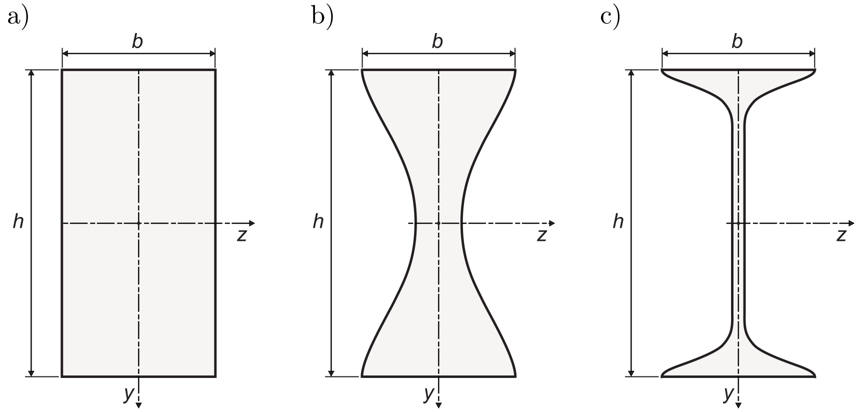

Fig. 5. Three example cross-sections of the beam:
a) CS-1 ($\beta_0 = 1$, $k_c = 0$), b) CS-2 ($\beta_0 = 0.3$, $k_c = 1$), c) CS-3 ($\beta_0 = 0.08$, $k_c = 10$).

The values of the coefficients β_{2j-1} $(j = 1, 2, 3)$ and C_m (3.12) calculated for the example cross-sections are specified in Table 2.

Table 2. The values of coefficients β_{2j-1} $(j = 1, 2, 3)$ and C_m for the example cross-sections.

	β_1	β_3	β_5	C_m
CS-1	1	0	0	0.1
CS-2	1.3118	−0.3701	0.006238	0.18604
CS-3	0.9345	−0.2296	1.0423	0.36972

The values of the shear coefficient C_{vs} (3.12) calculated for the beam with example cross-sections and selected load cases (2.5) are specified in Table 3.

The values of the shear coefficient C_{vs} increase with concentrated load – approach to three-point bending $(n \to \infty)$.

The deflection coefficient C_v (3.12) as a function of the exponent–natural number n is shown in Fig. 6.

Table 3. The values of the shear coefficient C_{vs} for the beam with example cross-sections.

n	0	10	100	400
C_{vs}/CS-1	2.4960	2.7557	2.9662	3.0377
C_{vs}/CS-2	4.6435	5.1266	5.5182	5.6512
C_{vs}/CS-3	9.2282	10.1882	10.9656	11.2310

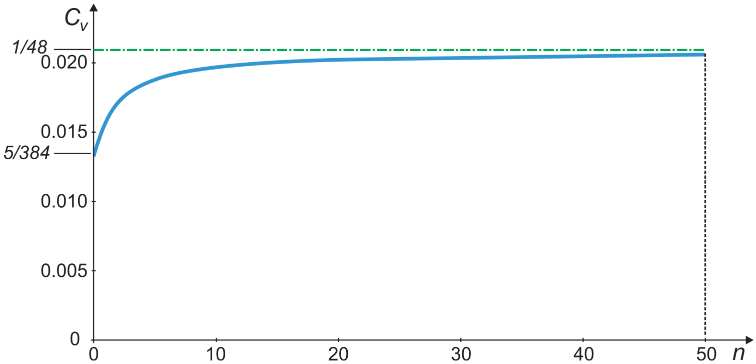

FIG. 6. Graph of the deflection coefficient C_v as a function of the natural number n.

The value of this coefficient for $n = 0$ (uniformly distributed load) is $C_v = 5/384$, while for $n \to \infty$ (three-point bending) is $C_v = 1/48$.

The graphs of the analytical (3.14) and Zhuravsky's (3.16) shear stresses for shear force $\widetilde{T}(0) = 1/2$ in the three example cross-sections of the beam are shown in Fig. 7.

The graphs of the analytical (3.14) and Zhuravsky's (3.16) shear stresses for the rectangular cross-section CS-1 are identical. Nevertheless, for the cross-sections CS-2 and CS-3 the graphs are similar, the differences between the maximum values of these stresses for $\eta = 0$ are 2.8% and 2.1%, respectively. These differences are a consequence of the approximation, i.e., a limited number of the coefficients of the series determining the dimensionless function of deformation (2.9) to seventh order.

By taking into account the engineering practice, the example analytical shaping of the bisymmetrical cross-section based on the function (2.1) is presented for the selected standard I-beam cross-sections. The values of the parameter β_0, the dimensionless area $\widetilde{A}^{(St)} = A^{(St)}/bh$ and the second moment $\widetilde{J}_z^{(St)} = J_z^{(St)}/bh^3$ of the standard I-beams are specified in Table 4.

