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Research Paper

Nonlinear Vibration of Multi-Walled Carbon Nanotubes with Initial Curvature Resting on Elastic Foundations in a Nonlinear Thermo-Magnetic Environment

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The present work focuses on nonlinear dynamics models of multi-walled carbon nanotubes with initial curvature resting on Winkler-Pasternak elastic foundations in a nonlinear thermomagnetic environment using nonlocal elasticity theory. The derived systems of nonlinear vibration models are solved with the aid of the Galerkin decomposition and the homotopy perturbation method. Effects of temperature, magnetic field, multi-layer, and other thermo-mechanical parameters on the dynamic responses of the slightly curved multi-walled carbon nanotubes are investigated and discussed. As the temperature increases, the frequency ratio decreases as the linear natural frequency of the system increases. The results reveal that the frequency ratios decrease as the number of nanotube walls, temperature, spring constants, magnetic field strength, and the ratio of the radius of curvature to the length of the slightly curved nanotubes increase. These trends are the same for all the boundary conditions considered. However, clamped-simple and clamped-clamped supported multi-walled nanotube have the highest and lowest frequency ratio, respectively. Also, from the parametric studies to control nonlinear vibration of the carbon nanotubes, it is shown that quadruple-walled carbon nanotubes can be taken as pure linear vibration even at any value of linear Winkler and Pasternak constants. Therefore, this can be used for the restraining of the chaos vibration in the objective structure. These research findings will assist the designers and manufacturers in developing multi-walled carbon nanotubes for various structural, electrical, mechanical, and biological applications, especially in the areas of designing nanoelectronics, nanodevices, nanomechanical systems, nanobiological devices, and nanocomposites, and particularly when they are subjected to thermal loads, magnetic fields and elastic foundations.

Key words: small-scale effects; multi-walled carbon nanotubes; nonlocal elastic theory; nonlinear thermal effect; magnetic field effect.

Notations

A – area of the nanotube,
c_i – coefficient of the van der Waals force between the i-th tube and the $(i-1)$-th tube,
F_i – van der Waals force between the i-th tube and the $(i-1)$-th tube,
E – Young modulus of elasticity,
EI – bending rigidity,
h_1, h_2, h_3 – Murnaghan's constants,
H_x – magnetic field strength,
I – layer number $i = 1, 2, 3, 4, ..., N$,
I – moment of area,
k_1, k_3 – spring constants/Winkler foundation constants,
k_p – spring constant/Pasternak foundation constant,
L – length of the nanotube,
m_c – mass of tube per unit length,
N – axial/longitudinal force,
r – radius of the nanotube,
ΔT – change in temperature,
t – time coordinate,
w – transverse displacement/deflection of the nanotube,
W – time-dependent parameter,
x – axial coordinate,
Z_o – initial curvature of the tube,
$\phi(x)$ – trial/comparison function,
$EA\alpha_x\Delta T$ – constant axial force due to thermal effects,
ηAH_x^2 – magnetic force per unit length due to Lorentz force,
α_x – coefficient of thermal expansion,
η – magnetic field permeability,
v – Poisson's ratio.

1. Introduction

The discovery of the novel nanostructure materials by Iijima [1] has led to the increasing number of various applications of nanomaterials for the developments of nanoelectronics, nanodevices, nanomechanical systems, nanobiological devices and nanocomposites. This is due to their excellent properties and high strength to weight ratio. However, the carbon nanotubes (Fig. 1) undergo large deformations within the elastic limit and vibrate at the frequency in the order of

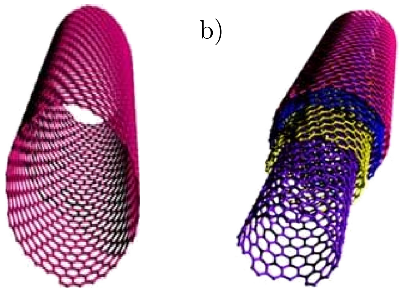


Fig. 1. Typical SWCNT (a), typical MWCNT (b).

GHz and THz. Consequently, there have been large volumes of research studies that investigated or provided insight into the dynamic behaviors of the novel structures [2–13]. The mechanics and buckling behaviors of the nanotubes under external influences were explored by Liew *et al.* [5], Pantano *et al.* [6, 7], Qian *et al.* [8] and Salvetat *et al.* [9], Sears and Batra [10], Yoon *et al.* [11], Wang and Cai [12], Wang *et al.* [13], Zhang *et al.* [14]. The nonlinear vibration analyses of the single-walled carbon nanotubes (SWCNTs) was presented by Elishakoff and Pentaras [15], Buks and Yurke [16], Postma *et al.* [17], Fu *et al.* [18], Dequesnes *et al.* [19], Ouakad and Younis [20], Zamanian *et al.* [21], and Belhadj *et al.* [22]. Theoretical investigations of double-walled carbon nanotubes (DWCNTs) have been conducted by Abdel-Rahman and Nayfeh [23], Hawwa and Al-Qahtani [24], Hajnayeb and Khadem [25], Xu and Guo [26], Lei *et al.* [27] and Ghorbanpour *et al.* [28]. The analyses of the carbon nanotubes were extended to multi-walled carbon nanotubes (MWCNTs) [29–33].

Studies on vibrations of SWCNTs, DWCNTs and MWCNTs presented in the literature use experimental measurements, density functional theory, molecular dynamics simulations, and continuum mechanics. There are some difficulties in the experiment investigations at the nanoscale level. Therefore, the majority of the past works are based on theoretical investigations using classical continuum models (which do not consider the small-scale effects). However, due to their scale-free models, as they cannot incorporate the small-scale effects in their formulations, the classical continuum theories are inadequate for accurate predictions of the nanotubes' dynamic behaviors. Such inadequacy in the classical continuum models is corrected in the work of Eringen [34–37], where the author developed nonlocal continuum mechanics based on nonlocal elasticity theory. Although some studies in the literature have used the nonlocal continuum mechanics to present some theoretical investigations, to the best of the authors' knowledge, a study on the simultaneous influences of nonlinear thermo-magneto-mechanical parameters on the nonlinear vibration of slightly curved multi-walled carbon nanotubes subjected to linear and nonlinear elastic foundations using nonlocal elasticity theory has not been presented in the past studies. Therefore, the present article, using the Galerkin decomposition-homotopy perturbation method, studies the impacts of nonlinear thermal loads, magnetic field, boundary conditions, linear and nonlinear elastic foundations on the nonlinear dynamic behavior of slightly curved multi-walled nanotubes. Parametric studies are presented and the results are discussed. The study aims to better design multi-walled nanoelectronics, nanodevices, nanomechanical systems, nanobiological devices and nanocomposites under the influences of linear and nonlinear thermal loads, magnetic fields, and linear and nonlinear elastic foundations.

2. Model development for SWCNT and MWCNT

Consider a slightly curved SWCNT and DWCNT subjected to stretching effects and resting on Winkler and Pasternak foundations in a thermomagnetic environment as depicted in Fig. 2. Applying Eringen's nonlocal elasticity theory, Euler-Bernoulli beam theory and Hamilton's principle, the equation of motion for the SWCNT under the influence of linear and nonlinear elastic foundations in a nonlinear thermal and magnetic environment is developed as

$$
\begin{aligned}
(2.1)\quad & EI\frac{\partial^4 w}{\partial x^4} + m_c\frac{\partial^2 w}{\partial t^2} + k_{w1}w + k_{w3}w^3 \\
& - \left[\frac{EA}{2L}\int_0^L \left(\frac{\partial Z_o}{\partial x}\frac{\partial w}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2\right)\mathrm{d}x\right]\left(\frac{\partial^2 Z_o}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) \\
& - \left[EA\alpha_x\Delta T + \left[h_1(1-2v) - 2h_2(v^2-1) + h_3v^2\right]A(\alpha_x)^2(\Delta T)^2 + \eta AH_x^2 + k_p\right]\frac{\partial^2 w}{\partial x^2} \\
& + (e_oa)^2\Bigg[m_c\frac{\partial^4 w}{\partial x^2\partial t^2} + k_p\frac{\partial^2 w}{\partial x^2} + 6k_3w\left(\frac{\partial w}{\partial x}\right)^2 + 3k_3w^2\left(\frac{\partial w}{\partial x}\right) \\
& - \left[\frac{EA}{2L}\int_0^L \left(\frac{\partial Z_o}{\partial x}\frac{\partial w}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2\right)\mathrm{d}x\right]\left(\frac{\partial^4 Z_o}{\partial x^4} + \frac{\partial^4 w}{\partial x^4}\right) \\
& - \left[EA\alpha_x\Delta T + \left[h_1(1-2v) - 2h_2(v^2-1) + h_3v^2\right]A(\alpha_x)^2(\Delta T)^2 \right. \\
& \left. + \eta AH_x^2 + k_p\right]\frac{\partial^4 w}{\partial x^4}\Bigg] = 0.
\end{aligned}
$$

Further works on developing the governing equation of motion for the SWCNT can be found in our previous studies [38–41]. Figure 3 shows the embedded slightly curved DWCNT on elastic foundations.

Following the derivation procedures in our previous studies [39–41], the governing equations of motion for the multi-walled carbon nanotube under the

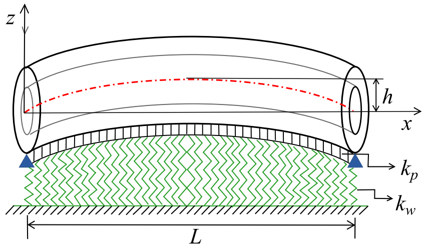


Fig. 2. Simply-supported embedded slightly curved SWCNT on the two-parameter elastic foundation [42].

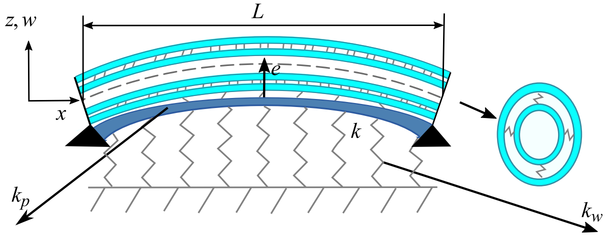


Fig. 3. Simply-supported embedded slightly curved DWCNT on elastic foundations [43].

influence of linear and nonlinear elastic foundations in a nonlinear thermal and magnetic environment is developed as

$$
\begin{aligned}
(2.2)\quad & EI\frac{\partial^4 w_i}{\partial x^4} + m_c\frac{\partial^2 w_i}{\partial t^2} + kw_i + k_w w_i^3 \\
& - \left[\frac{EA}{2L}\int_0^L \left(\frac{\partial Z_o}{\partial x}\frac{\partial w_i}{\partial x} + \left(\frac{\partial w_i}{\partial x}\right)^2\right) \mathrm{d}x\right]\left(\frac{\partial^2 Z_o}{\partial x^2} + \frac{\partial^2 w_i}{\partial x^2}\right) \\
& - \left[EA_i\alpha_x\Delta T + \left[h_{1i}(1-2v) - 2h_{2i}(v^2-1) + h_{3i}v^2\right]A_i(\alpha_x)^2(\Delta T)^2 + \eta A_i H_x^2\right]\frac{\partial^2 w_i}{\partial x^2} \\
& + (e_o a)^2\Bigg[m_c\frac{\partial^4 w_i}{\partial x^2\partial t^2} + k\frac{\partial^2 w_i}{\partial x^2} + 6k_w w_i\left(\frac{\partial w_i}{\partial x}\right)^2 + 3k_w w_i^2\left(\frac{\partial w_i}{\partial x}\right) \\
& - \left[\frac{EA}{2L}\int_0^L \left(\frac{\partial Z_o}{\partial x}\frac{\partial w_i}{\partial x} + \left(\frac{\partial w_i}{\partial x}\right)^2\right) \mathrm{d}x\right]\left(\frac{\partial^4 Z_o}{\partial x^4} + \frac{\partial^4 w_i}{\partial x^4}\right) \\
& - \left[EA_i\alpha_x\Delta T + \left[h_{1i}(1-2v) - 2h_{2i}(v^2-1) + h_{3i}v^2\right]A_i(\alpha_x)^2(\Delta T)^2 \right. \\
& \left. + \eta A_i H_x^2\right]\frac{\partial^4 w_i}{\partial x^4}\Bigg] + F_i = 0,
\end{aligned}
$$

where the number of layer is represented by i and $i = 1, 2, 3, 4, ..., N$, and F_i are terms that represent the interlayer interactions for the MWCNTs with N layers. The elastic foundations' terms k_p and $kw_i + k_w w_i^3$, will be included in the governing equation only for the outer tube that interacts with the elastic foundations.

The development of the expression for F_i is based on the fact that the pressure at any point between any two adjacent tubes depends on the difference of their deflections at that point. Consequently, using the linear form of the van der Waals forces, one can express the van der Waals force between the i-th nanotube and the $(i-1)$-th nanotube as

$$F_i = c_i(w_i - w_{i-1}). \tag{2.3}$$

It is assumed that all nested individual tubes of the MWCNT vibrate in the same plane. Using the van der Waals forces in Eq. (2.2), the developed nonlinear governing equations of vibration for the embedded MWCNT in a nonlinear thermal and magnetic environment with N layers are given in expanded form as

$$\begin{aligned} EI_1\frac{\partial^4 w_1}{\partial x^4}+m_{c_1}\frac{\partial^2 w_1}{\partial t^2}-\left[\frac{EA_1}{2L}\int_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_1}{\partial x}+\left(\frac{\partial w_1}{\partial x}\right)^2\right)\mathrm{d}x\right]\left(\frac{\partial^2 Z_o}{\partial x^2}+\frac{\partial^2 w_1}{\partial x^2}\right)\\ -\left[EA_1\alpha_x\Delta T+\left[h_{11}(1-2v)-2h_{21}(v^2-1)+h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1 H_x^2\right]\frac{\partial^2 w_1}{\partial x^2}\\ -(e_o a)^2\left[m_{c1}\frac{\partial^4 w_1}{\partial x^2\partial t^2}-\left[\frac{EA_1}{2L}\int_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_1}{\partial x}+\frac{\partial^2 w_1}{\partial x^2}\right)^2\mathrm{d}x\right]\left(\frac{\partial^4 Z_o}{\partial x^4}+\frac{\partial^4 w_1}{\partial x^4}\right)\right.\\ -\left[EA_1\alpha_x\Delta T+\left[h_{11}(1-2v)-2h_{21}(v^2-1)+h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1 H_x^2\right]\frac{\partial^4 w_1}{\partial x^4}\\ \left.+c_1\left(\frac{\partial^2 w_2}{\partial x^2}-\frac{\partial^2 w_1}{\partial x^2}\right)\right]+c_1(w_2-w_1)=0, \end{aligned} \tag{2.4}$$

$$\begin{aligned} EI_2\frac{\partial^4 w_2}{\partial x^4}+m_{c2}\frac{\partial^2 w_2}{\partial t^2}-\left[\frac{EA_2}{2L}\int_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_2}{\partial x}+\left(\frac{\partial w_2}{\partial x}\right)^2\right)\mathrm{d}x\right]\left(\frac{\partial^2 Z_o}{\partial x^2}+\frac{\partial^2 w_2}{\partial x^2}\right)\\ -\left[EA_2\alpha_x\Delta T+\left[h_{12}(1-2v)-2h_{22}(v^2-1)+h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2+\eta A_2 H_x^2\right]\frac{\partial^2 w_2}{\partial x^2}\\ -(e_o a)^2\left[m_{c2}\frac{\partial^4 w_2}{\partial x^2\partial t^2}-\left[\frac{EA_2}{2L}\int_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_2}{\partial x}+\left(\frac{\partial w_2}{\partial x}\right)^2\right)\mathrm{d}x\right]\left(\frac{\partial^4 Z_o}{\partial x^4}+\frac{\partial^4 w_2}{\partial x^4}\right)\right.\\ -\left[EA_2\alpha_x\Delta T+\left[h_{12}(1-2v)-2h_{22}(v^2-1)+h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2+\eta A_2 H_x^2\right]\frac{\partial^4 w_2}{\partial x^4}\\ \left.-c_2\left(\frac{\partial^2 w_2}{\partial x^2}-\frac{\partial^2 w_1}{\partial x^2}\right)+c_2\left(\frac{\partial^2 w_3}{\partial x^2}-\frac{\partial^2 w_2}{\partial x^2}\right)\right]+c_2(w_3-w_2)-c_2(w_2-w_1)=0, \end{aligned} \tag{2.5}$$

$$EI_3\frac{\partial^4 w_3}{\partial x^4}+m_{c3}\frac{\partial^2 w_3}{\partial t^2}-\left[\frac{EA_3}{2L}\int\limits_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_3}{\partial x}+\left(\frac{\partial w_3}{\partial x}\right)^2\right)\mathrm{d}x\right]\left(\frac{\partial^2 Z_o}{\partial x^2}+\frac{\partial^2 w_3}{\partial x^2}\right)$$
$$-\left[EA_3\alpha_x\Delta T+\left[h_{13}(1-2v)-2h_{23}(v^2-1)+h_{33}v^2\right]A_3(\alpha_x)^2(\Delta T)^2+\eta A_3H_x^2\right]\frac{\partial^2 w_3}{\partial x^2}$$
$$-(e_oa)^2\left[m_{c3}\frac{\partial^4 w_3}{\partial x^2\partial t^2}-\left[\frac{EA_3}{2L}\int\limits_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_3}{\partial x}+\left(\frac{\partial w_3}{\partial x}\right)^2\right)\mathrm{d}x\right]\left(\frac{\partial^4 Z_o}{\partial x^4}+\frac{\partial^4 w_3}{\partial x^4}\right)\right.$$
$$-\left[EA_3\alpha_x\Delta T+\left[h_{13}(1-2v)-2h_{23}(v^2-1)+h_{33}v^2\right]A_3(\alpha_x)^2(\Delta T)^2+\eta A_3H_x^2\right]\frac{\partial^4 w_3}{\partial x^4}$$
$$\left.-c_3\left(\frac{\partial^2 w_3}{\partial x^2}-\frac{\partial^2 w_2}{\partial x^2}\right)+c_3\left(\frac{\partial^2 w_4}{\partial x^2}-\frac{\partial^2 w_3}{\partial x^2}\right)\right]+c_3(w_4-w_3)-c_3(w_3-w_2)=0, \tag{2.6}$$

$$EI_4\frac{\partial^4 w_4}{\partial x^4}+m_{c4}\frac{\partial^2 w_4}{\partial t^2}-\left[\frac{EA_4}{2L}\int\limits_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_4}{\partial x}+\left(\frac{\partial w_4}{\partial x}\right)^2\right)\mathrm{d}x\right]\left(\frac{\partial^2 Z_o}{\partial x^2}+\frac{\partial^2 w_4}{\partial x^2}\right)$$
$$-\left[EA_4\alpha_x\Delta T+\left[h_{14}(1-2v)-2h_{24}(v^2-1)+h_{34}v^2\right]A_4(\alpha_x)^2(\Delta T)^2+\eta A_4H_x^2\right]\frac{\partial^2 w_4}{\partial x^2}$$
$$-(e_oa)^2\left[m_{c4}\frac{\partial^4 w_4}{\partial x^2\partial t^2}-\left[\frac{EA_4}{2L}\int\limits_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_4}{\partial x}+\left(\frac{\partial w_4}{\partial x}\right)^2\right)\mathrm{d}x\right]\left(\frac{\partial^4 Z_o}{\partial x^4}+\frac{\partial^4 w_4}{\partial x^4}\right)\right.$$
$$-\left[EA_4\alpha_x\Delta T+\left[h_{14}(1-2v)-2h_{24}(v^2-1)+h_{34}v^2\right]A_4(\alpha_x)^2(\Delta T)^2+\eta A_4H_x^2\right]\frac{\partial^4 w_4}{\partial x^4}$$
$$\left.-c_4\left(\frac{\partial^2 w_4}{\partial x^2}-\frac{\partial^2 w_3}{\partial x^2}\right)+c_4\left(\frac{\partial^2 w_5}{\partial x^2}-\frac{\partial^2 w_4}{\partial x^2}\right)\right]+c_4(w_5-w_4)-c_4(w_4-w_3)=0, \tag{2.7}$$

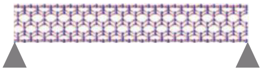
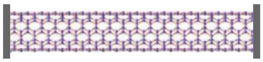
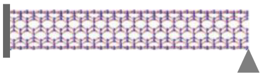
$$\vdots$$

$$EI_N\frac{\partial^4 w_N}{\partial x^4}+m_{c_N}\frac{\partial^2 w_N}{\partial t^2}+kw_N+k_ww_N^3$$
$$-\left[\frac{EA_N}{2L}\int\limits_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_N}{\partial x}+\left(\frac{\partial w_N}{\partial x}\right)^2\right)\mathrm{d}x\right]\left(\frac{\partial^2 Z_o}{\partial x^2}+\frac{\partial^2 w_N}{\partial x^2}\right)$$
$$-\left[EA_N\alpha_x\Delta T+\left[h_{1N}(1-2v)-2h_{2N}(v^2-1)+h_{3N}v^2\right]A_N(\alpha_x)^2(\Delta T)^2\right.$$
$$\left.+\eta A_NH_x^2+k_p\right]\frac{\partial^2 w_N}{\partial x^2}-(e_oa)^2\left[m_{cN}\frac{\partial^4 w_N}{\partial x^2\partial t^2}+k\frac{\partial^2 w_N}{\partial x^2}+6k_ww_N\left(\frac{\partial w_N}{\partial x}\right)^2\right.$$
$$+3k_ww_N^2\left(\frac{\partial w_N}{\partial x}\right)-\left[\frac{EA_N}{2L}\int\limits_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_N}{\partial x}+\left(\frac{\partial w_N}{\partial x}\right)^2\right)\mathrm{d}x\right]\left(\frac{\partial^4 Z_o}{\partial x^4}+\frac{\partial^4 w_N}{\partial x^4}\right)$$
$$-\left[EA_N\alpha_x\Delta T+\left[h_{1N}(1-2v)-2h_{2N}(v^2-1)+h_{3N}v^2\right]A_N(\alpha_x)^2(\Delta T)^2\right.$$
$$\left.\left.+\eta A_NH_x^2+k_p\right]\frac{\partial^4 w_N}{\partial x^4}+c_{N-1}\left(\frac{\partial^2 w_N}{\partial x^2}-\frac{\partial^2 w_{N-1}}{\partial x^2}\right)\right]+c_{N-1}(w_N-w_{N-1})=0. \tag{2.8}$$

It can be noted that the elastic foundations' parameters k_p, k, and k_w do not enter into the equations of the inner tubes. This is because only the outer tube interacts with the elastic foundations.

Table 1 presents the end conditions considered in this work.

Table 1. The basic functions corresponding to the above boundary conditions [39].

Cases	Mode shape, $\phi(x)$	Value of β
1. Simple-Simple support 	$\sin\left(\frac{\beta x}{L}\right)$	π
2. Clamped-Clamped support 	$\left(\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right) - \left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right)$	4.730041
3. Clamped-Simple support 	$\left(\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right) - \left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right)$	3.926602

The following boundary conditions for the multi-walled nanotubes are considered in this work:

- for simply supported (S-S) nanotube,

$$w_i(0,t)=0,\quad \frac{\partial^2 w_i(0,t)}{\partial^2 x}=0,\quad w_i(Lt)=0,\quad \frac{\partial^2 w_i(L,t))}{\partial^2 x}=0. \tag{2.9}$$

- for clamped-clamped supported (C-C) nanotube,

$$w_i(0,t)=0,\quad \frac{\partial w_i(0,t)}{\partial x}=0,\quad w_i((Lt)=0,\quad \frac{\partial w_i(L,t)}{\partial x}=0. \tag{2.10}$$

- for a clamped-simply supported (C-S) nanotube,

$$w_i(0,t)=0,\quad \frac{\partial w_i(0,t)}{\partial x}=0,\quad w_i(Lt)=0,\quad \frac{\partial^2 w_i((L,t)}{\partial^2 x}=0. \tag{2.11}$$

3. Solution methodology: Galerkin decomposition and homotopy perturbation method

The method of solution for the governing equation includes the Galerkin decomposition and homotopy perturbation methods. As the name implies, the

Galerkin decomposition method is used to decompose the governing partial differential equations of motion so they can be separated into spatial and temporal parts. The resulting temporal equations are solved using the homotopy perturbation method.

The procedures for the analysis of the equations are given in the proceeding sections as follows.

3.1. Galerkin decomposition method

With the application of the Galerkin decomposition procedure, the governing partial differential equations of motion can be separated into spatial and temporal parts of the lateral displacement functions as

$$w_i(x,t) = \phi(x)W_i(t), \quad i = 1,2,3,...,N. \tag{3.1}$$

Using a one-parameter Galerkin decomposition procedure, one arrives at

$$\int_0^L R_i(x,t)\phi(x)\,\mathrm{d}x = 0, \tag{3.2}$$

where $R_N(x,t)$ is the governing equation of motion for each tube of the multi-walled nanotubes. From Eq. (2.8), for the outer tube of multi-walled carbon nanotubes one can write that

$$\begin{aligned} R_N(x,t) &= EI_N\frac{\partial^4 w_N}{\partial x^4} + m_{c_N}\frac{\partial^2 w_N}{\partial t^2} + kw_N + k_w w_N^3 \\ &- \left[\frac{EA_N}{2L}\int_0^L \left(\frac{\partial Z_o}{\partial x}\frac{\partial w_N}{\partial x} + \left(\frac{\partial w_N}{\partial x}\right)^2\right)\mathrm{d}x\right]\left(\frac{\partial^2 Z_o}{\partial x^2} + \frac{\partial^2 w_N}{\partial x^2}\right) \\ &- \left[EA_N\alpha_x\Delta T + \left[h_{1N}(1-2v) - 2h_{2N}(v^2-1) + h_{3N}v^2\right]A_N(\alpha_x)^2(\Delta T)^2\right. \\ &\left.+ \eta A_N H_x^2 + k_p\right]\frac{\partial^2 w_N}{\partial x^2} - (e_o a)^2\left[m_{cN}\frac{\partial^4 w_N}{\partial x^2\partial t^2} + k\frac{\partial^2 w_N}{\partial x^2} + 6k_w w_N\left(\frac{\partial w_N}{\partial x}\right)^2\right. \\ &+ 3k_w w_N^2\left(\frac{\partial w_N}{\partial x}\right) - \left[\frac{EA_N}{2L}\int_0^L \left(\frac{\partial Z_o}{\partial x}\frac{\partial w_N}{\partial x} + \left(\frac{\partial w_N}{\partial x}\right)^2\right)\mathrm{d}x\right]\left(\frac{\partial^4 Z_o}{\partial x^4} + \frac{\partial^4 w_N}{\partial x^4}\right) \\ &- \left[EA_N\alpha_x\Delta T + \left[h_{1N}(1-2v) - 2h_{2N}(v^2-1) + h_{3N}v^2\right]A_N(\alpha_x)^2(\Delta T)^2\right. \\ &\left.\left.+ \eta A_N H_x^2 + k_p\right]\frac{\partial^4 w_N}{\partial x^4} + c_{N-1}\left(\frac{\partial^2 w_N}{\partial x^2} - \frac{\partial^2 w_{N-1}}{\partial x^2}\right)\right] = -c_{N-1}(w_N - w_{N-1}). \end{aligned} \tag{3.3}$$

From Eqs (3.1) and (3.2), one arrives at

$$
\begin{aligned}
(3.4)\quad & \alpha_1 EI_N W_N + \alpha_2 m_{CN}\frac{\mathrm{d}^2 W_N}{\mathrm{d}t^2} + \alpha_2 k W_N + \alpha_3 k_w W_N^3 - \alpha_4\frac{EA_N}{2L}W_N^3 \\
& - \alpha_5\left[EA_N\alpha_x\Delta T + [h_{1N}(1-2v) - 2h_{2N}(v^2-1) + h_{3N}v^2]A_N(\alpha_x)^2(\Delta T)^2\right. \\
& \left. + \eta A_N H_x^2 + k_p\right]W_N - (e_0a)^2\alpha_5 m_{cN}\frac{\mathrm{d}^2 W_N}{\mathrm{d}t^2} - (e_0a)^2\alpha_5 k W_N \\
& - 6\alpha_6(e_0a)^2 k_w W_N^3 - 3\alpha_7(e_0a)^2 k_w W_N^3 + \alpha_8(e_0a)^2\frac{EA_N}{2L}W_N^3 \\
& + \alpha_1(e_0a)^2\left[EA_N\alpha_x\Delta T + [h_{1N}(1-2v) - 2h_{2N}(v^2-1) + h_{3N}v^2]A_N(\alpha_x)^2(\Delta T)^2\right. \\
& \left. + \eta A_N H_x^2 + k_p\right]W_N - \alpha_5(e_0a)^2 C_{N-1}(W_N - W_{N-1}) + \alpha_2 C_{N-1}(W_N - W_{N-1}) = 0.
\end{aligned}
$$

After collecting like terms, we have

$$
\begin{aligned}
(3.5)\quad & \frac{\mathrm{d}^2 W_N}{\mathrm{d}t^2} + \frac{N^*_{(3.5)}}{(\alpha_2 - \alpha_5\mu)\,\rho A_N}W_N - \frac{c_{N-1}}{\rho A_N}W_{N-1} \\
& + \frac{\alpha_3 k_w - \alpha_4\frac{EA_N}{2L} - 6\alpha_6\mu k_w - 3\alpha_7\mu k_w + \alpha_8\mu\frac{EA_N}{2L}}{(\alpha_2 - \alpha_5\mu)\,\rho A_N}W_N^3 = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
N^*_{(3.5)} = {} & \alpha_1 EI_N + \alpha_2 k - \alpha_5\mu k + (\alpha_1\mu - \alpha_5)\Big(EA_N\alpha_x\Delta T \\
& + \left[h_{1N}(1-2v) - 2h_{2N}\left(v^2-1\right) + h_{3N}v^2\right]A_N(\alpha_x)^2(\Delta T)^2 + \eta A_N H_x^2 + k_p\Big) \\
& + (\alpha_2 - \alpha_5\mu)c_{N-1},
\end{aligned}
$$

$$
\begin{aligned}
& \alpha_1 = \int_0^L \phi(x)\frac{\mathrm{d}^4\phi(x)}{\mathrm{d}x^4}\,\mathrm{d}x, \qquad \alpha_2 = \int_0^L \phi^2(x)\,\mathrm{d}x, \qquad \alpha_3 = \int_0^L \phi^4(x)\,\mathrm{d}x, \\
& \alpha_4 = \int_0^L \left(\left(\int_0^L \phi(x)\left(\frac{\partial Z_o}{\partial x}\frac{\partial\phi(x)}{\partial x} + \left(\frac{\partial\phi(x)}{\partial x}\right)^2\right)\mathrm{d}x\right)\left(\frac{\partial^2 Z_o}{\partial x^2} + \frac{\partial^2\phi(x)}{\partial x^2}\right)\right)\mathrm{d}x, \\
(3.6)\quad & \\
& \alpha_5 = \int_0^L \phi(x)\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2}\,\mathrm{d}x, \qquad \alpha_6 = \int_0^L \phi^2(x)\left(\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right)^2\mathrm{d}x, \\
& \alpha_7 = \int_0^L \phi^3(x)\frac{\mathrm{d}\phi(x)}{dx}\,\mathrm{d}x, \qquad \mu = (e_0a)^2, \qquad m_{cN} = \rho A_N.
\end{aligned}
$$

Similarly, the same procedure is applied to the other inner tubes of the MWCNTs, appropriately.

From the above and based on the previous information, the temporal parts of the governing equations of motion for each nanotube in the multi-walled carbon nanotubes can be written as

$$\frac{\mathrm{d}^2W_1}{\mathrm{d}t^2}+\frac{N^*_{(3.7)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}W_1-\frac{c_1}{\rho A_1}W_2+\frac{(\alpha_8\mu-\alpha_4)\,\frac{EA_1}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}W_1^3=0, \tag{3.7}$$

where

$$N^*_{(3.7)}=\alpha_1EI_1+(\alpha_1\mu-\alpha_5)\Big(EA_N\alpha_x\Delta T+\left[h_{11}\,(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1\,(\alpha_x)^2\,(\Delta T)^2+\eta A_1H_x^2+k_p\Big)+(\alpha_2-\alpha_5\mu)c_1,$$

$$\frac{\mathrm{d}^2W_2}{\mathrm{d}t^2}+\frac{N^*_{(3.8)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}W_2-\frac{c_1}{\rho A_2}W_1-\frac{c_2}{\rho A_2}W_3+\frac{(\alpha_8\mu-\alpha_4)\,\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}W_2^3=0, \tag{3.8}$$

where

$$N^*_{(3.8)}=\alpha_1EI_2+(\alpha_1\mu-\alpha_5)\Big(EA_N\alpha_x\Delta T+\left[h_{12}\,(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2\,(\alpha_x)^2\,(\Delta T)^2+\eta A_2H_x^2+k_p\Big)+(\alpha_2-\alpha_5\mu)\,(c_1+c_2)\,,$$

$$\frac{\mathrm{d}^2W_3}{\mathrm{d}t^2}+\frac{N^*_{(3.9)}}{(\alpha_2-\alpha_5\mu)\,\rho A_3}W_3-\frac{c_2}{\rho A_3}W_2-\frac{c_3}{\rho A_3}W_4+\frac{(\alpha_8\mu-\alpha_4)\,\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_3}W_3^3=0, \tag{3.9}$$

where

$$N^*_{(3.9)}=\alpha_1EI_3+(\alpha_1\mu-\alpha_5)\Big(EA_N\alpha_x\Delta T+\left[h_{13}\,(1-2v)-2h_{23}\left(v^2-1\right)+h_{33}v^2\right]A_3\,(\alpha_x)^2\,(\Delta T)^2+\eta A_3H_x^2+k_p\Big)+(\alpha_2-\alpha_5\mu)\,(c_2+c_3)\,,$$

$$\frac{\mathrm{d}^2W_4}{\mathrm{d}t^2}+\frac{N^*_{(3.10)}}{(\alpha_2-\alpha_5\mu)\,\rho A_4}W_4-\frac{c_3}{\rho A_4}W_3-\frac{c_4}{\rho A_4}W_5+\frac{(\alpha_8\mu-\alpha_4)\,\frac{EA_4}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_4}W_4^3=0, \tag{3.10}$$

where

$$N^*_{(3.10)} = \alpha_1 EI_4 + (\alpha_1\mu - \alpha_5)\Big(EA_N\alpha_x\Delta T + \left[h_{14}(1-2v) - 2h_{24}\left(v^2-1\right) + h_{34}v^2\right]A_4(\alpha_x)^2(\Delta T)^2 + \eta A_4 H_x^2 + k_p\Big) + (\alpha_2 - \alpha_5\mu)(c_3 + c_4),$$

$$\vdots$$

$$\frac{d^2W_N}{dt^2} + \frac{N^*_{(3.11)}}{(\alpha_2-\alpha_5\mu)\rho A_N}W_N - \frac{c_{N-1}}{\rho A_N}W_{N-1} + \frac{\alpha_3 k_w - \alpha_4\frac{EA_N}{2L} - 6\alpha_6\mu k_w - 3\alpha_7\mu k_w + \alpha_8\mu\frac{EA_N}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_N}W_N^3 = 0, \tag{3.11}$$

where

$$N^*_{(3.11)} = \alpha_1 EI_N + \alpha_2 k - \alpha_5\mu k + (\alpha_1\mu - \alpha_5)\Big(EA_N\alpha_x\Delta T + \left[h_{1N}(1-2v) - 2h_{2N}\left(v^2-1\right) + h_{3N}v^2\right]A_N(\alpha_x)^2(\Delta T)^2 + \eta A_N H_x^2 + k_p\Big) + (\alpha_2 - \alpha_5\mu)c_{N-1},$$

and the initial conditions are

$$W_i(0) = X \quad \text{and} \quad \frac{dW_i(0)}{dt} = 0, \qquad i = 1, 2, 3, ..., N. \tag{3.12}$$

3.2. Homotopy perturbation method

Considering the nonlinear terms in Eqs (3.5)–(3.8), the development of the closed-form solution becomes highly involving. In order to generate symbolic solutions for the nonlinear equations, we adopt the homotopy perturbation method. The principle and the procedures of the homotopy perturbation method can be found in our previous works [60, 61].

3.2.1. Analysis of SWCNT ($N = 1$). For SWCNT, the dynamic equation of motion is given by

$$\frac{d^2W}{dt^2} + \left(\frac{N^*_{(3.13)}}{(\alpha_2-\alpha_5\mu)\rho A_N}\right)W + \left(\frac{\alpha_3 k_3 - \alpha_4\frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}}{(\alpha_2-\alpha_5\mu)\rho A}\right)W^3 = 0, \tag{3.13}$$

where

$$N^*_{(3.13)} = \alpha_1 EI + \alpha_2 k_1 - \alpha_5 \mu k_1 + (\alpha_1 \mu - \alpha_5)\Big(EA\alpha_x \Delta T + \left[h_1(1-2v) - 2h_2\left(v^2-1\right) + h_3 v^2\right] A\left(\alpha_x\right)^2 (\Delta T)^2 + \eta A H_x^2 + k_p\Big),$$

Introducing the following dimensionless parameters to Eq. (3.13):

$$r = \sqrt{\frac{I}{A}}, \qquad \tau = \omega_0 t, \qquad a = \frac{W}{r}, \tag{3.14}$$

one arrives at

$$\omega_0^2 \frac{\mathrm{d}^2 a}{\mathrm{d}\tau^2} + f_1 a + f_2 a^3 = 0, \tag{3.15}$$

where

$$f_1 = \frac{N^*_{(3.16)}}{(\alpha_2 - \alpha_5 \mu)\rho A} = \omega^2, \tag{3.16}$$

$$f_2 = \left(\frac{\alpha_3 k_3 - \alpha_4 \frac{EA}{2L} - 6\alpha_6 \mu k_3 - 3\alpha_7 \mu k_3 + \alpha_8 \mu \frac{EA}{2L}}{(\alpha_2 - \alpha_5 \mu)\rho A}\right)\left(\frac{I}{A}\right), \tag{3.17}$$

and

$$N^*_{(3.16)} = \alpha_1 EI + \alpha_2 k_1 - \alpha_5 \mu k_1 + (\alpha_1 \mu - \alpha_5)\Big(EA\alpha_x \Delta T + \left[h_1(1-2v) - 2h_2\left(v^2-1\right) + h_3 v^2\right] A\left(\alpha_x\right)^2 (\Delta T)^2 + \eta A H_x^2 + k_p\Big).$$

The initial conditions of Eq. (3.15) are given as

$$a(0) = X \qquad \text{and} \qquad \frac{\mathrm{d}a(0)}{\mathrm{d}\tau} = 0. \tag{3.18}$$

It can be derived from Eq. (3.12) that

$$\omega = \sqrt{f_1} = \sqrt{\frac{N^*_{(3.19)}}{(\alpha_2 - \alpha_5 \mu)\rho A}}, \tag{3.19}$$

where

$$N^*_{(3.19)} = \alpha_1 EI + \alpha_2 k - \alpha_5 \mu k + (\alpha_1 \mu - \alpha_5)\Big(EA_N \alpha_x \Delta T + \left[h_1(1-2v) - 2h_2\left(v^2-1\right) + h_3 v^2\right] A\left(\alpha_x\right)^2 (\Delta T)^2 + \eta A H_x^2 + k_p\Big),$$

and ω and ω_0 are the linear forced vibration and unknown nonlinear angular frequencies, respectively. It should be stated that ω_0 is to be determined.