The results of the analytical calculations of the value of the exponent k_c (2.2) with consideration of the condition $\widetilde{J}_z^{(An)} = \widetilde{J}_z^{(St)}$ are specified in Table 5.

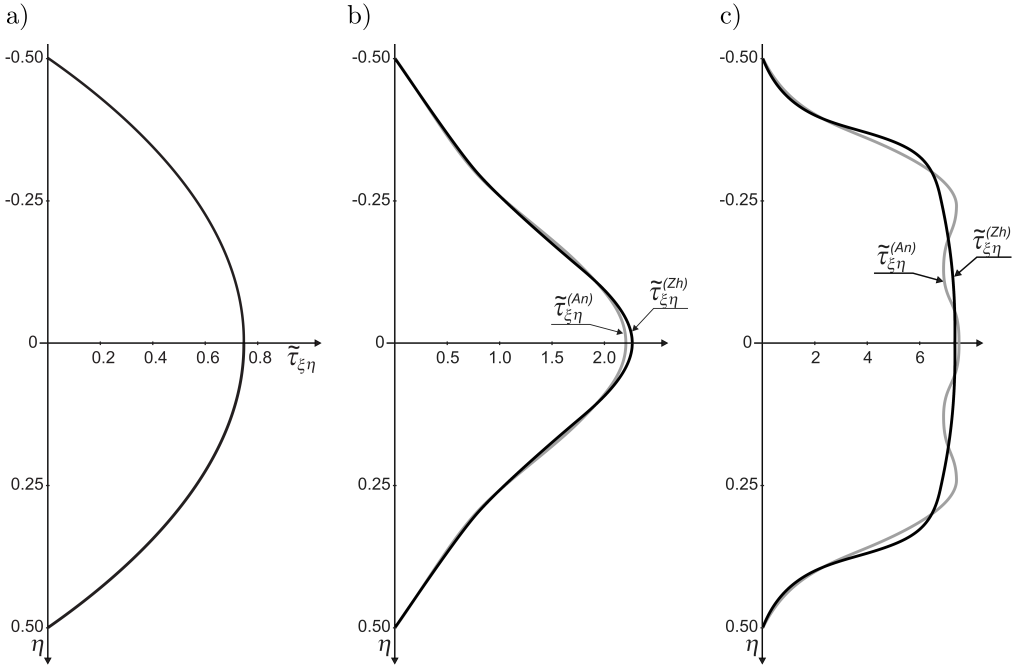

FIG. 7. Graphs of the analytical (3.14) and Zhuravsky's (3.16) shear stresses in the cross-sections: a) CS-1, b) CS-2, c) CS-3.

Table 4. The dimensionless values of the area and second moment of the standard I-beams.

Cross-section	I-100	I-200	I-300	I-400	I-500
β_0	4.5/50	7.5/90	10.8/125	14.4/155	18.0/185
$\widetilde{A}^{(St)}$	0.2120	0.1861	0.1843	0.1903	0.1946
$\widetilde{J}_z^{(St)}$	0.03420	0.02972	0.02904	0.02945	0.02973

Table 5. The dimensionless values of the area and second moment – analytical calculations.

Cross-section	I-100	I-200	I-300	I-400	I-500
k_c	8.127	11.80	12.84	12.81	12.79
$\widetilde{A}^{(An)}$	0.2191	0.1896	0.1876	0.1935	0.1975
$\widetilde{J}_z^{(An)}$	0.03420	0.02972	0.02904	0.02945	0.02973
$\Delta\widetilde{A}/\widetilde{A}^{(St)}$ [%]	3.3	1.9	1.8	1.7	1.5

The relative differences between the values of the dimensionless area of the standard cross-section and the results obtained analytically ($\Delta\widetilde{A}/\widetilde{A}^{(St)}$, where $\Delta\widetilde{A} = \widetilde{A}^{(An)} - \widetilde{A}^{(St)}$) for I-100 beam amount to 3.3%, while for I-beams they are smaller (1.9–1.5%).

5. Conclusions

The analytical studies of bending of the simply-supported prismatic beams with bisymmetrical cross-sections under non-uniformly distributed load enable formulating the following notes:

- the function (2.2) with the parameter β_0 and exponent k_c – positive real number shapes the bisymmetrical cross-sections from the rectangular to I-beams,
- the proposed individual seventh-order shear deformation theory of planar cross-sections of the beam (2.9) is novel and describes the displacements and shear stresses.

The presented analytical model of bending beam with consideration of the shear effect is purely theoretical. Nevertheless, it can also be used in the practical computation of the beams. The proposed individual shear deformation theory of the beam (2.9) may be improved in the future.

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