With the application of the homotopy perturbation method (presented in Appendix A) to Eq. (3.15), we have the following solutions:

- **S-S support**

(3.20)
$$w(x,t) = \left(X\cos\omega_0 t + \frac{X^3\left(\alpha_3 k_3 - \alpha_4\frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}\right)\left(\frac{I}{A}\right)}{D^*_{(3.20)}} \cdot (\cos 3\omega_0 t - \cos\omega_0 t)\right)\cdot\sqrt{\frac{I}{A}}\sin\left(\frac{n\pi x}{l}\right),$$

where

$$D^*_{(3.20)} = 32\Big(\alpha_1 EI + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5) \cdot \Big(EA\alpha_x\Delta T + \left[h_1(1-2v) - 2h_2\left(v^2-1\right) + h_3 v^2\right]A(\alpha_x)^2(\Delta T)^2 + \eta A H_x^2 + k_p\Big)\Big) + 24X^2\left(\alpha_3 k_3 - \alpha_4\frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}\right)\left(\frac{I}{A}\right).$$

- **C-C support**

(3.21)
$$w(x,t) = \left(X\cos\omega_0 t + \frac{X^3\left(\alpha_3 k_3 - \alpha_4\frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}\right)\left(\frac{I}{A}\right)}{D^*_{(3.21)}} \cdot (\cos 3\omega_0 t - \cos\omega_0 t)\right)\cdot\sqrt{\frac{I}{A}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cosh\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\sinh\beta + \sin\beta}{\cosh\beta - \cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.21)} = 32\Big(\alpha_1 EI + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5) \cdot \Big(EA\alpha_x\Delta T + \left[h_1(1-2v) - 2h_2\left(v^2-1\right) + h_3 v^2\right]A(\alpha_x)^2(\Delta T)^2 + \eta A H_x^2 + k_p\Big)\Big) + 24X^2\left(\alpha_3 k_3 - \alpha_4\frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}\right)\left(\frac{I}{A}\right).$$

- **C-S support**

(3.22)

$$w(x,t) = \left(X\cos\omega_0 t + \frac{X^3\left(\alpha_3 k_3 - \alpha_4 \frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}\right)\left(\frac{I}{A}\right)}{D^*_{(3.22)}} \cdot (\cos 3\omega_0 t - \cos\omega_0 t)\right)\sqrt{\frac{I}{A}} \cdot \left\{\left(\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right) - \left(\frac{\cosh\beta - \cos\beta}{\sinh\beta - \sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.22)} = 32\Big(\alpha_1 EI + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5) \cdot\Big(EA\alpha_x\Delta T + \left[h_1(1-2v) - 2h_2\left(v^2-1\right) + h_3 v^2\right] A(\alpha_x)^2(\Delta T)^2 + \eta A H_x^2 + k_p\Big)\Big) + 24X^2\left(\alpha_3 k_3 - \alpha_4\frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}\right)\left(\frac{I}{A}\right).$$

3.2.2. Analysis of DWCNT ($N = 2$). The DWCNT governing equations of motion for the temporal part are given by

$$\frac{\mathrm{d}^2 W_1}{\mathrm{d}t^2} + \frac{N^*_{(3.23)}}{(\alpha_2 - \alpha_5\mu)\rho A_1} W_1 - \frac{c_1}{\rho A_1} W_2 + \left(\frac{(\alpha_8\mu - \alpha_4)\left(\frac{EA_1}{2L}\right)}{(\alpha_2 - \alpha_5\mu)\rho A_1}\right) W_1^3 = 0, \tag{3.23}$$

where

$$N^*_{(3.23)} = \alpha_1 EI_1 + (\alpha_1\mu - \alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v) - 2h_{21}\left(v^2-1\right) + h_{31}v^2\right] \cdot A_1(\alpha_x)^2(\Delta T)^2 + \eta A_1 H_x^2 + k_p\Big) + (\alpha_2 - \alpha_5\mu)c_1,$$

$$\frac{\mathrm{d}^2 W_2}{\mathrm{d}t^2} + \frac{N^*_{(3.24)}}{(\alpha_2 - \alpha_5\mu)\rho A_2} W_2 - \frac{c_1}{\rho A_2} W_1 + \left(\frac{\alpha_3 k_3 - \alpha_4\frac{EA_2}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2 - \alpha_5\mu)\rho A_2}\right) W_2^3 = 0, \tag{3.24}$$

where

$$N^*_{(3.24)} = \alpha_1 EI_2 + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v) - 2h_{22}\left(v^2-1\right) + h_{32}v^2\right] A_2(\alpha_x)^2(\Delta T)^2 + \eta A_2 H_x^2 + k_p\Big) + (\alpha_2 - \alpha_5\mu)c_1.$$

Applying the following dimensionless parameters:

$$r=\sqrt{\frac{I_1}{A_1}},\qquad a_1=\frac{W_1}{r},\qquad a_2=\frac{W_2}{r}\quad\text{and}\quad \tau=\omega_0 t \tag{3.25}$$

into Eqs (3.23) and (3.24), the following dimensionless nonlinear system of equations is derived

$$\omega_0^2\frac{\mathrm{d}^2a_1}{\mathrm{d}\tau^2}+f_1a_1+f_2a_1^3-f_3a_2=0, \tag{3.26}$$

$$\omega_0^2\frac{\mathrm{d}^2a_2}{\mathrm{d}\tau^2}+g_1a_2+g_2a_2^3-g_3a_1=0, \tag{3.27}$$

where

$$\begin{aligned}
f_1&=\frac{N^*_{(3.28f)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}+\frac{c_1}{\rho A_1},\\
f_2&=\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1},\\
f_3&=\frac{c_1}{\rho A_1},\\
g_1&=\frac{N^*_{(3.28g)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}+\frac{c_1}{\rho A_2},\\
g_2&=\frac{\left(\alpha_3k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}\right)}{(\alpha_2-\alpha_5\mu)\,\rho A_2}\left(\frac{I_1}{A_1}\right),\\
g_3&=\frac{c_1}{\rho A_2},
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
N^*_{(3.28f)}&=\alpha_1EI_1+(\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T\\
&\quad+\left[h_{11}(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1H_x^2+k_p\Big),\\
N^*_{(3.28g)}&=\alpha_1EI_2+\alpha_2k_1-\alpha_5\mu k_1+(\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T\\
&\quad+\left[h_{12}(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2+\eta A_2H_x^2+k_p\Big).
\end{aligned}$$

The solutions to the natural frequency and Eqs (3.26) and (3.27) using the homotopy perturbation method are presented in Appendix B. We arrived at the solutions.

- **S-S support**

(3.29)
$$w_1(x,t) = \left(X_1 \cos\omega_0 t + \frac{\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right) X^3 \left(\cos 3\omega_0 t - \cos\omega_0 t\right)}{32\left(\frac{D^*_{(3.29)}}{(\alpha_2-\alpha_5\mu)\rho A_1} + \frac{c_1}{\rho A_1}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)} \right) \cdot \sqrt{\frac{I_1}{A_1}} \sin\left(\frac{n\pi x}{l}\right),$$

where

$$D^*_{(3.29)} = \alpha_1 EI_1 + (\alpha_1\mu - \alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v) - 2h_{21}\left(v^2-1\right) + h_{31}v^2\right] A_1 (\alpha_x)^2 (\Delta T)^2 + \eta A_1 H_x^2 + k_p\Big).$$

(3.30)
$$w_2(x,t) = \left(X_2 \cos\omega_0 t + \frac{N^*_{(3.30)}}{D^*_{(3.30)}} \right) \cdot \sqrt{\frac{I_1}{A_1}} \sin\left(\frac{n\pi x}{l}\right),$$

where

$$N^*_{(3.30)} = X_2^3 \left(\frac{\alpha_3 k_3 - \alpha_4\frac{EA_2}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2} \right)\left(\frac{I_1}{A_1}\right)(\cos 3\omega_0 t - \cos\omega_0 t),$$

$$D^*_{(3.30)} = 32\left(\frac{\alpha_1 EI_2 + \alpha_2 k_1 - \alpha_5\mu k_1}{(\alpha_2-\alpha_5\mu)\rho A_2} + \frac{(\alpha_1\mu-\alpha_5)\big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v) - 2h_{22}(v^2-1) + h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2 + \eta A_2 H_x^2 + k_p\big)}{(\alpha_2-\alpha_5\mu)\rho A_2} + \frac{c_1}{\rho A_2}\right) + 24\left(\frac{\alpha_3 k_3 - \alpha_4\frac{EA_2}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2} \right)\left(\frac{I_1}{A_1}\right).$$

- **C-C support**

(3.31)
$$w_1(x,t) = \frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\tau - \cos\tau)}{32\left(\frac{D^*_{(3.31)}}{(\alpha_2-\alpha_5\mu)\rho A_1} + \frac{c_1}{\rho A_1}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)} \sqrt{\frac{I_1}{A_1}} \cdot \left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\sinh\beta + \sin\beta}{\cosh\beta - \cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^{*}_{(3.31)}=\alpha_1 EI_1+(\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T$$
$$+\left[h_{11}\left(1-2v\right)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1\left(\alpha_x\right)^2\left(\Delta T\right)^2+\eta A_1H_x^2+k_p\Big),$$

$$w_2\left(x,t\right)=\left(X_2\cos\omega_0 t+\frac{N^{*}_{(3.32)}}{D^{*}_{(3.32)}}\right)\sqrt{\frac{I_1}{A_1}}$$
$$\cdot\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\}, \tag{3.32}$$

where

$$N^{*}_{(3.32)}=X_2^3\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)\left(\frac{I_1}{A_1}\right)(\cos 3\omega_0 t-\cos\omega_0 t),$$

$$D^{*}_{(3.32)}=32\left(\frac{\alpha_1EI_2+\alpha_2k_1-\alpha_5\mu k_1}{(\alpha_2-\alpha_5\mu)\rho A_2}\right.$$
$$+\frac{(\alpha_1\mu-\alpha_5)\left(EA_2\alpha_x\Delta T+\left[h_{12}(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2+\eta A_2H_x^2+k_p\right)}{(\alpha_2-\alpha_5\mu)\rho A_2}$$
$$\left.+\frac{c_1}{\rho A_2}\right)+24\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)\left(\frac{I_1}{A_1}\right).$$

- **C-S support**

$$w_1\left(x,t\right)=\left(X_1\cos\omega_0 t+\frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\tau-\cos\tau)}{32\left(\frac{D^{*}_{(3.33)}}{(\alpha_2-\alpha_5\mu)\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)}\right)\sqrt{\frac{I_1}{A_1}}$$
$$\cdot\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\}, \tag{3.33}$$

where

$$D^{*}_{(3.33)}=\alpha_1 EI_1+(\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T$$
$$+\left[h_{11}\left(1-2v\right)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1\left(\alpha_x\right)^2\left(\Delta T\right)^2+\eta A_1H_x^2+k_p\Big),$$

$$(3.34)\quad w_2(x,t) = \left(X_2\cos\omega_0 t + \frac{N^*_{(3.34)}}{D^*_{(3.21)}}\right)\sqrt{\frac{I_1}{A_1}}$$

$$\cdot\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$N^*_{(3.34)} = X_2^3\left(\frac{\alpha_3 k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)\left(\frac{I_1}{A_1}\right)(\cos 3\omega_0 t-\cos\omega_0 t),$$

$$D^*_{(3.34)} = 32\left(\frac{\alpha_1 EI_2+\alpha_2 k_1-\alpha_5\mu k_1}{(\alpha_2-\alpha_5\mu)\rho A_2}\right.$$

$$+\frac{(\alpha_1\mu-\alpha_5)\left(EA_2\alpha_x\Delta T+\left[h_{12}(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2+\eta A_2 H_x^2+k_p\right)}{(\alpha_2-\alpha_5\mu)\rho A_2}$$

$$\left.+\frac{c_1}{\rho A_2}\right)+24\left(\frac{\alpha_3 k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)\left(\frac{I_1}{A_1}\right).$$

3.2.3. Analysis of triple-walled carbon nanotube ($N = 3$). The nonlinear vibration dynamic equation for the triple-walled carbon nanotube (TWCNT) is given by Eqs (3.35)–(3.37)

$$(3.35)\quad \frac{d^2W_1}{dt^2}+\left(\frac{N^*_{(3.35)}}{(\alpha_2-\alpha_5\mu)\rho A_1}\right)W_1-\frac{c_1}{\rho A_1}W_2+\left(\frac{(\alpha_8\mu-\alpha_4)\frac{EA_1}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_1}\right)W_1^3=0,$$

where

$$N^*_{(3.35)} = \alpha_1 EI_1+(\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T$$

$$+\left[h_{11}(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1 H_x^2+k_p\Big)$$

$$+(\alpha_2-\alpha_5\mu)c_1,$$

$$(3.36)\quad \frac{d^2W_2}{dt^2}+\left(\frac{N^*_{(3.36)}}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)W_2-\frac{c_1}{\rho A_2}W_1-\frac{c_2}{\rho A_2}W_3$$

$$+\left(\frac{(\alpha_8\mu-\alpha_4)\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)W_2^3=0,$$

where

$$N^*_{(3.36)} = \alpha_1 EI_2 + (\alpha_1\mu - \alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v) - 2h_{22}\left(v^2-1\right) + h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2 + \eta A_2 H_x^2 + k_p\Big) + (\alpha_2 - \alpha_5\mu)c_1 + (\alpha_2 - \alpha_5\mu)c_2,$$

$$\frac{\mathrm{d}^2 W_3}{\mathrm{d}t^2} + \left(\frac{N^*_{(3.37)}}{(\alpha_2-\alpha_5\mu)\rho A_3}\right)W_3 - \frac{c_3}{\rho A_3}W_2 + \left(\frac{\left(\alpha_3 k_3 - \alpha_4\frac{EA_3}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\right)\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_3}\right)W_3^3 = 0, \tag{3.37}$$

where

$$N^*_{(3.37)} = \alpha_1 EI_3 + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}(1-2v) - 2h_{23}\left(v^2-1\right) + h_{33}v^2\right]A_3(\alpha_x)^2(\Delta T)^2 + \eta A_3 H_x^2 + k_p\Big) + (\alpha_2 - \alpha_5\mu).$$

With the aid of the following dimensionless parameters:

$$r = \sqrt{\frac{I_1}{A_1}}, \qquad a_1 = \frac{W_1}{r}, \qquad a_2 = \frac{W_2}{r}, \qquad a_3 = \frac{W_3}{r}, \qquad \tau = \omega_0 t, \tag{3.38}$$

Eqs (3.35)–(3.37) are transformed into the following dimensionless nonlinear system of equations:

$$\omega_0^2\frac{\mathrm{d}^2 a_1}{\mathrm{d}\tau^2} + f_1 a_1 + f_2 a_1^3 - f_3 a_2 = 0, \tag{3.39}$$

$$\omega_0^2\frac{\mathrm{d}^2 a_2}{d\tau^2} + g_1 a_2 + g_2 a_2^3 - g_3 a_1 - g_4 a_3 = 0, \tag{3.40}$$

$$\omega_0^2\frac{\mathrm{d}^2 a_3}{\mathrm{d}\tau^2} + h_1 a_3 + h_2 a_3^3 - h_3 a_2 = 0, \tag{3.41}$$

where

$$f_1 = \frac{N^*_{(3.42f)}}{(\alpha_2 - \alpha_5\mu)\,\rho A_1} + \frac{c_1}{\rho A_1},$$

$$f_2 = \frac{(\alpha_8\mu - \alpha_4)EI_1}{2L\,(\alpha_2 - \alpha_5\mu)\,\rho A_1}, \qquad f_3 = \frac{c_1}{\rho A_1},$$

$$g_1 = \frac{N^*_{(3.42g)}}{(\alpha_2 - \alpha_5\mu)\,\rho A_2} + \frac{c_1 + c_2}{\rho A_2},$$

$$g_2 = \frac{(\alpha_8\mu - \alpha_4)EI_2}{2L\,(\alpha_2 - \alpha_5\mu)\,\rho A_2}, \qquad g_3 = \frac{c_1}{\rho A_2}, \qquad g_4 = \frac{c_2}{\rho A_2}, \tag{3.42}$$

$$\hbar_1 = \frac{N^*_{(3.42\hbar)}}{(\alpha_2 - \alpha_5\mu)\,\rho A_3} + \frac{c_2}{\rho A_3},$$

$$\hbar_2 = \frac{\left(\alpha_3 k_3 - \alpha_4\frac{EA_3}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA_3}{2L}\right)}{(\alpha_2 - \alpha_5\mu)\,\rho A_3}\left(\frac{I_1}{A_1}\right),$$

$$\hbar_3 = \frac{c_2}{\rho A_3}$$

and

$$N^*_{(3.42f)} = \alpha_1 EI_1 + (\alpha_1\mu - \alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v) - 2h_{21}\left(v^2-1\right) + h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2 + \eta A_1 H_x^2 + k_p\Big),$$

$$N^*_{(3.42g)} = \alpha_1 EI_2 + (\alpha_1\mu - \alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v) - 2h_{22}\left(v^2-1\right) + h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2 + \eta A_2 H_x^2 + k_p\Big),$$

$$N^*_{(3.42\hbar)} = \alpha_1 EI_3 + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}(1-2v) - 2h_{23}\left(v^2-1\right) + h_{33}v^2\right]A_3(\alpha_x)^2(\Delta T)^2 + \eta A_3 H_x^2 + k_p\Big).$$

The procedure of the homotopy perturbation analysis is presented in Appendix C. The displacements of the TWCNTs are expressed as follows.

- **S-S support**

$$w_1(x,t)=\left(X_1\cos\omega_0 t+\frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\,\omega_0 t-\cos\omega_0 t)}{32\left(\frac{D^*_{(3.43)}}{(\alpha_2-\alpha_5\mu)\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)}\right)\cdot\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right), \tag{3.43}$$

where

$$D^*_{(3.43)}=\alpha_1 EI_1+(\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T+\left[h_{11}(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1 H_x^2+k_p\Big),$$

$$w_2(x,t)=\left(X_2\cos\omega_0 t+\frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\,\omega_0 t-\cos\omega_0 t)}{D^*_{(3.44)}}\right)\cdot\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right), \tag{3.44}$$

where

$$D^*_{(3.44)}=32\left(\frac{N^*_{(3.44)}}{(\alpha_2-\alpha_5\mu)\rho A_2}+\frac{c_1+c_2}{\rho A_2}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right),$$

$$N^*_{(3.44)}=\alpha_1 EI_2+(\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T+\left[h_{12}(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2+\eta A_2 H_x^2+k_p\Big).$$

$$w_3(x,t)=\left(X_3\cos\omega_0 t+\frac{X_3^3\left(\frac{\alpha_3 k_3-\alpha_4\frac{EA_3}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_3}\right)}{D^*_{(3.45)}}\cdot\left(\frac{I_1}{A_1}\right)(\cos 3\,\omega_0 t-\cos\omega_0 t)\right)\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right), \tag{3.45}$$

where

$$D^*_{(3.45)} = 32\left(\frac{N^*_{(3.45)}}{(\alpha_2-\alpha_5\mu)\rho A_3} + \frac{c_2}{\rho A_3}\right) + 24\left(\frac{\alpha_3 k_3 - \alpha_4\frac{EA_3}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_3}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(3.45)} = \alpha_1 EI_3 + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5)\Big(EA_3\alpha_x\Delta T$$
$$+ \left[h_{13}(1-2v) - 2h_{23}\left(v^2-1\right) + h_{33}v^2\right] A_3 (\alpha_x)^2 (\Delta T)^2 + \eta A_3 H_x^2 + k_p\Big).$$

- **C-C support**

$$(3.46)\quad w_1(x,t) = \left(X_1\cos\omega_0 t + \frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\,\omega_0 t - \cos\omega_0 t)}{D^*_{(3.46)}}\right)$$
$$\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.46)} = 32\left(\frac{N^*_{(3.46)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1} + \frac{c_1}{\rho A_1}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(3.46)} = \alpha_1 EI_1 + (\alpha_1\mu - \alpha_5)\Big(EA_1\alpha_x\Delta T$$
$$+ \left[h_{11}(1-2v) - 2h_{21}\left(v^2-1\right) + h_{31}v^2\right] A_1 (\alpha_x)^2 (\Delta T)^2 + \eta A_1 H_x^2 + k_p\Big),$$

$$(3.47)\quad w_2(x,t) = \left(X_2\cos\omega_0 t + \frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\,\omega_0 t - \cos\omega_0 t)}{D^*_{(3.47)}}\right)$$
$$\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.47)} = 32\left(\frac{N^*_{(3.47)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2} + \frac{c_1}{\rho A_2} + \frac{c_2}{\rho A_2}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_2}\right),$$

$$N^*_{(3.47)} = \alpha_1 EI_2 + (\alpha_1\mu - \alpha_5)\Big(EA_2\alpha_x\Delta T$$
$$+ \left[h_{12}(1-2v) - 2h_{22}\left(v^2-1\right) + h_{32}v^2\right] A_2 (\alpha_x)^2 (\Delta T)^2 + \eta A_2 H_x^2 + k_p\Big),$$

(3.48)

$$w_3(x,t)=\left(X_3\cos\omega_0 t+\frac{X_3^3\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_3}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_3}\right)\left(\frac{I_1}{A_1}\right)(\cos 3\omega_0 t-\cos\omega_0 t)}{D^*_{(3.48)}}\right)$$

$$\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.48)}=32\left(\frac{N^*_{(3.48)}}{(\alpha_2-\alpha_5\mu)\rho A_3}+\frac{c_2}{\rho A_3}\right)+24\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_3}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_3}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(3.48)}=\alpha_1EI_3+\alpha_2k_1-\alpha_5\mu k_1+(\alpha_1\mu-\alpha_5)\Big(EA_3\alpha_x\Delta T+\left[h_{13}(1-2v)-2h_{23}\left(v^2-1\right)+h_{33}v^2\right]A_3(\alpha_x)^2(\Delta T)^2+\eta A_3H_x^2+k_p\Big).$$

- **C-S support**

$$w_1(x,t)=\left(X_1\cos\omega_0 t+\frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\omega_0 t-\cos\omega_0 t)}{D^*_{(3.49)}}\right) \tag{3.49}$$

$$\cdot\sqrt{\frac{I}{A}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.49)}=32\left(\frac{N^*_{(3.49)}}{(\alpha_2-\alpha_5\mu)\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right),$$

$$N^*_{(3.49)}=\alpha_1EI_1+(\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T+\left[h_{11}(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1H_x^2+k_p\Big),$$

$$w_2(x,t)=\left(X_2\cos\omega_0 t+\frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\omega_0 t-\cos\omega_0 t)}{D^*_{(3.50)}}\right) \tag{3.50}$$

$$\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.50)} = 32\left(\frac{N^*_{(3.50)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}+\frac{c_1}{\rho A_2}+\frac{c_2}{\rho A_2}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_2}\right),$$

$$N^*_{(3.50)} = \alpha_1 EI_2 + (\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}\,(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2\,(\alpha_x)^2\,(\Delta T)^2+\eta A_2H_x^2+k_p\Big),$$

(3.51)

$$w_3(x,t) = \left(X_3\cos\omega_0 t + \frac{X_3^3\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_3}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_3}\right)\left(\frac{I_3}{A_3}\right)(\cos 3\omega_0 t-\cos\omega_0 t)}{D^*_{(3.51)}}\right)$$

$$\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.51)} = 32\left(\frac{N^*_{(3.51)}}{(\alpha_2-\alpha_5\mu)\,\rho A_3}+\frac{c_2}{\rho A_3}\right) + 24\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_3}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_3}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(3.51)} = \alpha_1 EI_3+\alpha_2k_1-\alpha_5\mu k_1+(\alpha_1\mu-\alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}\,(1-2v)-2h_{23}\left(v^2-1\right)+h_{33}v^2\right]A_3\,(\alpha_x)^2\,(\Delta T)^2+\eta A_3H_x^2+k_p\Big).$$

3.2.4. Analysis of quadruple-walled carbon nanotube ($N = 4$). The governing dynamic equation of motion for the quadruple-walled carbon nanotube (QWCNT) is given by

$$\frac{\mathrm{d}^2W_1}{\mathrm{d}t^2}+\left(\frac{N^*_{(3.52)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}\right)W_1-\frac{c_1}{\rho A_1}W_2+\left(\frac{(\alpha_8\mu-\alpha_4)\,\frac{EA_1}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}\right)W_1^3=0, \tag{3.52}$$

where

$$N^*_{(3.52)} = \alpha_1 EI_1 + (\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}\,(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1\,(\alpha_x)^2\,(\Delta T)^2+\eta A_1H_x^2+k_p\Big) + (\alpha_2-\alpha_5\mu)\,c_1,$$

$$\frac{\mathrm{d}^2W_2}{\mathrm{d}t^2} + \left(\frac{N^*_{(3.53)}}{(\alpha_2 - \alpha_5\mu)\,\rho A_2}\right) W_2 - \frac{c_1}{\rho A_2} W_1 - \frac{c_2}{\rho A_2} W_3 + \left(\frac{(\alpha_8\mu - \alpha_4)\,\frac{EA_2}{2L}}{(\alpha_2 - \alpha_5\mu)\,\rho A_2}\right) W_2^3 = 0, \tag{3.53}$$

where

$$N^*_{(3.53)} = \alpha_1 EI_2 + (\alpha_1\mu - \alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v) - 2h_{22}\left(v^2-1\right) + h_{32}v^2\right] A_2(\alpha_x)^2(\Delta T)^2 + \eta A_2 H_x^2 + k_p\Big) + (\alpha_2 - \alpha_5\mu)\,c_1 + (\alpha_2 - \alpha_5\mu)\,c_2,$$

$$\frac{\mathrm{d}^2W_3}{\mathrm{d}t^2} + \left(\frac{N^*_{(3.54)}}{(\alpha_2 - \alpha_5\mu)\,\rho A_3}\right) W_3 - \frac{c_2}{\rho A_3} W_2 - \frac{c_3}{\rho A_3} W_4 + \left(\frac{(\alpha_8\mu - \alpha_4)\,\frac{EA_3}{2L}}{(\alpha_2 - \alpha_5\mu)\,\rho A_3}\right) W_3^3 = 0, \tag{3.54}$$

where

$$N^*_{(3.54)} = \alpha_1 EI_3 + (\alpha_1\mu - \alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}(1-2v) - 2h_{23}\left(v^2-1\right) + h_{33}v^2\right] A_3(\alpha_x)^2(\Delta T)^2 + \eta A_3 H_x^2 + k_p\Big) + (\alpha_2 - \alpha_5\mu)\,c_2 + (\alpha_2 - \alpha_5\mu)\,c_3,$$

$$\frac{\mathrm{d}^2W_4}{\mathrm{d}t^2} + \left(\frac{N^*_{(3.55)}}{(\alpha_2 - \alpha_5\mu)\,\rho A_4}\right) W_4 - \frac{c_3}{\rho A_4} W_3 + \left(\frac{\alpha_3 k_3 - \alpha_4\frac{EA_4}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA_4}{2L}}{(\alpha_2 - \alpha_5\mu)\,\rho A_4}\right) W_4^3 = 0. \tag{3.55}$$

where

$$N^*_{(3.55)} = \alpha_1 EI_4 + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5)\Big(EA_4\alpha_x\Delta T + \left[h_{14}(1-2v) - 2h_{24}\left(v^2-1\right) + h_{34}v^2\right] A_4(\alpha_x)^2(\Delta T)^2 + \eta A_4 H_x^2 + k_p\Big) + (\alpha_2 - \alpha_5\mu)\,c_3.$$

Using the following dimensionless parameters:

$$r = \sqrt{\frac{I_1}{A_1}}, \quad a_1 = \frac{W_1}{r} \quad a_2 = \frac{W_2}{r} \quad a_3 = \frac{W_3}{r} \quad a_4 = \frac{W_4}{r}, \quad \tau = \omega_0 t, \tag{3.56}$$

the following dimensionless nonlinear systems of equations are developed:

$$\omega_0^2 \frac{\mathrm{d}^2 a_1}{\mathrm{d}\tau^2} + f_1 a_1 + f_2 a_1^3 - f_3 a_2 = 0, \tag{3.57}$$

$$\omega_0^2 \frac{\mathrm{d}^2 a_2}{\mathrm{d}\tau^2} + g_1 a_2 + g_2 a_2^3 - g_3 a_1 - g_4 a_3 = 0, \tag{3.58}$$

$$\omega_0^2 \frac{\mathrm{d}^2 a_3}{\mathrm{d}\tau^2} + \hbar_1 a_3 + \hbar_2 a_3^3 - \hbar_3 a_1 - \hbar_4 a_4 = 0, \tag{3.59}$$

$$\omega_0^2 \frac{\mathrm{d}^2 a_4}{\mathrm{d}\tau^2} + r_1 a_4 + r_2 a_4^3 - r_3 a_3 = 0, \tag{3.60}$$

where

$$\begin{gathered}
f_1 = \frac{N^*_{(3.61f)}}{(\alpha_2 - \alpha_5\mu)\,\rho A_1} + \frac{c_1}{\rho A_1}, \\
f_2 = \frac{(\alpha_8\mu - \alpha_4) E I_1}{2L\,(\alpha_2 - \alpha_5\mu)\,\rho A_1}, \qquad f_3 = \frac{c_1}{\rho A_1}, \\
g_1 = \frac{N^*_{(3.61g)}}{(\alpha_2 - \alpha_5\mu)\,\rho A_2} + \frac{c_1 + c_2}{\rho A_2}, \\
g_2 = \frac{(\alpha_8\mu - \alpha_4) E I_2}{2L\,(\alpha_2 - \alpha_5\mu)\,\rho A_2}, \qquad g_3 = \frac{c_1}{\rho A_2}, \qquad g_4 = \frac{c_2}{\rho A_2}, \\
\hbar_1 = \frac{N^*_{(3.61\hbar)}}{(\alpha_2 - \alpha_5\mu)\,\rho A_3} + \frac{c_2 + c_3}{\rho A_3}, \\
\hbar_2 = \frac{(\alpha_8\mu - \alpha_4) E I_3}{2L\,(\alpha_2 - \alpha_5\mu)\,\rho A_3}, \qquad \hbar_3 = \frac{c_2}{\rho A_3}, \qquad \hbar_4 = \frac{c_3}{\rho A_3}, \\
r_1 = \frac{N^*_{(3.61r)}}{(\alpha_2 - \alpha_5\mu)\,\rho A_4} + \frac{c_3}{\rho A_4}, \\
r_2 = \frac{\left(\alpha_3 k_3 - \alpha_4 \frac{E A_4}{2L} - 6\alpha_6 \mu k_3 - 3\alpha_7 \mu k_3 + \alpha_8 \mu \frac{E A_4}{2L}\right)}{(\alpha_2 - \alpha_5\mu)\,\rho A_4} \left(\frac{I_1}{A_1}\right), \\
r_3 = \frac{c_3}{\rho A_4},
\end{gathered} \tag{3.61}$$

and

$$N^*_{(3.61f)} = \alpha_1 EI_1 + (\alpha_1\mu - \alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v) - 2h_{21}\left(v^2-1\right) + h_{31}v^2\right] A_1(\alpha_x)^2(\Delta T)^2 + \eta A_1 H_x^2 + k_p\Big),$$

$$N^*_{(3.61g)} = \alpha_1 EI_2 + (\alpha_1\mu - \alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v) - 2h_{22}\left(v^2-1\right) + h_{32}v^2\right] A_2(\alpha_x)^2(\Delta T)^2 + \eta A_2 H_x^2 + k_p\Big),$$

$$N^*_{(3.61\hbar)} = \alpha_1 EI_3 + (\alpha_1\mu - \alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}(1-2v) - 2h_{23}\left(v^2-1\right) + h_{33}v^2\right] A_3(\alpha_x)^2(\Delta T)^2 + \eta A_3 H_x^2 + k_p\Big),$$

$$N^*_{(3.61r)} = \alpha_1 EI_4 + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5)\Big(EA_4\alpha_x\Delta T + \left[h_{14}(1-2v) - 2h_{24}\left(v^2-1\right) + h_{34}v^2\right] A_4(\alpha_x)^2(\Delta T)^2 + \eta A_4 H_x^2 + k_p\Big).$$

The homotopy perturbation method is shown in Appendix D. The solutions of Eqs (3.57)–(3.60) presenting the displacements of the QWCNTs are expressed as follows.

- **S-S support**

(3.62)

$$w_1(x,t) = \left(X_1\cos\tau + \frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\tau - \cos\tau)}{D^*_{(3.62)}}\right)\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right),$$

where

$$D^*_{(3.62)} = 32\left(\frac{N^*_{(3.62)}}{(\alpha_2-\alpha_5\mu)\rho A_1} + \frac{c_1}{\rho A_1}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right),$$

$$N^*_{(3.62)} = \alpha_1 EI_1 + (\alpha_1\mu - \alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v) - 2h_{21}\left(v^2-1\right) + h_{31}v^2\right] A_1(\alpha_x)^2(\Delta T)^2 + \eta A_1 H_x^2 + k_p\Big),$$

(3.63)

$$w_2(x,t) = \left(X_2\cos\tau + \frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\tau - \cos\tau)}{D^*_{(3.63)}}\right)\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right),$$

where

$$D^*_{(3.63)} = 32\left(\frac{N^*_{(3.63)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2} + \frac{c_1}{\rho A_2} + \frac{c_2}{\rho A_2}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_2}\right),$$

$$N^*_{(3.63)} = \alpha_1 EI_2 + (\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}\,(1-2v) - 2h_{22}\left(v^2-1\right) + h_{32}v^2\right] A_2\,(\alpha_x)^2\,(\Delta T)^2 + \eta A_2 H_x^2 + k_p\Big),$$

(3.64)

$$w_3\,(x,t) = \left(X_3\cos\tau + \frac{X_3^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_3}{2L(\alpha_2-\alpha_5\mu)\rho A_3}\right)(\cos 3\tau - \cos\tau)}{D^*_{(3.64)}}\right)\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right),$$

where

$$D^*_{(3.64)} = 32\left(\frac{N^*_{(3.64)}}{(\alpha_2-\alpha_5\mu)\,\rho A_3} + \frac{c_2}{\rho A_3} + \frac{c_3}{\rho A_3}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_3}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_3}\right),$$

$$N^*_{(3.64)} = \alpha_1 EI_3 + (\alpha_1\mu-\alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}\,(1-2v) - 2h_{23}\left(v^2-1\right) + h_{33}v^2\right] A_3\,(\alpha_x)^2\,(\Delta T)^2 + \eta A_3 H_x^2 + k_p\Big),$$

$$(3.65)\quad w_4\,(x,t) = \left(X_4\cos\tau + \frac{X_3^3\left(\frac{\left(\alpha_3 k_3 - \alpha_4\frac{EA_4}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA_4}{2L}\right)}{(\alpha_2-\alpha_5\mu)\rho A_4}\right)}{D^*_{(3.65)}} \cdot \left(\frac{I_1}{A_1}\right)(\cos 3\tau - \cos\tau)\right)\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right),$$

where

$$D^*_{(3.65)} = 32\left(\frac{N^*_{(3.65)}}{(\alpha_2-\alpha_5\mu)\,\rho A_4} + \frac{c_3}{\rho A_4}\right) + 24\left(\frac{\alpha_3 k_3 - \alpha_4\frac{EA_4}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA_4}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_4}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(3.65)} = \alpha_1 EI_4 + \alpha_2 k_1 - \alpha_5 \mu k_1 + (\alpha_1\mu - \alpha_5)\Big(EA_4\alpha_x\Delta T + \left[h_{14}(1-2v) - 2h_{24}\left(v^2-1\right) + h_{34}v^2\right] A_4 (\alpha_x)^2 (\Delta T)^2 + \eta A_4 H_x^2 + k_p\Big).$$

- **C-C support**

$$(3.66)\quad w_1(x,t) = \left(X_1\cos\tau + \frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\tau - \cos\tau)}{D^*_{(3.66)}}\right) \cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.66)} = 32\left(\frac{N^*_{(3.66)}}{(\alpha_2-\alpha_5\mu)\rho A_1} + \frac{c_1}{\rho A_1}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right),$$

$$N^*_{(3.66)} = \alpha_1 EI_1 + (\alpha_1\mu - \alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v) - 2h_{21}\left(v^2-1\right) + h_{31}v^2\right] A_1 (\alpha_x)^2 (\Delta T)^2 + \eta A_1 H_x^2 + k_p\Big),$$

$$(3.67)\quad w_2(x,t) = \left(X_2\cos\tau + \frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\tau - \cos\tau)}{D^*_{(3.67)}}\right) \cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.67)} = 32\left(\frac{N^*_{(3.670}}{(\alpha_2-\alpha_5\mu)\rho A_2} + \frac{c_1}{\rho A_2} + \frac{c_2}{\rho A_2}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right),$$

$$N^*_{(3.67)} = \alpha_1 EI_2 + (\alpha_1\mu - \alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v) - 2h_{22}\left(v^2-1\right) + h_{32}v^2\right] A_2 (\alpha_x)^2 (\Delta T)^2 + \eta A_2 H_x^2 + k_p\Big),$$

$$(3.68)\quad w_3(x,t) = \left(X_3\cos\tau + \frac{X_3^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_3}{2L(\alpha_2-\alpha_5\mu)\rho A_3}\right)(\cos 3\tau - \cos\tau)}{D^*_{(3.68)}}\right) \cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.68)} = 32\left(\frac{N^*_{(3.68)}}{(\alpha_2-\alpha_5\mu)\,\rho A_3} + \frac{c_2}{\rho A_3} + \frac{c_3}{\rho A_3}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_3}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_3}\right),$$

$$N^*_{(3.68)} = \alpha_1 EI_3 + (\alpha_1\mu-\alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}(1-2v) - 2h_{23}\left(v^2-1\right) + h_{33}v^2\right] A_3(\alpha_x)^2(\Delta T)^2 + \eta A_3 H_x^2 + k_p\Big),$$

$$\text{(3.69)}\quad w_4(x,t) = \left(X_4\cos\tau + \frac{X_3^3\left(\frac{\left(\alpha_3k_3-\alpha_4\frac{EA_4}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_4}{2L}\right)}{(\alpha_2-\alpha_5\mu)\rho A_4}\right)\cdot\left(\frac{I_1}{A_1}\right)(\cos 3\tau-\cos\tau)}{D^*_{(3.69)}}\right)\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.69)} = 32\left(\frac{N^*_{(3.69)}}{(\alpha_2-\alpha_5\mu)\,\rho A_4}\right) + \frac{c_3}{\rho A_4} + 24\left(\frac{\left(\alpha_3k_3-\alpha_4\frac{EA_4}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_4}{2L}\right)}{(\alpha_2-\alpha_5\mu)\,\rho A_4}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(3.69)} = \alpha_1 EI_4 + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu-\alpha_5)\Big(EA_4\alpha_x\Delta T + \left[h_{14}(1-2v) - 2h_{24}\left(v^2-1\right) + h_{34}v^2\right] A_4(\alpha_x)^2(\Delta T)^2 + \eta A_4 H_x^2 + k_p\Big).$$

- **C-S support**

$$\text{(3.70)}\quad w_1(x,t) = \left(X_1\cos\tau + \frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\tau-\cos\tau)}{D^*_{(3.70)}}\right)\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.70)} = 32\left(\frac{N^*_{(3.70)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(3.70)} = \alpha_1 EI_1+(\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}\,(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1\,(\alpha_x)^2\,(\Delta T)^2+\eta A_1H_x^2+k_p\Big),$$

$$(3.71)\quad w_2\,(x,t)=\left(X_2\cos\tau+\frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\tau-\cos\tau)}{D^*_{(3.71)}}\right)\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.71)} = 32\left(\frac{N^*_{(3.71)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}+\frac{c_1+c_2}{\rho A_2}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_2}\right),$$

$$N^*_{(3.71)} = \alpha_1 EI_2+(\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}\,(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2\,(\alpha_x)^2\,(\Delta T)^2+\eta A_2H_x^2+k_p\Big),$$

$$(3.72)\quad w_3\,(x,t)=\left(X_3\cos\tau+\frac{X_3^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_3}{2L(\alpha_2-\alpha_5\mu)\rho A_3}\right)(\cos 3\tau-\cos\tau)}{D^*_{(3.72)}}\right)\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(3.72)} = 32\left(\frac{N^*_{(3.72)}}{(\alpha_2-\alpha_5\mu)\,\rho A_3}+\frac{c_2}{\rho A_3}+\frac{c_3}{\rho A_3}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_3}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_3}\right),$$

$$N^*_{(3.72)} = \alpha_1 EI_3+(\alpha_1\mu-\alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}\,(1-2v)-2h_{23}\left(v^2-1\right)+h_{33}v^2\right]A_3\,(\alpha_x)^2\,(\Delta T)^2+\eta A_3H_x^2+k_p\Big),$$

$$w_4(x,t) = \left(X_4 \cos\tau + \frac{X_3^3 \left(\frac{\left(\alpha_3 k_3 - \alpha_4 \frac{EA_4}{2L} - 6\alpha_6 \mu k_3 - 3\alpha_7 \mu k_3 + \alpha_8 \mu \frac{EA_4}{2L}\right)}{(\alpha_2 - \alpha_5 \mu)\rho A_4} \right)}{D^*_{(3.73)}} \cdot \left(\frac{I_1}{A_1}\right)(\cos 3\tau - \cos\tau) \right) \sqrt{\frac{I_1}{A_1}} \left\{ \left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right) \right] - \left(\frac{\cosh\beta - \cos\beta}{\sinh\beta - \sin\beta}\right) \left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right) \right] \right\}, \tag{3.73}$$

where

$$D^*_{(3.73)} = 32\left(\frac{N^*_{(3.73)}}{(\alpha_2 - \alpha_5\mu)\,\rho A_4}\right) + \frac{c_3}{\rho A_4} + 24\left(\frac{\left(\alpha_3 k_3 - \alpha_4 \frac{EA_4}{2L} - 6\alpha_6 \mu k_3 - 3\alpha_7 \mu k_3 + \alpha_8 \mu \frac{EA_4}{2L}\right)}{(\alpha_2 - \alpha_5\mu)\,\rho A_4}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(3.73)} = \alpha_1 EI_4 + \alpha_2 k_1 - \alpha_5 \mu k_1 + (\alpha_1\mu - \alpha_5)\Big(EA_4 \alpha_x \Delta T + \left[h_{14}(1-2v) - 2h_{24}\left(v^2 - 1\right) + h_{34} v^2\right] A_4 (\alpha_x)^2 (\Delta T)^2 + \eta A_4 H_x^2 + k_p\Big).$$

In the same manner, the analysis of N layers nanotube can be carried out.

4. Results and discussion

Using the values of the parameters in Table 2, the effects of the various parameters on the dynamic responses of the SWCNT, DWCNT, and MWCNT were studied. The results are presented and discussed.

4.1. Effects of boundary conditions on the frequency ratio

Figures 4–7 show the effects of boundary conditions or supports on the frequency ratio for the nonlinear dynamic responses of SWCNTs and MWCNTs, respectively operating in thermal and magnetic environments and resting on Winkler and Pasternak foundations. The figures show that the frequency ratio for all supports or the boundary conditions considered decreases as the number of walls increases. This response is because the carbon nanotubes generally have weak shear interactions between adjacent tubes and such interaction becomes more predominant as the number of walls increases. It could be inferred that the

Table 2. Parameters used for simulations.

Parameters	Symbol	Value
Young modulus	E	1.1 TPa
Density	ρ	1300 kg/m^3
Length	l	45 nm
Outer diameter	d_o	3 nm
Thickness of each layer	h	0.34 nm
Linear spring stiffness constant	k_1	10^7 N/m^2
Nonlinear spring stiffness constant	k_3	10^8 N/m^2
Magnetic field strength	H_x	107 A/m
Nonlocal parameter	e_oa	$1.5 \cdot 10^{-9}$ m
Inter layer constants	$c_1 = c_2 = c_3$	$0.3 \cdot 10^{12}$N/m^2

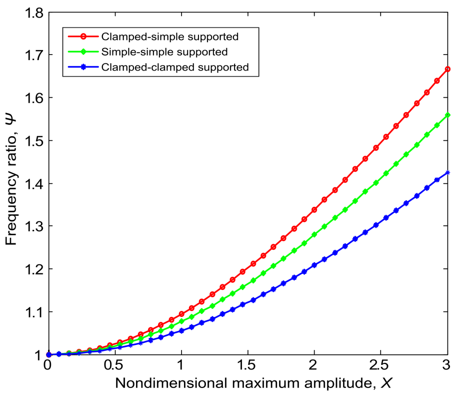


Fig. 4. Frequency ratio versus non-dimensional amplitude for SWCNT under various boundary conditions.

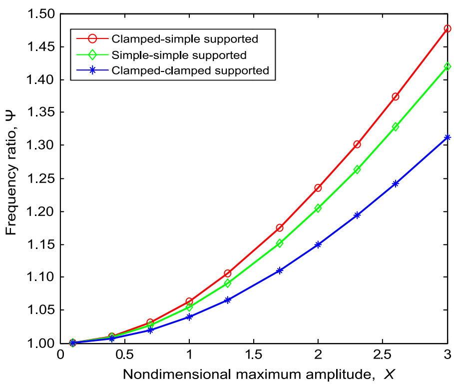


Fig. 5. Frequency ratio versus non-dimensional amplitude for DWCNT under various boundary conditions.

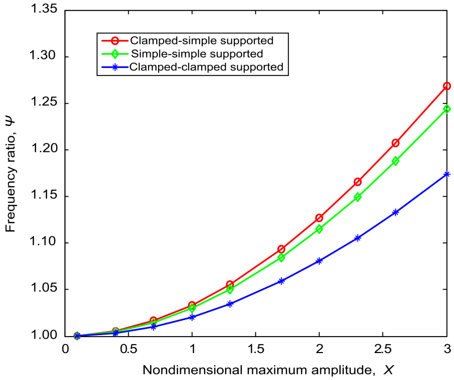


FIG. 6. Frequency ratio versus non-dimensional amplitude for TWCNT under various boundary conditions.

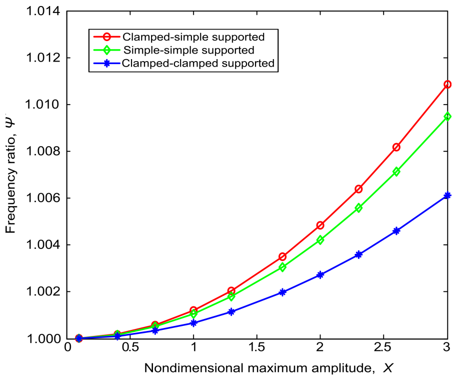


FIG. 7. Frequency ratio versus non-dimensional amplitude for QWCNT under various boundary conditions.

approach of increasing layers (with the same geometry and size) of the carbon nanotubes can be used to achieve the system stability as it changes the nonlinear vibration to linear vibration.

It is established from all the number of walls considered that the frequency ratio is minimum for C-C supported nanotubes and maximum for C-S supported nanotubes. This is because the C-C supported system provides the best grip (support) for the nanotubes. Therefore, the C-C supported system can be used to control the nonlinear vibration of the system.

4.2. Effects of spring stiffness (k_1) *on the frequency ratio*

The effects of the spring stiffness (k_1) on the SWCNT, DWCNT, TWCNT, and MWCNT frequency ratio are shown in Figs 8–11. From the results, the frequency ratio decreases with increases in the spring constant (k_1) because of

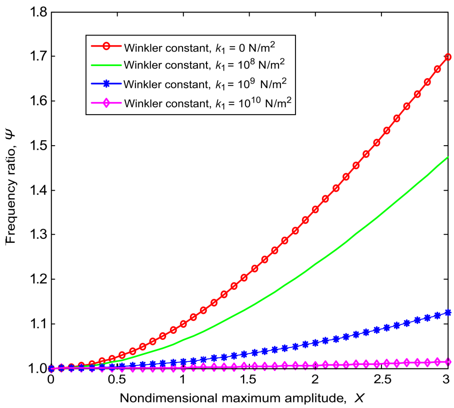


FIG. 8. Effect of Winkler constant (k_1) on the amplitude-frequency ratio curve for SWCN.

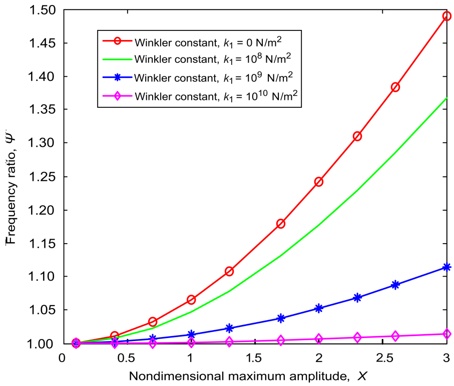


FIG. 9. Effect of Winkler constant (k_1) on the amplitude-frequency ratio curve for DWCNT.

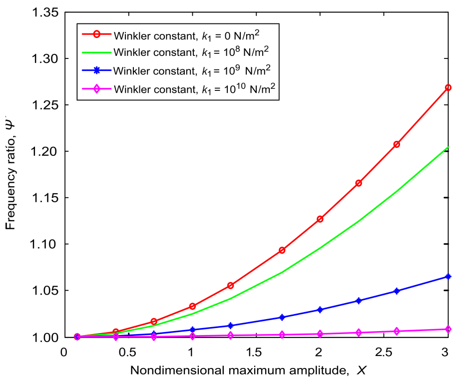


FIG. 10. Effect of Winkler constant (k_1) on the amplitude-frequency ratio curve for TWCNT.

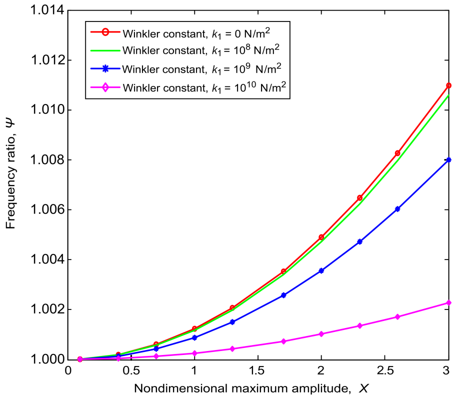


FIG. 11. Effect of Winkler constant (k_1) on the amplitude-frequency ratio curve for QWCNT.

the increase in the linear frequency. Also, it was recorded that at the large value of the spring constant ($k_1 = 10^{10}$ N/m^2), the system vibration becomes stable, and such a large value of the spring constant can be used as a good measure in controlling the nonlinear vibration of the system.

4.3. Effects of nonlinear spring stiffness (k_3) and van der Waals forces on the frequency ratio

The influence of nonlinear spring stiffness (k_3) and **van der Waals force** on the frequency ratio of outer walled of embedded DWCNT, TWCNT and QWCNT is presented in Figs 12–15. The frequency ratio increases with increases in the value of the nonlinear spring constant, but it decreases with an

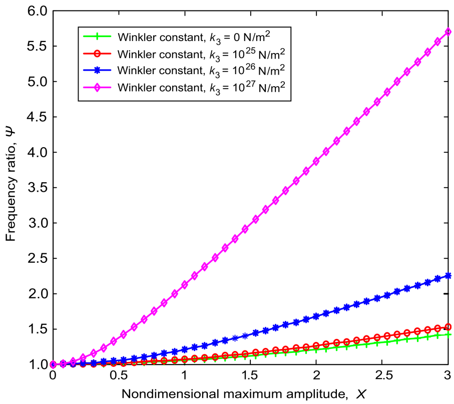


FIG. 12. Effect of the nonlinear spring constant (k_3) on the amplitude-frequency ratio curve for DWCNT.

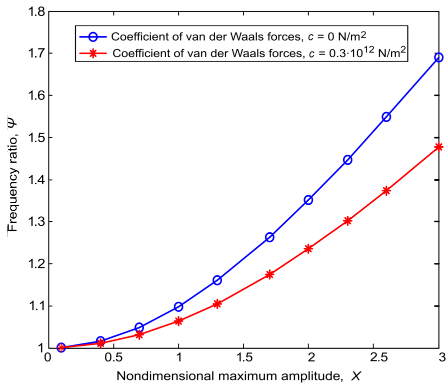


FIG. 13. Effect of van der Waals force on the amplitude-frequency ratio curve for DWCNT.

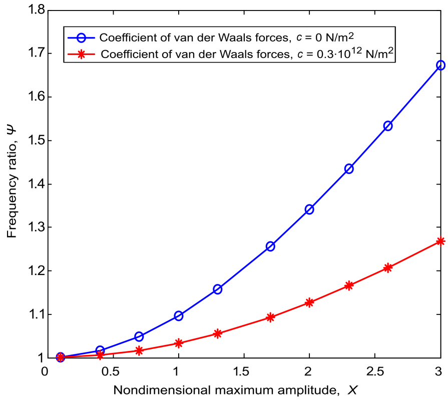


FIG. 14. Effect of van der Waals force on the amplitude-frequency ratio curve for TWCNT.

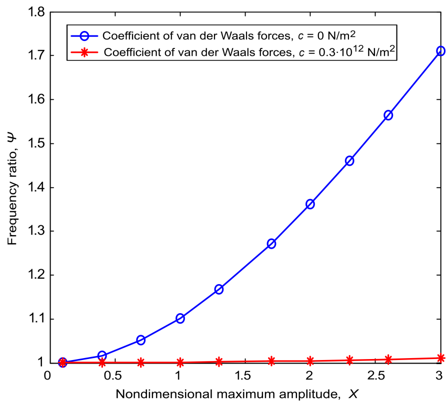


FIG. 15. Effect of van der Waals force on the amplitude-frequency ratio curve for QWCNT.

increase in the number of walls. The frequency ratio increases with increasing value of the nonlinear spring constant because the nonlinear frequency increases as the value of the nonlinear spring constant increases without producing any effect on the linear frequency. Therefore, the value of the nonlinear spring constant can be kept as low as possible and the number of walls can be raised to four and above, and these can be used to control the nonlinear vibration and instability of the system.

4.4. Effects of environmental temperature change and magnetic force strength on the frequency ratio of the outer wall of DWCNT

The effects of temperature and magnetic force strength on the frequency ratio on the outer wall of DWCNT are presented in Figs 16 and 17, respec-

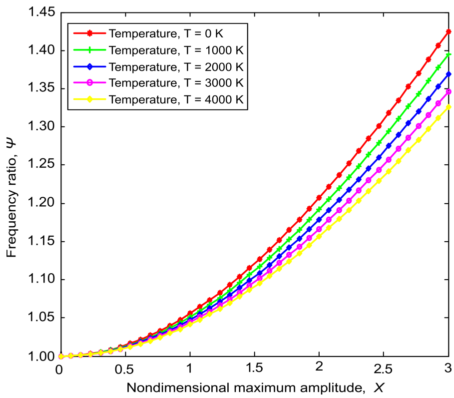


FIG. 16. Effect of magnetic force strength on the amplitude-frequency ratio curve on the outer wall of DWCNT.

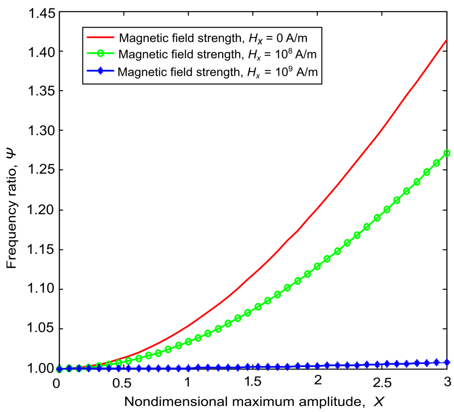


FIG. 17. Effect of magnetic force strength on the amplitude-frequency ratio curve on the outer wall of DWCNT.

tively. These figures show that the frequency ratio decreases with an increase in temperature and the magnetic field strength decreases. A further investigation revealed the same trend in SWCNT, TWCNT, and QWCNT. Such a response is because the linear natural frequency of the structure increases as the temperature of the nanotube increases. It should be noted that the vibration of the system approaches linear vibration at a higher value of magnetic force strength, $H = 10^9$ A/m. Such a range of values can be used to control the nonlinear vibration and instability of the system.

4.5. The linear and nonlinear vibration deflection-time curve of the outer wall of DWCNT

Although Fig. 18 presents the difference in the linear vibration with nonlinear vibration of the DWCNT only, the same trend is recorded for SWCNT, TWCNT, and QWCNT. The results show that the discrepancy between the linear and nonlinear maximum displacement increases as the vibration time progresses.

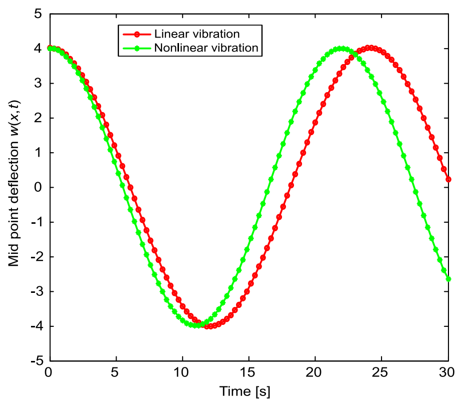


FIG. 18. The linear and nonlinear vibration deflection-time curve of the outer wall of DWCNT.

4.6. Effects of curvature on the frequency ratio of the single- and multi-walled nanotubes

The impacts of the radius of curvature on the frequency ratio of the SWCNT and MWCNT are presented in Fig. 19. The figure shows that as the ratio of the radius of curvature to the length of the nanotubes increases, the frequency ratio decreases. Therefore, the ratio of the radius of curvature to the length of the nanotubes can be used to control the nonlinear vibration and instability of the system.

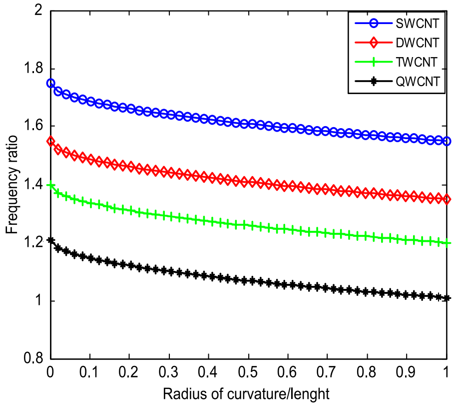


FIG. 19. Effects of the radius of curvature on the frequency ratio.

4.7. *Effects of the nonlocal parameter on the elastic foundations of the multi-walled nanotubes*

Figures 20 and 21 present the influences of nonlocal, Winkler and Pasternak foundation parameters on the elastic critical initial height of curvature of MWCNT. The Winkler and Pasternak foundation parameters increase as the initial height of the curvature increases. It is observed in Fig. 20 that the initial height of the curvature increases as the nonlocal parameter increases in the Winkler foundation. However, the initial height of the curvature decreases as the nonlocal parameter increases in the Pasternak foundation, as shown in Fig. 21. Such an initial height of the curvature causes a buckling of the nanotubes.

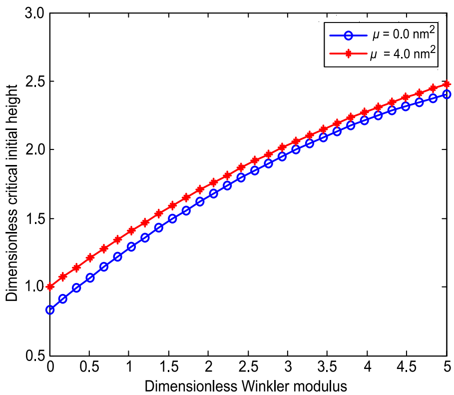


FIG. 20. Effect of the nonlocal parameter and Winkler parameter on the initial critical height.

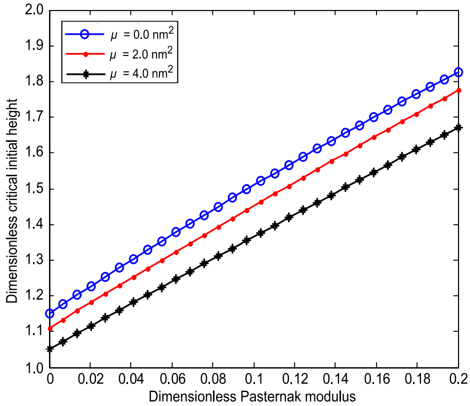


Fig. 21. Effect of the nonlocal parameter and Pasternak parameter on the initial critical height.

5. Conclusion

In this work, nonlocal elasticity theory was used to develop systems of nonlinear dynamics models of slightly curved multi-walled carbon nanotubes resting on linear and nonlinear foundations in a nonlinear thermal and magnetic environment. Parametric studies were carried on the effects of thermo-mechanical parameters on the dynamic responses of the slightly curved multi-walled carbon nanotubes. It was established that

(i) The frequency ratios decrease as the number of nanotube walls, temperature, spring constants, magnetic field strength and the ratio of the radius of curvature to the length of the slightly curved nanotubes increase. These trends were the same for all the boundary conditions considered. Therefore, the increasing layers (with the same geometry and size) of the carbon nanotubes, magnetic field, and temperature change can be used to achieve system stability or control the system's instability.

(ii) The C-S and C-C supported multi-walled nanotube have the highest and lowest frequency ratio, respectively. Therefore, a C-C supported multi-walled nanotube is recommended to achieve the system's better stability.

(iii) The initial amplitude increases the nonlinear natural frequencies of the multi-walled nanotubes. Therefore, the initial amplitude should be minimized for better system stability.

(iv) As the temperature change and magnetic field increase, the frequency ratio decreases as the linear natural frequency of the system increases. The significant effects of these parameters show the potentials in controlling the nonlinear vibration and instability of the system.

(v) The frequency ratio increases as the dimensionless amplitude increases. This is due to the "hardening spring" behavior of the nanotube.

(vi) To control nonlinear vibration of the carbon nanotubes, QWCNT can be taken as pure linear vibration even at any value of linear Winkler and Pasternak constants, and this can be used for restraining the chaos vibration in the objective structure.

The above results will greatly help in designing and manufacturing multi-walled nanoelectronics, nanodevices, nanomechanical systems, nanobiological devices and nanocomposites when subjected to thermal loads, magnetic fields and elastic foundations.

Appendix A. Homotopy perturbation method for the single-walled nanotube

To provide a solution to Eq. (3.15), a homotopy with ω_0 as the initial approximation for the angular nonlinear frequency was constructed as

$$(1-p)\left\{\omega_0^2\left(\frac{\mathrm{d}^2 a}{\mathrm{d}\tau^2}+a\right)\right\}+p\left(\omega_0^2\frac{\mathrm{d}^2 a}{\mathrm{d}\tau^2}+f_1 a+f_2 a^3\right)=0. \tag{A.1}$$

Assuming that the solution of Eq. (3.20) takes the form of:

$$a(\tau)=a_0(\tau)+pa_1(\tau)+p^2 a_2(\tau)+p^3 a_3(\tau)+\ldots, \tag{A.2}$$

and

$$\omega_0=\omega_0+p\omega_1+p^2\omega_2+p^3\omega_3+\ldots \tag{A.3}$$

On substituting Eq. (A.2) into the homotopy Eq. (A.1) and rearranging the coefficients of the terms with identical powers of p, one obtains series of linear differential equations as

$$p^0:\omega_0^2\left(\frac{\mathrm{d}^2 a_0}{\mathrm{d}\tau^2}+a_0\right)=0, \tag{A.4}$$

with the conditions

$$a_0(0)=X \qquad \text{and} \qquad \frac{\mathrm{d}a_0(0)}{\mathrm{d}\tau}=0, \tag{A.5}$$

$$p^1:\omega_0^2\left[\frac{\mathrm{d}^2 a_1}{\mathrm{d}\tau^2}+a_1-\left(\frac{\mathrm{d}^2 a_0}{\mathrm{d}\tau^2}+a_0\right)\right]+\omega_0^2\frac{\mathrm{d}^2 a_0}{\mathrm{d}\tau^2}+f_1 a_0+f_2 a_0^3=0, \tag{A.6}$$

with corresponding initial conditions

$$a_1(0)=0 \qquad \text{and} \qquad \frac{\mathrm{d}a_1(0)}{\mathrm{d}\tau}=0, \tag{A.7}$$

$$p^2:\omega_0^2\left(\frac{\mathrm{d}^2 a_2}{\mathrm{d}\tau^2}+a_2\right)-\omega_0^2 a_1+2\omega_0\omega_1\frac{\mathrm{d}^2 a_0}{\mathrm{d}\tau^2}+f_1 a_1-f_2 a_0^2 a_1=0, \tag{A.8}$$

and the initial conditions are

$$a_2(0)=0 \qquad \text{and} \qquad \frac{\mathrm{d}a_2(0)}{\mathrm{d}\tau}=0. \tag{A.9}$$

Since $\frac{\mathrm{d}^2 a_0}{\mathrm{d}\tau^2} + a_0 = 0$, which implies that $\frac{\mathrm{d}^2 a_0(0)}{\mathrm{d}\tau^2} = -a_0$ from Eq. (A.4), we can write Eq. (A.6) as

$$\omega_0^2\left(\frac{\mathrm{d}^2 a_1}{\mathrm{d}\tau^2} + a_1\right) - \omega_0^2 a_0 + f_1 a_0 + f_2 a_0^3 = 0. \tag{A.10}$$

The solution of the initial zeroth approximation is given by

$$a_0 = X\cos\tau. \tag{A.11}$$

On substituting Eq. (A.11) into the first approximation equation in Eq. (A.10), one obtains

$$\omega_0^2\left(\frac{\mathrm{d}^2 a_1}{\mathrm{d}\tau^2} + a_1\right) - \omega_0^2 X\cos\tau + f_1 X\cos\tau + f_2\left(X\cos\tau\right)^3 = 0. \tag{A.12}$$

Using trigonometry identities to the fourth-term on the left-hand side of Eq. (A.12), then

$$\omega_0^2\left(\frac{\mathrm{d}^2 a_1}{\mathrm{d}\tau^2} + a_1\right) - \omega_0^2 X\cos\tau + f_1 X\cos\tau + \frac{3}{4} f_2 X^3\cos\tau + \frac{1}{4} f_2 X^3 \cos 3\tau = 0. \tag{A.13}$$

The coefficient of $\cos\tau$ in Eq. (A.13) is set to zero in order to eliminate the secular terms

$$-\omega_0^2 X\cos\tau + f_1 X\cos\tau + \frac{3}{4} f_2 X^3\cos\tau = 0. \tag{A.14}$$

From Eq. (A.14), the nonlinear natural frequency is given as

$$\omega_0 = \sqrt{f_1 + \frac{3}{4} f_2 X^2}. \tag{A.15}$$

The frequency ratio is given as $\psi = \frac{\omega_0}{\omega}$. Therefore, from Eq. (A.5) and Eq. (3.20), the frequency ratio can be written as

$$\psi = \frac{\sqrt{f_1 + \frac{3}{4} f_2 X^2}}{\sqrt{f_1}} = \sqrt{1 + \frac{3}{4}\frac{f_2}{f_1} X^2}. \tag{A.16}$$

On substituting Eqs (3.16) and (3.17) into Eqs (A.7), the frequency ratio is developed as

$$\psi = \sqrt{1 + \frac{3}{4}\frac{\left(\alpha_3 k_3 - \alpha_4 \frac{EA}{2L} - 6\alpha_6 \mu k_3 - 3\alpha_7 \mu k_3 + \alpha_8 \mu \frac{EA}{2L}\right)\left(\frac{I}{A}\right)}{D^*_{(\mathrm{A}.17)}} X^2}, \tag{A.17}$$

where

$$D^*_{(\mathrm{A}.17)} = \alpha_1 EI + \alpha_2 k_1 - \alpha_5 \mu k_1 + (\alpha_1 \mu - \alpha_5)\Big(EA\alpha_x \Delta T + \left[h_1(1-2v) - 2h_2\left(v^2 - 1\right) + h_3 v^2\right] A\left(\alpha_x\right)^2 \left(\Delta T\right)^2 + \eta A H_x^2 + k_p\Big).$$

The solution of Eq. (3.32) gives

$$a_1(\tau) = \frac{f_2 X^3}{32 f_1 + 24 f_2 X^2}\left(\cos 3\tau - \cos\tau\right). \tag{A.18}$$

Therefore, the first approximate solution of Eq. (A.6) is

$$a(\tau) = a_0(\tau) + a_1(\tau) = X\cos\tau + \frac{f_2 X^3}{32 f_1 + 24 f_2 X^2}\left(\cos 3\tau - \cos\tau\right). \tag{A.19}$$

After substitution of Eq. (3.16) and (3.17), we arrived at

$$a(\tau) = X\cos\tau + \frac{N^*_{(A.20)}}{D^*_{(A.20)}}\left(\cos 3\tau - \cos\tau\right). \tag{A.20}$$

where

$$N^*_{(A.20)} = X^3\left(\alpha_3 k_3 - \alpha_4\frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}\right)\left(\frac{I}{A}\right),$$

$$\begin{aligned} D^*_{(A.20)} = 32\Big(&\alpha_1 EI + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5)\left(EA\alpha_x\Delta T\right.\\ &\left. + \left[h_1(1-2v) - 2h_2\left(v^2-1\right) + h_3 v^2\right]A(\alpha_x)^2(\Delta T)^2 + \eta A H_x^2 + k_p\right)\Big)\\ &+ 24X^2\left(\alpha_3 k_3 - \alpha_4\frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}\right)\left(\frac{I}{A}\right). \end{aligned}$$

From Eqs. (3.1) and (3.14), we have

$$w(x,t) = \phi(x)\,W(t), \qquad W = a(\tau)\sqrt{\frac{I}{A}}. \tag{A.21}$$

Therefore, the displacement of the nanotube is expressed as

$$w(x,t) = \phi(x)\,a(t)\sqrt{\frac{I}{A}}. \tag{A.22}$$

Substituting Eq. (A.19) and the shape functions in Table 1 into Eq. (A.23), we have for:

- **S-S support**

$$w(x,t) = \left(X\cos\omega_0 t + \frac{N^*_{(A.23)}}{D^*_{(A.23)}}\left(\cos 3\omega_0 t - \cos\omega_0 t\right)\right)\sqrt{\frac{I}{A}}\sin\left(\frac{n\pi x}{l}\right), \tag{A.23}$$

where

$$N^*_{(A.23)} = X^3\left(\alpha_3 k_3 - \alpha_4\frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}\right)\left(\frac{I}{A}\right),$$

$$\begin{aligned} D^*_{(A.23)} = 32\Big(&\alpha_1 EI + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5)\big(EA\alpha_x\Delta T\\ &+ \left[h_1(1-2v) - 2h_2\left(v^2-1\right) + h_3 v^2\right]A(\alpha_x)^2(\Delta T)^2 + \eta A H_x^2 + k_p\big)\Big)\\ &+ 24X^2\left(\alpha_3 k_3 - \alpha_4\frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}\right)\left(\frac{I}{A}\right). \end{aligned}$$

- **C-C support**

$$\begin{aligned} w(x,t) = &\left(X\cos\omega_0 t + \frac{N^*_{(A.24)}}{D^*_{(A.24)}}\left(\cos 3\omega_0 t - \cos\omega_0 t\right)\right)\\ &\cdot\sqrt{\frac{I}{A}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cosh\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\sinh\beta + \sin\beta}{\cosh\beta - \cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\}, \end{aligned} \tag{A.24}$$

where

$$N^*_{(A.24)} = X^3\left(\alpha_3 k_3 - \alpha_4 \frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}\right)\left(\frac{I}{A}\right),$$

$$D^*_{(A.24)} = 32\Big(\alpha_1 EI + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5)\big(EA\alpha_x\Delta T + \left[h_1(1-2v) - 2h_2\left(v^2-1\right) + h_3 v^2\right]A(\alpha_x)^2(\Delta T)^2 + \eta A H_x^2 + k_p\big)\Big) + 24X^2\left(\alpha_3 k_3 - \alpha_4\frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}\right)\left(\frac{I}{A}\right).$$

- **C-S support**

(A.25)
$$w(x,t) = \left(X\cos\omega_0 t + \frac{N^*_{(A.25)}}{D^*_{(A.25)}}(\cos 3\omega_0 t - \cos\omega_0 t)\right) \cdot\sqrt{\frac{I}{A}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\cosh\beta - \cos\beta}{\sinh\beta - \sin\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right)\right\},$$

where

$$N^*_{(A.25)} = X^3\left(\alpha_3 k_3 - \alpha_4 \frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}\right)\left(\frac{I}{A}\right),$$

$$D^*_{(A.25)} = 32\Big(\alpha_1 EI + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5)\big(EA\alpha_x\Delta T + \left[h_1(1-2v) - 2h_2\left(v^2-1\right) + h_3 v^2\right]A(\alpha_x)^2(\Delta T)^2 + \eta A H_x^2 + k_p\big)\Big) + 24X^2\left(\alpha_3 k_3 - \alpha_4\frac{EA}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA}{2L}\right)\left(\frac{I}{A}\right).$$

Appendix B. Homotopy perturbation method for the double-walled nanotube

Following similar procedures for the analysis of SWCNT, a homotopy is constructed for Eqs (3.26) and (3.27) as

(B.1)
$$(1-p)\left\{\omega_0^2\left(\frac{\mathrm{d}^2 a_1}{\mathrm{d}\tau^2} + a_1\right)\right\} + p\left\{\omega_0^2\frac{\mathrm{d}^2 a_1}{\mathrm{d}\tau^2} + f_1 a_1 + f_2 a_1^3 - f_3 a_2\right\} = 0,$$

(B.2)
$$(1-p)\left\{\omega_0^2\left(\frac{\mathrm{d}^2 a_2}{\mathrm{d}\tau^2} + a_2\right)\right\} + p\left\{\omega_0^2\frac{\mathrm{d}^2 a_2}{\mathrm{d}\tau^2} + g_1 a_2 + g_2 a_2^3 - g_3 a_1\right\} = 0.$$

Taking the solutions of Eqs (3.48) and (3.49) as

(B.3)
$$a_1(\tau) = a_{10}(\tau) + p\,a_{11}(\tau) + p^2 a_{12}(\tau) + p^3 a_{13}(\tau) + ...,$$

(B.4)
$$a_2(\tau) = a_{20}(\tau) + p\,a_{21}(\tau) + p^2 a_{22}(\tau) + p^3 a_{23}(\tau) + ...,$$

(B.5)
$$\omega = \omega_0 + p\,\omega_1 + p^2\omega_2 + p^3\omega_3 + ...$$

After the substitutions of Eqs (B.3)–(B.5) into the homotopy in Eqs (B.1) and (B.2), a series of linear differential equations is generated upon collecting and rearranging the coefficients of the terms with identical powers of p,

$$\text{(B.6)}\qquad p^0 : \begin{cases} \frac{\mathrm{d}^2 a_{10}}{\mathrm{d}\tau^2} + a_{10} = 0, \quad a_{10}(0) = X_1, \quad \frac{\mathrm{d} a_{10}(0)}{\mathrm{d}\tau} = 0, \\ \frac{\mathrm{d}^2 a_{20}}{\mathrm{d}\tau^2} + a_{20} = 0, \quad a_{20}(0) = X_1, \quad \frac{\mathrm{d} a_{20}(0)}{\mathrm{d}\tau} = 0, \end{cases}$$

$$\text{(B.7)}\qquad p^1 : \begin{cases} \omega_0^2 \left\{ \frac{\mathrm{d}^2 a_{11}}{\mathrm{d}\tau^2} + a_{11} \right\} - \omega_0^2 a_{10} + f_1 a_{10} + f_2 a_{10}^3 - f_3 a_{20} = 0, \\ \qquad a_{11}(0) = 0, \quad \frac{\mathrm{d} a_{11}(0)}{\mathrm{d}\tau} = 0, \\ \omega_0^2 \left\{ \frac{\mathrm{d}^2 a_{21}}{\mathrm{d}\tau^2} + a_{21} \right\} - \omega_0^2 a_{20} + g_1 a_{20} + g_2 a_{20}^3 - g_3 a_{10} = 0, \\ \qquad a_{21}(0) = 0, \quad \frac{\mathrm{d} a_{21}(0)}{\mathrm{d}\tau} = 0, \end{cases}$$

$$\text{(B.8)}\qquad p^2 : \begin{cases} \omega_0^2 \left\{ \frac{\mathrm{d}^2 a_{12}}{\mathrm{d}\tau^2} + a_{12} \right\} - \omega_0^2 a_{11} + 2\omega_0\omega_1 \frac{\mathrm{d}^2 a_{10}}{\mathrm{d}\tau^2} + f_1 a_{11} + 3 f_2 a_{10}^2 a_{11} - f_3 a_{21} = 0, \\ \qquad a_{12}(0) = 0, \quad \frac{\mathrm{d} a_{12}(0)}{\mathrm{d}\tau} = 0, \\ \omega_0^2 \left\{ \frac{\mathrm{d}^2 a_{22}}{\mathrm{d}\tau^2} + a_{22} \right\} - \omega_0^2 a_{21} + 2\omega_0\omega_1 \frac{\mathrm{d}^2 a_{20}}{\mathrm{d}\tau^2} + g_1 a_{21} + 3 g_2 a_{20}^2 a_{21} - g_3 a_{11} = 0, \\ \qquad a_{22}(0) = 0, \quad \frac{\mathrm{d} a_{22}(0)}{\mathrm{d}\tau} = 0, \end{cases}$$

$$\vdots$$

The solution of the initial zeroth approximation in Eq. (B.6) is given by

$$\text{(B.9)}\qquad a_{10}(0) = X_1 \cos\tau,$$

$$\text{(B.10)}\qquad a_{20}(0) = X_2 \cos\tau.$$

On substituting Eqs (B.9) and (B.10) into the first approximation in Eq. (B.7), after the elimination of the secular terms, the following nonlinear systems of equations are developed:

$$\text{(B.11)}\qquad -X_1\omega_0^2 + f_1 X_1 + \frac{3}{4} f_2 X_1^3 - f_3 X_2 = 0,$$

$$\text{(B.12)}\qquad -X_2\omega_0^2 + g_1 X_2 + \frac{3}{4} g_2 X_2^3 - g_3 X_1 = 0.$$

From Eq. (B.11),

$$\text{(B.13)}\qquad X_2 = \frac{-X_1\omega_0^2 + f_1 X_1 + \frac{3}{4} f_2 X_1^3}{f_3}.$$

After the substitution of Eq. (B.13) into Eq. (B.12), the resulting equation is

$$\text{(B.14)}\qquad -\omega_0^2 \left(\frac{-X_1\omega_0^2 + f_1 X_1 + \frac{3}{4} f_2 X_1^3}{f_3} \right) + g_1 \left(\frac{-X_1\omega_0^2 + f_1 X_1 + \frac{3}{4} f_2 X_1^3}{f_3} \right) + \frac{3}{4} g_2 \left(\frac{-X_1\omega_0^2 + f_1 X_1 + \frac{3}{4} f_2 X_1^3}{f_3} \right)^3 - g_3 X_1 = 0.$$

Expanding Eq. (B.14) and collecting like terms, one gets

$$\text{(B.15)}\quad \frac{3}{4}\frac{g_2}{f_3^3}X_1^3\omega_0^6+\left(\frac{X_1}{f_3}-\frac{3}{4}\frac{g_2}{f_3^3}X_1^2\left(f_1X_1+\frac{3}{2}f_2X_1^3\right)-\frac{9}{8}\frac{g_2}{f_3^3}f_1f_2X_1^8\right)\omega_0^4$$
$$+\left(\frac{f_1X_1+\frac{3}{4}f_2X_1^3+g_1X_1}{f_3}+\frac{3}{4}\frac{g_2}{f_3^3}f_1^2X_1^3+\frac{9}{16}f_2^2X_1^7+2X_1\left(f_1^2X_1^2+\frac{9}{16}f_2^2X_1^6\right)+\frac{9}{2}f_1f_2X_1^5\right)\omega_0^2$$
$$+g_3X_1-g_1\left(\frac{f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)-\frac{3}{4}\frac{g_2}{f_3^3}\left[f_1^3X_1^3+\frac{27}{64}f_2^3X_1^6+\frac{3}{4}f_1^2f_2X_1^5\right.$$
$$\left.+\frac{9}{16}f_1f_2^2X_1^7+2\left(\frac{3}{4}f_1^2f_2X_1^5+\frac{9}{16}f_1f_2^2X_1^7\right)\right]=0.$$

Equation (B.15) can be written as

$$\text{(B.16)}\qquad \lambda_1\omega_0^6+\lambda_2\omega_0^4+\lambda_3\omega_0^2+\lambda_4=0,$$

where

$$\text{(B.17)}\qquad \begin{aligned}
\lambda_1&=\frac{3}{4}\frac{g_2}{f_3^3}X_1^3,\\
\lambda_2&=\frac{X_1}{f_3}-\frac{3}{4}\frac{g_2}{f_3^3}X_1^2\left(f_1X_1+\frac{3}{2}f_2X_1^3\right)-\frac{9}{8}\frac{g_2}{f_3^3}f_1f_2X_1^8,\\
\lambda_3&=\frac{f_1X_1+\frac{3}{4}f_2X_1^3+g_1X_1}{f_3}+\frac{3}{4}\frac{g_2}{f_3^3}f_1^2X_1^3+\frac{9}{16}f_2^2X_1^7\\
&\qquad+2X_1\left(f_1^2X_1^2+\frac{9}{16}f_2^2X_1^6\right)+\frac{9}{2}f_1f_2X_1^5,\\
\lambda_4&=g_3X_1-g_1\left(\frac{f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)-\frac{3}{4}\frac{g_2}{f_3^3}\left[f_1^3X_1^3+\frac{27}{64}f_2^3X_1^6+\frac{3}{4}f_1^2f_2X_1^5\right.\\
&\qquad\left.+\frac{9}{16}f_1f_2^2X_1^7+2\left(\frac{3}{4}f_1^2f_2X_1^5+\frac{9}{16}f_1f_2^2X_1^7\right)\right].
\end{aligned}$$

The roots of the sextic equation are

$$\text{(B.18)}\qquad (\omega_0)_1=\sqrt{A^*_{(\mathrm{B.18})-(\mathrm{B.23})}-\frac{\lambda_2}{3\lambda_1}},$$

$$\text{(B.19)}\qquad (\omega_0)_2=-\sqrt{A^*_{(\mathrm{B.18})-(\mathrm{B.23})}-\frac{\lambda_2}{3\lambda_1}},$$

$$\text{(B.20)}\qquad (\omega_0)_3=\sqrt{\frac{-A^*_{(\mathrm{B.18})-(\mathrm{B.23})}}{2\lambda_1}+\frac{\sqrt{-3}A^*_{(\mathrm{B.18})-(\mathrm{B.23})}}{2\lambda_1}-\frac{\lambda_2}{3\lambda_1}},$$

$$\text{(B.21)}\qquad (\omega_0)_4=-\sqrt{\frac{-A^*_{(\mathrm{B.18})-(\mathrm{B.23})}}{2\lambda_1}+\frac{\sqrt{-3}A^*_{(\mathrm{B.18})-(\mathrm{B.23})}}{2\lambda_1}-\frac{\lambda_2}{3\lambda_1}},$$

$$\text{(B.22)}\qquad (\omega_0)_5=\sqrt{\frac{-A^*_{(\mathrm{B.18})-(\mathrm{B.23})}}{2\lambda_1}-\frac{\sqrt{-3}A^*_{(\mathrm{B.18})-(\mathrm{B.23})}}{2\lambda_1}-\frac{\lambda_2}{3\lambda_1}},$$

$$(\omega_0)_6 = -\sqrt{\frac{-A^*_{(B.18)-(B.23)}}{2\lambda_1} - \frac{\sqrt{-3}A^*_{(B.18)-(B.23)}}{2\lambda_1} - \frac{\lambda_2}{3\lambda_1}}, \tag{B.23}$$

where

$$A^*_{(B.18)-(B.23)} = \sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) + \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}}$$
$$+ \sqrt[3]{\left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right) - \sqrt{\left(\frac{\lambda_3}{3\lambda_1} - \frac{\lambda_2^2}{9\lambda_1^2}\right)^3 + \left(\frac{-\lambda_2^3}{27\lambda_1^3} + \frac{\lambda_2\lambda_3}{6\lambda_1^2} - \frac{\lambda_4}{2\lambda_1}\right)^2}}.$$

The smallest real value of ω_0 is the nonlinear natural frequency for DWCNT. Therefore,

$$\omega_0 = \sqrt{A^*_{(B.18)-(B.23)} - \frac{\lambda_2}{3\lambda_1}}. \tag{B.24}$$

To find the expression for the linear natural frequencies for DWNT, substitute

$$a_{10}(\tau) = X_1 \cos\tau, \tag{B.25}$$

$$a_{20}(\tau) = X_2 \cos\tau, \tag{B.26}$$

into Eqs (B.11) and (B.12) after neglecting the nonlinear terms, which gives

$$-X_1\omega^2 + f_1X_1 - f_3X_2 = 0, \tag{B.27}$$

$$-X_2\omega^2 + g_1X_2 - g_3X_1 = 0, \tag{B.28}$$

which can be written in matrix form as

$$\begin{bmatrix} -\omega^2 + f_1 & -f_1 \\ -g_3 & -\omega^2 + g_1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{B.29}$$

For the nontrivial case to occur,

$$\begin{bmatrix} -\omega^2 + f_1 & -f_1 \\ -g_3 & -\omega^2 + g_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{B.30}$$

The determinant of the matrix in Eq. (B.30) is equal to zero and the frequency characteristic equation is developed as

$$\omega^4 - (f_1 + g_1)\,\omega^2 + f_1g_1 - f_3g_3 = 0, \tag{B.31}$$

where the roots of the quartic equation are

$$(\omega)_1 = \sqrt{\frac{(f_1+g_1)+\sqrt{(f_1+g_1)^2-4(f_1g_1-f_3g_3)}}{2}}, \tag{B.32}$$

$$(\omega)_2 = -\sqrt{\frac{(f_1+g_1)+\sqrt{(f_1+g_1)^2-4(f_1g_1-f_3g_3)}}{2}}, \tag{B.33}$$

$$(\omega)_3 = \sqrt{\frac{(f_1+g_1)-\sqrt{(f_1+g_1)^2-4(f_1g_1-f_3g_3)}}{2}}, \tag{B.34}$$

$$(\omega)_4 = -\sqrt{\frac{(f_1+g_1)-\sqrt{(f_1+g_1)^2-4(f_1g_1-f_3g_3)}}{2}}. \tag{B.35}$$

The linear natural frequency of DWNT is the lowest root, therefore,

$$\omega = \sqrt{\frac{(f_1+g_1)-\sqrt{(f_1+g_1)^2-4(f_1g_1-f_3g_3)}}{2}}. \tag{B.36}$$

Recall that the frequency ratio is defined as $\psi = \frac{\omega_0}{\omega}$

$$\psi = \sqrt{\frac{2A^*_{(\mathrm{B}.18)-(\mathrm{B}.23)} - \frac{\lambda_2}{3\lambda_1}}{\left((f_1+g_1)-\sqrt{(f_1+g_1)^2-4(f_1g_1-f_3g_3)}\right)}}. \tag{B.37}$$

On substituting Eqs (B.25) and (B.26) into Eqs (B.7) and (B.8), we have

$$\begin{cases} \omega_0^2\left\{\frac{\mathrm{d}^2a_{11}}{\mathrm{d}\tau^2}+a_{11}\right\} - \omega_0^2X_1\cos\tau + f_1X_1\cos\tau + f_2(X_1\cos\tau)^3 - f_3X_2\cos\tau = 0, \\ \qquad a_{11}(0)=0, \quad \frac{\mathrm{d}a_{11}(0)}{\mathrm{d}\tau}=0, \\ \omega_0^2\left\{\frac{\mathrm{d}^2a_{21}}{\mathrm{d}\tau^2}+a_{21}\right\} - \omega_0^2X_2\cos\tau + g_1X_2\cos\tau + g_2(X_2\cos\tau)^3 - g_3X_1\cos\tau = 0, \\ \qquad a_{21}(0)=0, \quad \frac{\mathrm{d}a_{21}(0)}{\mathrm{d}\tau}=0. \end{cases} \tag{B.38}$$

After the application of trigonometry identities to the fourth-term on the left-hand side we have

$$\begin{cases} \omega_0^2\left\{\frac{\mathrm{d}^2a_{11}}{\mathrm{d}\tau^2}+a_{11}\right\} - \omega_0^2X_1\cos\tau + f_1X_1\cos\tau + \frac{3}{4}f_2X_1^3\cos\tau + \frac{1}{4}f_2X_1^3\cos3\tau \\ \qquad -f_3X_2\cos\tau = 0, \quad a_{11}(0)=0, \quad \frac{\mathrm{d}a_{11}(0)}{\mathrm{d}\tau}=0, \\ \omega_0^2\left\{\frac{\mathrm{d}^2a_{21}}{\mathrm{d}\tau^2}+a_{21}\right\} - \omega_0^2X_2\cos\tau + g_1X_2\cos\tau + \frac{3}{4}g_2X_2^3\cos\tau + \frac{1}{4}g_2X_2^3\cos3\tau \\ \qquad -g_3X_1\cos\tau = 0, \quad a_{21}(0)=0, \quad \frac{\mathrm{d}a_{21}(0)}{\mathrm{d}\tau}=0. \end{cases} \tag{B.39}$$

It can be easily shown that the solutions of Eq. (B.39) are

$$a_{11}(\tau) = \frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos3\tau-\cos\tau)}{D^*_{(\mathrm{B}.40)}}, \tag{B.40}$$

where

$$D^*_{(B.40)} = 32\left(\frac{N^*_{(B.40)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)\,EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(B.40)} = \alpha_1 EI_1 + (\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1 H_x^2+k_p\Big),$$

(B.41)
$$a_{21}(\tau) = \frac{X_2^3\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)\left(\frac{I_1}{A_1}\right)(\cos 3\tau-\cos\tau)}{D^*_{(B.41)}},$$

where

$$D^*_{(B.41)} = 32\left(\frac{N^*_{(B.41)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}+\frac{c_1}{\rho A_2}\right) + 24\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(B.41)} = \alpha_1 EI_2+\alpha_2k_1-\alpha_5\mu k_1 + (\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2+\eta A_2 H_x^2+k_p\Big).$$

Therefore, the first approximate solutions of Eqs (37) and (38) are:

(B.42)
$$a_1(x,\tau) = X_1\cos\tau + \frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\tau-\cos\tau)}{D^*_{(B.42)}},$$

where

$$D^*_{(B.42)} = 32\left(\frac{N^*_{(B.42)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(B.42)} = \alpha_1 EI_1 + (\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1 H_x^2+k_p\Big),$$

(B.43)
$$a_2(x,\tau)=X_2\cos\tau+\frac{X_2^3\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)\left(\frac{I_1}{A_1}\right)(\cos 3\tau-\cos\tau)}{D^*_{(B.43)}},$$

where

$$D^*_{(\mathrm{B}.43)} = 32\left(\frac{N^*_{(\mathrm{B}.43)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}+\frac{c_1}{\rho A_2}\right) + 24\left(\frac{\alpha_3 k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(\mathrm{B}.43)} = \alpha_1 EI_2+\alpha_2 k_1-\alpha_5\mu k_1+(\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2+\eta A_2H_x^2+k_p\Big).$$

The displacements of the nanotubes are expressed as follows.

- **S-S support**

$$w_1(x,t)=\left(X_1\cos\omega_0 t+\frac{\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)X^3(\cos 3\omega_0 t-\cos\omega_0 t)}{D^*_{(\mathrm{B}.44)}}\right)\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right), \tag{B.44}$$

where

$$D^*_{(\mathrm{B}.44)} = 32\left(\frac{N^*_{(\mathrm{B}.44)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(\mathrm{B}.44)} = \alpha_1 EI_1+(\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1H_x^2+k_p\Big),$$

$$w_2(x,t)=\left(X_2\cos\omega_0 t+\frac{X_2^3\left(\frac{\alpha_3 k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)}{D^*_{(\mathrm{B}.45)}}\cdot\left(\frac{I_1}{A_1}\right)(\cos 3\omega_0 t-\cos\omega_0 t)\right)\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right), \tag{B.45}$$

where

$$D^*_{(\mathrm{B}.45)} = 32\left(\frac{N^*_{(\mathrm{B}.45)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}+\frac{c_1}{\rho A_2}\right) + 24\left(\frac{\alpha_3 k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(\mathrm{B}.45)} = \alpha_1 EI_2+\alpha_2 k_1-\alpha_5\mu k_1+(\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2+\eta A_2H_x^2+k_p\Big).$$

- **C-C support**

$$w_1(x,t) = \frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\tau - \cos\tau)}{D^*_{(\mathrm{B.46})}} \cdot \sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\}, \tag{B.46}$$

where

$$D^*_{(\mathrm{B.46})} = 32\left(\frac{N^*_{(\mathrm{B.46})}}{(\alpha_2-\alpha_5\mu)\,\rho A_1} + \frac{c_1}{\rho A_1}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(\mathrm{B.46})} = \alpha_1 EI_1 + (\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v) - 2h_{21}\left(v^2-1\right) + h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2 + \eta A_1 H_x^2 + k_p\Big),$$

$$w_2(x,t) = \left(X_2\cos\omega_0 t + \frac{X_2^3\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)}{D^*_{(\mathrm{B.47})}} \cdot \left(\frac{I_1}{A_1}\right)(\cos 3\omega_0 t - \cos\omega_0 t)\right)\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\}, \tag{B.47}$$

where

$$D^*_{(\mathrm{B.47})} = 32\left(\frac{N^*_{(\mathrm{B.47})}}{(\alpha_2-\alpha_5\mu)\,\rho A_2} + \frac{c_1}{\rho A_2}\right) + 24\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(\mathrm{B.47})} = \alpha_1 EI_2 + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v) - 2h_{22}\left(v^2-1\right) + h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2 + \eta A_2 H_x^2 + k_p\Big).$$

- **C-S support**

$$w_1(x,t) = \left(X_1\cos\omega_0 t + \frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\tau - \cos\tau)}{D^*_{(\mathrm{B.48})}}\right)\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\}, \tag{B.48}$$

where

$$D^*_{(B.48)} = 32\left(\frac{N^*_{(B.48)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(B.48)} = \alpha_1 EI_1 + (\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1 H_x^2+k_p\Big),$$

(B.49)
$$w_2(x,t) = \left(X_2\cos\omega_0 t + \frac{X_2^3\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_2}\right)}{D^*_{(B.49)}}\cdot\left(\frac{I_1}{A_1}\right)(\cos 3\omega_0 t-\cos\omega_0 t)\right)\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(B.49)} = 32\left(\frac{N^*_{(B.49)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}+\frac{c_1}{\rho A_2}\right) + 24\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_2}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_2}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(B.49)} = \alpha_1 EI_2+\alpha_2k_1-\alpha_5\mu k_1 + (\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2+\eta A_2H_x^2+k_p\Big).$$

Appendix C. Homotopy perturbation method for the tripled-walled nanotube

Following the same procedure as in the previous analysis, a homotopy is constructed on Eqs (3.39)–(3.41) as follows:

(C.1)
$$(1-p)\left\{\omega_0^2\left(\frac{\mathrm{d}^2a_1}{\mathrm{d}\tau^2}+a_1\right)\right\}+p\left\{\omega_0^2\frac{\mathrm{d}^2a_1}{\mathrm{d}\tau^2}+f_1a_1+f_2a_1^3-f_3a_2\right\}=0,$$

(C.2)
$$(1-p)\left\{\omega_0^2\left(\frac{\mathrm{d}^2a_2}{\mathrm{d}\tau^2}+a_2\right)\right\}+p\left\{\omega_0^2\frac{\mathrm{d}^2a_2}{\mathrm{d}\tau^2}+g_1a_2+g_2a_2^3-g_3a_1\right\}=0,$$

(C.3)
$$(1-p)\left\{\omega_0^2\left(\frac{d^2a_3}{d\tau^2}+a_3\right)\right\}+p\left\{\omega_0^2\frac{d^2a_3}{d\tau^2}+\hbar_1a_3+\hbar_2a_3^3-\hbar_3a_2\right\}=0.$$

We assume that the solutions of Eqs (3.39)–(3.41) are in the following forms:

$$a_1(\tau) = a_{10}(\tau) + pa_{11}(\tau) + p^2 a_{12}(\tau) + p^3 a_{13}(\tau) + \ldots, \tag{C.4}$$

$$a_2(\tau) = a_{20}(\tau) + pa_{21}(\tau) + p^2 a_{22}(\tau) + p^3 a_{23}(\tau) + \ldots, \tag{C.5}$$

$$a_3(\tau) = a_{30}(\tau) + pa_{31}(\tau) + p^2 a_{32}(\tau) + p^3 a_{33}(\tau) + \ldots, \tag{C.6}$$

$$\omega = \omega_0 + p\omega_1 + p^2\omega_2 + p^3\omega_3 + \ldots \tag{C.7}$$

After the substitutions of Eqs (C.4)–(C.7) into the homotopy in Eqs (C.1)–(C.3), a series of linear differential equations is generated after collecting and rearranging the coefficients of the terms with identical powers of p,

$$p^0 : \begin{cases} \frac{d^2 a_{10}}{d\tau^2} + a_{10} = 0, \quad a_{10}(0) = X_1, \quad \frac{d a_{10}(0)}{d\tau} = 0, \\ \frac{d^2 a_{20}}{d\tau^2} + a_{20} = 0, \quad a_{20}(0) = X_1, \quad \frac{d a_{20}(0)}{d\tau} = 0, \\ \frac{d^2 a_{30}}{d\tau^2} + a_{30} = 0, \quad a_{30}(0) = X_1, \quad \frac{d a_{30}(0)}{d\tau} = 0, \end{cases} \tag{C.8}$$

$$p^1 : \begin{cases} \omega_0^2 \left\{ \frac{d^2 a_{11}}{d\tau^2} + a_{11} \right\} - \omega_0^2 a_{10} + f_1 a_{10} + f_2 a_{10}^3 - f_3 a_{20} = 0, \\ \qquad a_{11}(0) = 0, \quad \frac{d a_{11}(0)}{d\tau} = 0, \\ \omega_0^2 \left\{ \frac{d^2 a_{21}}{d\tau^2} + a_{21} \right\} - \omega_0^2 a_{20} + g_1 a_{20} + g_2 a_{20}^3 - g_3 a_{10} = 0, \\ \qquad a_{21}(0) = 0, \quad \frac{d a_{21}(0)}{d\tau} = 0, \\ \omega_0^2 \left\{ \frac{d^2 a_{31}}{d\tau^2} + a_{31} \right\} - \omega_0^2 a_{30} + h_1 a_{30} + h_2 a_{30}^3 - h_3 a_{20} = 0, \\ \qquad a_{31}(0) = 0, \quad \frac{d a_{31}(0)}{d\tau} = 0, \end{cases} \tag{C.9}$$

$$p^2 : \begin{cases} \omega_0^2 \left\{ \frac{d^2 a_{12}}{d\tau^2} + a_{12} \right\} - \omega_0^2 a_{11} + 2\omega_0\omega_1 \frac{d^2 a_{10}}{d\tau^2} + f_1 a_{11} + 3 f_2 a_{10}^2 a_{11} - f_3 a_{21} = 0, \\ \qquad a_{12}(0) = 0, \quad \frac{d a_{12}(0)}{d\tau} = 0, \\ \omega_0^2 \left\{ \frac{d^2 a_{22}}{d\tau^2} + a_{22} \right\} - \omega_0^2 a_{21} + 2\omega_0\omega_1 \frac{d^2 a_{20}}{d\tau^2} + g_1 a_{21} + 3 g_2 a_{20}^2 a_{21} - g_3 a_{11} = 0, \\ \qquad a_{22}(0) = 0, \quad \frac{d a_{22}(0)}{d\tau} = 0, \\ \omega_0^2 \left\{ \frac{d^2 a_{32}}{d\tau^2} + a_{32} \right\} - \omega_0^2 a_{31} + 2\omega_0\omega_1 \frac{d^2 a_{30}}{d\tau^2} + \hbar_1 a_{31} + 3 \hbar_2 a_{30}^2 a_{31} - \hbar_3 a_{21} = 0, \\ \qquad a_{32}(0) = 0, \quad \frac{d a_{32}(0)}{d\tau} = 0, \end{cases} \tag{C.10}$$

$$\vdots$$

In order to calculate the nonlinear natural frequencies for TWCNT, initial zeroth approximations of the following forms are assumed

$$a_{10}(\tau) = X_1 \cos\tau, \tag{C.11}$$

$$a_{20}(\tau) = X_2 \cos\tau, \tag{C.12}$$

$$a_{30}(\tau) = X_3 \cos\tau. \tag{C.13}$$

After substitution of Eqs (C.11)–(C.13) into Eqs (C.9), and setting the coefficient of $\cos\tau$ to zero in order to eliminate the secular terms, the following nonlinear systems of equations are established

$$-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3 - f_3X_2 = 0, \tag{C.14}$$

$$-X_2\omega_0^2 + g_1X_2 + \frac{3}{4}g_2X_2^3 - g_3X_1 - g_4X_3 = 0, \tag{C.15}$$

$$-X_1\omega_0^2 + \hbar_1X_3 + \frac{3}{4}\hbar_2X_3^3 - \hbar_3X_2 = 0. \tag{C.16}$$

From Eq. (C.14)

$$X_2 = \frac{-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}. \tag{C.17}$$

Substitution of X_2 in Eq. (C.17) into Eq. (C.15) makes X_3 to be

$$X_3 = \frac{-\omega_0^2\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right) + g_1\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)}{g_4} + \frac{\frac{3}{4}g_2\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)^3 - g_3X_1}{g_4}. \tag{C.18}$$

Also, the substitution of Eq. (C.17) and (C.18) into Eq. (C.16) gives

$$-\omega_0^2\left(\frac{N^*_{(\mathrm{C.19})}}{g_4}\right) + \hbar_1\left(\frac{N^*_{(\mathrm{C.19})}}{g_4}\right) + \frac{3}{4}\hbar_1\left(\frac{N^*_{(\mathrm{C.19})}}{g_4}\right)^3 - \hbar_3\left(\frac{-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}\right), \tag{C.19}$$

where

$$N^*_{(\mathrm{C.19})} = -\omega_0^2\left(\frac{-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}\right) + g_1\left(\frac{-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}\right) + \frac{3}{4}g_2\left(\frac{-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}\right)^3 - g_3X_1.$$

Equation (C.19) gives a nonlinear algebraic equation of degree 18. Developing an exact analytical expression or solution of the roots finding is very difficult. Therefore, the equation is solved numerically using the Newton-Raphson method. The smallest real value of ω_0 obtained from the solutions is the nonlinear natural frequency for the TWCNT.

The analysis of the linear natural frequencies for TWCNT is presented as follows.

Neglecting the nonlinear terms in Eqs (C.14)–(C.16) results in

$$-X_1\omega^2 + f_1X_1 - f_3X_2 = 0, \tag{C.20}$$

$$-X_2\omega^2 + g_1X_2 - g_3X_1 = 0, \tag{C.21}$$

$$-X_1\omega^2 + h_1X_2 - h_3X_1 = 0. \tag{C.22}$$

The above equations can be written in matrix form as

$$\begin{bmatrix} -\omega^2 + f_1 & -f_3 & 0 \\ -g_3 & -\omega^2 + g_1 & g_4 \\ 0 & \hbar_3 & -\omega^2 + \hbar_1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{C.23}$$

Since $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ cannot be equal to zero, for the nontrivial case, then

$$\begin{bmatrix} -\omega^2 + f_1 & -f_3 & 0 \\ -g_3 & -\omega^2 + g_1 & g_4 \\ 0 & h_3 & -\omega^2 + h_1 \end{bmatrix} = 0. \tag{C.24}$$

The frequency characteristic is obtained when the determinant of the matrix of Eq. (C.24) is set to zero as

$$\omega^6 - (f_1 + g_1 + \hbar_1)\,\omega^4 + (2f_1\hbar_1 - f_1g_1 - f_3g_3 - \hbar_3g_4)\,\omega^2 - (f_1g_1\hbar_1 - f_1\hbar_3g_4 - f_3\hbar_1g_3) = 0. \tag{C.25}$$

Equation (C.25) can be written as

$$\omega^6 + \gamma_1\omega^4 + \gamma_2\omega^2 + \gamma_3 = 0, \tag{C.26}$$

where

$$\gamma_1 = -\left(f_1 + g_1 + \hbar_1\right),$$

$$\gamma_2 = \left(2f_1\hbar_1 - f_1g_1 - f_3g_3 - \hbar_3g_4\right),$$

$$\gamma_3 = -\left(f_1g_1\hbar_1 - f_1\hbar_3g_4 - f_3\hbar_1g_3\right).$$

The roots of the sextic equations are

$$\omega_1 = \sqrt{A^* - \frac{\gamma_1}{3}}, \tag{C.27}$$

$$\omega_2 = -\sqrt{A^* - \frac{\gamma_1}{3}}, \tag{C.28}$$

$$\omega_3 = \sqrt{\frac{-\sqrt{A}^*}{2\lambda_1} + \frac{\sqrt{-3}}{2\lambda_1}\sqrt{\left(A^* - \frac{\gamma_1}{3}\right)} - \frac{\gamma_1}{3}}, \tag{C.29}$$

$$\omega_4 = -\sqrt{\frac{-\sqrt{A}^*}{2\lambda_1} + \frac{\sqrt{-3}}{2\lambda_1}\sqrt{\left(A^* - \frac{\gamma_1}{3}\right)} - \frac{\gamma_1}{3}}, \tag{C.30}$$

$$\omega_5 = \sqrt{\frac{-\sqrt{A}^*}{2\lambda_1} - \frac{\sqrt{-3}\sqrt{A}^*}{2\lambda_1} - \frac{\gamma_1}{3}}, \tag{C.31}$$

$$\omega_6 = \sqrt{\frac{-\sqrt{A}^*}{2\lambda_1} - \frac{\sqrt{-3}\sqrt{A}^*}{2\lambda_1} - \frac{\gamma_1}{3}}, \tag{C.32}$$

where

$$A^* = \sqrt[3]{\left(\frac{-\gamma_1^3}{27} + \frac{\gamma_1\gamma_3}{6} - \frac{\gamma_3}{2}\right) + \sqrt{\left(\frac{\gamma_2}{3} - \frac{\gamma_1^2}{9}\right)^3 + \left(\frac{-\gamma_1^3}{27} + \frac{\gamma_1\gamma_3}{6} - \frac{\gamma_3}{2}\right)^2}} + \sqrt[3]{\left(\frac{-\gamma_1^3}{27} + \frac{\gamma_1\gamma_3}{6} - \frac{\gamma_3}{2}\right) - \sqrt{\left(\frac{\gamma_2}{3} - \frac{\gamma_1^2}{9}\right)^3 + \left(\frac{-\gamma_1^3}{27} + \frac{\gamma_1\gamma_3}{6} - \frac{\gamma_3}{2}\right)^2}}.$$

The fundamental linear vibration frequency of TWNT is the lowest root of Eq. (C.26), which is

$$\omega = \sqrt{A^* - \frac{\gamma_1}{3}}, \tag{C.33}$$

where (as above)

$$A^* = \sqrt[3]{\left(\frac{-\gamma_1^3}{27} + \frac{\gamma_1\gamma_3}{6} - \frac{\gamma_3}{2}\right) + \sqrt{\left(\frac{\gamma_2}{3} - \frac{\gamma_1^2}{9}\right)^3 + \left(\frac{-\gamma_1^3}{27} + \frac{\gamma_1\gamma_3}{6} - \frac{\gamma_3}{2}\right)^2}} + \sqrt[3]{\left(\frac{-\gamma_1^3}{27} + \frac{\gamma_1\gamma_3}{6} - \frac{\gamma_3}{2}\right) - \sqrt{\left(\frac{\gamma_2}{3} - \frac{\gamma_1^2}{9}\right)^3 + \left(\frac{-\gamma_1^3}{27} + \frac{\gamma_1\gamma_3}{6} - \frac{\gamma_3}{2}\right)^2}}.$$

Recall that the frequency ratio is defined as $\psi = \frac{\omega_0}{\omega}$, and ω_0 is found numerically from Eq. (C.19) using the Newton-Raphson method.

Substituting Eqs (C.11)–(C.13) into Eqs (C.9) and (C.10) results in

$$p^1: \begin{cases} \omega_0^2\left\{\frac{\mathrm{d}^2 a_{11}}{\mathrm{d}\tau^2} + a_{11}\right\} - \omega_0^2 X_{10}\cos\tau + f_1 X_{10}\cos\tau + f_2\left(X_{10}\cos\tau\right)^3 - f_3 X_{20}\cos\tau = 0, \\ \qquad a_{11}(0) = 0, \quad \frac{\mathrm{d}a_{11}(0)}{\mathrm{d}\tau} = 0, \\ \omega_0^2\left\{\frac{\mathrm{d}^2 a_{21}}{\mathrm{d}\tau^2} + a_{21}\right\} - \omega_0^2 X_{20}\cos\tau + g_1 X_{20}\cos\tau + g_2\left(X_{20}\cos\tau\right)^3 - g_3 X_{10}\cos\tau = 0, \\ \qquad a_{21}(0) = 0, \quad \frac{\mathrm{d}a_{21}(0)}{\mathrm{d}\tau} = 0, \\ \omega_0^2\left\{\frac{\mathrm{d}^2 a_{31}}{\mathrm{d}\tau^2} + a_{31}\right\} - \omega_0^2 X_{30}\cos\tau + \hbar_1 X_{30}\cos\tau + \hbar_2\left(X_{30}\cos\tau\right)^3 - \hbar_3 X_{20}\cos\tau = 0, \\ \qquad a_{31}(0) = 0, \quad \frac{\mathrm{d}a_{31}(0)}{\mathrm{d}\tau} = 0, \end{cases} \tag{C.34}$$

After the application of trigonometry identities to the fourth-term on the left-hand side

$$p^1: \begin{cases} \omega_0^2\left\{\frac{d^2 a_{11}}{d\tau^2} + a_{11}\right\} - \omega_0^2 X_{10}\cos\tau + f_1 X_{10}\cos\tau + \frac{3}{4} f_2 X_{10}^3\cos\tau \\ \qquad + \frac{1}{4} f_2 X_{10}^3 \cos 3\tau - f_3 X_{20}\cos\tau = 0, \quad a_{11}(0) = 0, \quad \frac{\mathrm{d}a_{11}(0)}{\mathrm{d}\tau} = 0, \\ \omega_0^2\left\{\frac{\mathrm{d}^2 a_{21}}{\mathrm{d}\tau^2} + a_{21}\right\} - \omega_0^2 X_{20}\cos\tau + g_1 X_{20}\cos\tau + \frac{3}{4} g_2 X_{20}^3\cos\tau \\ \qquad + \frac{1}{4} g_2 X_{20}^3 \cos 3\tau - g_3 X_{10}\cos\tau = 0, \quad a_{21}(0) = 0, \quad \frac{\mathrm{d}a_{21}(0)}{\mathrm{d}\tau} = 0, \\ \omega_0^2\left\{\frac{\mathrm{d}^2 a_{31}}{\mathrm{d}\tau^2} + a_{31}\right\} - \omega_0^2 X_{30}\cos\tau + \hbar_1 X_{30}\cos\tau + \frac{3}{4} \hbar_2 X_{30}^3\cos\tau \\ \qquad + \frac{1}{4} \hbar_2 X_{30}^3 \cos 3\tau - \hbar_3 X_{20}\cos\tau = 0, \quad a_{31}(0) = 0, \quad \frac{\mathrm{d}a_{31}(0)}{\mathrm{d}\tau} = 0, \end{cases} \tag{C.35}$$

It can be easily shown that the solutions of Eqs (C.34) are

$$a_{11}(\tau) = \frac{X_1^3\left(\frac{(\alpha_8\mu - \alpha_4)EI_1}{2L(\alpha_2 - \alpha_5\mu)\rho A_1}\right)(\cos 3\tau - \cos\tau)}{D^*_{(\mathrm{C}.36)}}, \tag{C.36}$$

where

$$D^*_{(C.36)} = 32\left(\frac{N^*_{(C.36)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(C.36)} = \alpha_1 EI_1 + (\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1 H_x^2+k_p\Big),$$

(C.37)
$$a_{21}(\tau) = \frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\tau-\cos\tau)}{D^*_{(C.37)}},$$

where

$$D^*_{(C.37)} = 32\left(\frac{N^*_{(C.37)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}+\frac{c_1+c_2}{\rho A_2}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_2}\right),$$

$$N^*_{(C.37)} = \alpha_1 EI_2 + (\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2+\eta A_2 H_x^2+k_p\Big),$$

(C.38)
$$a_{31}(\tau) = \frac{X_3^3\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_3}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_3}\right)\left(\frac{I_1}{A_1}\right)(\cos 3\tau-\cos\tau)}{D^*_{(C.38)}},$$

where

$$D^*_{(C.38)} = 32\left(\frac{N^*_{(C.38)}}{(\alpha_2-\alpha_5\mu)\,\rho A_3}+\frac{c_2}{\rho A_3}\right) + 24\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_3}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_3}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(C.38)} = \alpha_1 EI_3 + \alpha_2k_1-\alpha_5\mu k_1 + (\alpha_1\mu-\alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}(1-2v)-2h_{23}\left(v^2-1\right)+h_{33}v^2\right]A_3(\alpha_x)^2(\Delta T)^2+\eta A_3 H_x^2+k_p\Big).$$

Therefore, the first approximate solutions of Eqs (50)–(52) are

(C.39)
$$a_1(x,\tau) = X_1\cos\tau + \frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\tau-\cos\tau)}{D^*_{(C.39)}},$$

where

$$D^*_{(C.39)} = 32\left(\frac{N^*_{(C.39)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(C.39)} = \alpha_1 EI_1 + (\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1 H_x^2+k_p\Big),$$

$$a_2(x,\tau) = X_2\cos\tau + \frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\tau - \cos\tau)}{D^*_{(C.40)}}, \tag{C.40}$$

where

$$D^*_{(C.40)} = 32\left(\frac{N^*_{(C.40)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2} + \frac{c_1+c_2}{\rho A_2}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_2}\right),$$

$$N^*_{(C.40)} = \alpha_1 EI_2 + (\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v) - 2h_{22}\left(v^2-1\right) + h_{32}v^2\right] A_2(\alpha_x)^2(\Delta T)^2 + \eta A_2 H_x^2 + k_p\Big),$$

$$a_3(x,\tau) = X_3\cos\tau + \frac{X_3^3\left(\frac{\alpha_3 k_3 - \alpha_4\frac{EA_3}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_3}\right)\left(\frac{I_1}{A_1}\right)(\cos 3\tau - \cos\tau)}{D^*_{(C.41)}}, \tag{C.41}$$

where

$$D^*_{(C.41)} = 32\left(\frac{N^*_{(C.41)}}{(\alpha_2-\alpha_5\mu)\,\rho A_3} + \frac{c_2}{\rho A_3}\right) + 24\left(\frac{\alpha_3 k_3 - \alpha_4\frac{EA_3}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_3}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(C.41)} = \alpha_1 EI_3 + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu-\alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}(1-2v) - 2h_{23}\left(v^2-1\right) + h_{33}v^2\right] A_3(\alpha_x)^2(\Delta T)^2 + \eta A_3 H_x^2 + k_p\Big).$$

The displacements of the TWCNT are expressed as:

- **S-S support**

$$w_1(x,t) = \left(X_1\cos\omega_0 t + \frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\omega_0 t - \cos\omega_0 t)}{D^*_{(C.42)}}\right)\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right), \tag{C.42}$$

where

$$D^*_{(C.42)} = 32\left(\frac{N^*_{(C.42)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1} + \frac{c_1}{\rho A_1}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(C.42)} = \alpha_1 EI_1 + (\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v) - 2h_{21}\left(v^2-1\right) + h_{31}v^2\right] A_1(\alpha_x)^2(\Delta T)^2 + \eta A_1 H_x^2 + k_p\Big),$$

$$w_2(x,t) = \left(X_2\cos\omega_0 t + \frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\omega_0 t - \cos\omega_0 t)}{D^*_{(C.43)}}\right)\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right), \tag{C.43}$$

where

$$D^*_{(\mathrm{C}.43)} = 32\left(\frac{N^*_{(\mathrm{C}.43)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}+\frac{c_1+c_2}{\rho A_2}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_2}\right),$$

$$N^*_{(\mathrm{C}.43)} = \alpha_1 EI_2+(\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}\,(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2\,(\alpha_x)^2\,(\Delta T)^2+\eta A_2 H_x^2+k_p\Big),$$

$$w_3\,(x,t) = \left(X_3\cos\omega_0 t+\frac{X_3^3\left(\frac{\alpha_3 k_3-\alpha_4\frac{EA_3}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_3}\right)}{D^*_{(\mathrm{C}.44)}}\cdot\left(\frac{I_1}{A_1}\right)(\cos 3\omega_0 t-\cos\omega_0 t)\right)\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right), \tag{C.44}$$

where

$$D^*_{(\mathrm{C}.44)} = 32\left(\frac{N^*_{(\mathrm{C}.44)}}{(\alpha_2-\alpha_5\mu)\,\rho A_3}+\frac{c_2}{\rho A_3}\right) + 24\left(\frac{\alpha_3 k_3-\alpha_4\frac{EA_3}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_3}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(\mathrm{C}.44)} = \alpha_1 EI_3+\alpha_2 k_1-\alpha_5\mu k_1+(\alpha_1\mu-\alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}\,(1-2v)-2h_{23}\left(v^2-1\right)+h_{33}v^2\right]A_3\,(\alpha_x)^2\,(\Delta T)^2+\eta A_3 H_x^2+k_p\Big).$$

- **C-C support**

$$w_1\,(x,t) = \left(X_1\cos\omega_0 t+\frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\omega_0 t-\cos\omega_0 t)}{D^*_{(\mathrm{C}.45)}}\right)\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\}, \tag{C.45}$$

where

$$D^*_{(\mathrm{C}.45)} = 32\left(\frac{N^*_{(\mathrm{C}.45)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(\mathrm{C}.45)} = \alpha_1 EI_1+(\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}\,(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1\,(\alpha_x)^2\,(\Delta T)^2+\eta A_1 H_x^2+k_p\Big),$$

$$\text{(C.46)}\quad w_2(x,t) = \left(X_2 \cos\omega_0 t + \frac{X_2^3 \left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\omega_0 t - \cos\omega_0 t)}{D^*_{(C.46)}} \right) \cdot \sqrt{\frac{I_1}{A_1}} \left\{ \left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right) \right] - \left(\frac{\sinh\beta + \sin\beta}{\cosh\beta - \cos\beta} \right) \left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right) \right] \right\},$$

where

$$D^*_{(C.46)} = 32 \left(\frac{N^*_{(C.46)}}{(\alpha_2 - \alpha_5\mu)\rho A_2} + \frac{c_1 + c_2}{\rho A_2} \right) + 24 \left(\frac{(\alpha_8\mu - \alpha_4)EI_2}{2L(\alpha_2 - \alpha_5\mu)\rho A_2} \right),$$

$$N^*_{(C.46)} = \alpha_1 EI_2 + (\alpha_1\mu - \alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v) - 2h_{22}\left(v^2 - 1\right) + h_{32}v^2 \right] A_2 (\alpha_x)^2 (\Delta T)^2 + \eta A_2 H_x^2 + k_p \Big),$$

$$\text{(C.47)}\quad w_3(x,t) = \left(X_3 \cos\omega_0 t + \frac{X_3^3 \left(\frac{\alpha_3 k_3 - \alpha_4 \frac{EA_3}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_3} \right)}{D^*_{(C.47)}} \cdot \left(\frac{I_1}{A_1}\right)(\cos 3\omega_0 t - \cos\omega_0 t) \right) \sqrt{\frac{I_1}{A_1}} \left\{ \left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right) \right] - \left(\frac{\sinh\beta + \sin\beta}{\cosh\beta - \cos\beta} \right) \left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right) \right] \right\},$$

where

$$D^*_{(C.47)} = 32 \left(\frac{N^*_{(C.47)}}{(\alpha_2 - \alpha_5\mu)\rho A_3} + \frac{c_2}{\rho A_3} \right) + 24 \left(\frac{\alpha_3 k_3 - \alpha_4 \frac{EA_3}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2 - \alpha_5\mu)\rho A_3} \right) \left(\frac{I_1}{A_1}\right).$$

$$N^*_{(C.47)} = \alpha_1 EI_3 + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}(1-2v) - 2h_{23}\left(v^2 - 1\right) + h_{33}v^2 \right] A_3 (\alpha_x)^2 (\Delta T)^2 + \eta A_3 H_x^2 + k_p \Big).$$

- **C-S support**

$$\text{(C.48)}\quad w_1(x,t) = \left(X_1 \cos\omega_0 t + \frac{X_1^3 \left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\omega_0 t - \cos\omega_0 t)}{D^*_{(C.48)}} \right) \cdot \sqrt{\frac{I}{A}} \left\{ \left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right) \right] - \left(\frac{\cosh\beta - \cos\beta}{\sinh\beta - \sin\beta} \right) \left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right) \right] \right\},$$

where

$$D^*_{(C.48)} = 32\left(\frac{N^*_{(C.48)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(C.48)} = \alpha_1 EI_1+(\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1 H_x^2+k_p\Big),$$

$$\text{(C.49)}\quad w_2(x,t)=\left(X_2\cos\omega_0 t+\frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\omega_0 t-\cos\omega_0 t)}{D^*_{(C.49)}}\right)\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(C.49)} = 32\left(\frac{N^*_{(C.49)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}+\frac{c_1+c_2}{\rho A_2}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_2}\right),$$

$$N^*_{(C.49)} = \alpha_1 EI_2+(\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2+\eta A_2 H_x^2+k_p\Big),$$

$$\text{(C.50)}\quad w_3(x,t)=\left(X_3\cos\omega_0 t+\frac{X_3^3\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_3}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_3}\right)}{D^*_{(C.50)}}\cdot\left(\frac{I_3}{A_3}\right)(\cos 3\omega_0 t-\cos\omega_0 t)\right)\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(C.50)} = 32\left(\frac{N^*_{(C.50)}}{(\alpha_2-\alpha_5\mu)\,\rho A_3}+\frac{c_2}{\rho A_3}\right)+24\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_3}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_3}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_3}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(C.50)} = \alpha_1 EI_3+\alpha_2k_1-\alpha_5\mu k_1+(\alpha_1\mu-\alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}(1-2v)-2h_{23}\left(v^2-1\right)+h_{33}v^2\right]A_3(\alpha_x)^2(\Delta T)^2+\eta A_3 H_x^2+k_p\Big).$$

Appendix D. Homotopy perturbation method for the quadruple-walled nanotube

Following the same procedure, a homotopy is constructed on Eqs (68)–(71) as follows:

(D.1) $$(1-p)\left\{\omega_0^2\left(\frac{\mathrm{d}^2a_1}{\mathrm{d}\tau^2}+a_1\right)\right\}+p\left\{\omega_0^2\frac{\mathrm{d}^2a_1}{\mathrm{d}\tau^2}+f_1a_1+f_2a_1^3-f_3a_2\right\}=0,$$

(D.2) $$(1-p)\left\{\omega_0^2\left(\frac{\mathrm{d}^2a_2}{\mathrm{d}\tau^2}+a_2\right)\right\}+p\left\{\omega_0^2\frac{\mathrm{d}^2a_2}{\mathrm{d}\tau^2}+g_1a_2+g_2a_2^3-g_3a_1-g_4a_3\right\}=0,$$

(D.3) $$(1-p)\left\{\omega_0^2\left(\frac{\mathrm{d}^2a_3}{\mathrm{d}\tau^2}+a_3\right)\right\}+p\left\{\omega_0^2\frac{\mathrm{d}^2a_3}{\mathrm{d}\tau^2}+\hbar_1a_3+\hbar_2a_3^3-\hbar_3a_1-\hbar_4a_4\right\}=0,$$

(D.4) $$(1-p)\left\{\omega_0^2\left(\frac{\mathrm{d}^2a_4}{\mathrm{d}\tau^2}+a_4\right)\right\}+p\left\{\omega_0^2\frac{\mathrm{d}^2a_4}{\mathrm{d}\tau^2}+r_1a_4+r_2a_4^3-r_3a_3\right\}=0.$$

Taking the solution of Eqs (3.57)–(3.60) to be in the following form:

(D.5) $$a_1(\tau)=a_{10}(\tau)+p\,a_{11}(\tau)+p^2a_{12}(\tau)+p^3a_{13}(\tau)+...,$$

(D.6) $$a_2(\tau)=a_{20}(\tau)+p\,a_{21}(\tau)+p^2a_{22}(\tau)+p^3a_{23}(\tau)+...,$$

(D.7) $$a_3(\tau)=a_{30}(\tau)+p\,a_{31}(\tau)+p^2a_{32}(\tau)+p^3a_{33}(\tau)+...,$$

(D.8) $$a_4(\tau)=a_{40}(\tau)+p\,a_{41}(\tau)+p^2a_{42}(\tau)+p^3a_{34}(\tau)+...,$$

(D.9) $$\omega=\omega_0+p\,\omega_1+p^2\omega_2+p^3\omega_3+...$$

On substituting Eqs (D.5)–(D.9) into the homotopy in Eqs (D.1)–(D.4), collecting and rearranging the coefficients of the terms with identical powers of p, a series of linear differential equations is developed as

(D.10) $$p^0:\begin{cases}\frac{\mathrm{d}^2a_{10}}{\mathrm{d}\tau^2}+a_{10}=0, & a_{10}(0)=X_1, \quad \frac{\mathrm{d}a_{10}(0)}{\mathrm{d}\tau}=0,\\ \frac{\mathrm{d}^2a_{20}}{\mathrm{d}\tau^2}+a_{20}=0, & a_{20}(0)=X_1, \quad \frac{\mathrm{d}a_{20}(0)}{\mathrm{d}\tau}=0,\\ \frac{\mathrm{d}^2a_{30}}{\mathrm{d}\tau^2}+a_{30}=0, & a_{30}(0)=X_1, \quad \frac{\mathrm{d}a_{30}(0)}{\mathrm{d}\tau}=0,\\ \frac{\mathrm{d}^2a_{40}}{\mathrm{d}\tau^2}+a_{40}=0, & a_{40}(0)=X_1, \quad \frac{\mathrm{d}a_{40}(0)}{\mathrm{d}\tau}=0,\end{cases}$$

(D.11)
$$p^1:\begin{cases}\omega_0^2\left\{\frac{\mathrm{d}^2a_{11}}{\mathrm{d}\tau^2}+a_{11}\right\}-\omega_0^2a_{10}+f_1a_{10}+f_2a_{10}^3-f_3a_{20}=0,\\ \qquad a_{11}(0)=0,\quad \frac{\mathrm{d}a_{11}(0)}{\mathrm{d}\tau}=0,\\ \omega_0^2\left\{\frac{\mathrm{d}^2a_{21}}{\mathrm{d}\tau^2}+a_{21}\right\}-\omega_0^2a_{20}+g_1a_{20}+g_2a_{20}^3-g_3a_{10}-g_4a_{30}=0,\\ \qquad a_{21}(0)=0,\quad \frac{\mathrm{d}a_{21}(0)}{\mathrm{d}\tau}=0,\\ \omega_0^2\left\{\frac{\mathrm{d}^2a_{31}}{\mathrm{d}\tau^2}+a_{31}\right\}-\omega_0^2a_{30}+\hbar_1a_{30}+\hbar_2a_{30}^3-\hbar_3a_{20}-\hbar_4a_{40}=0,\\ \qquad a_{31}(0)=0,\quad \frac{\mathrm{d}a_{31}(0)}{\mathrm{d}\tau}=0,\\ \omega_0^2\left\{\frac{\mathrm{d}^2a_{41}}{\mathrm{d}\tau^2}+a_{41}\right\}-\omega_0^2a_{40}+r_1a_{40}+r_2a_{40}^3-r_3a_{30}=0,\\ \qquad a_{41}(0)=0,\quad \frac{\mathrm{d}a_{41}(0)}{\mathrm{d}\tau}=0,\end{cases}$$

(D.12)
$$p^2:\begin{cases}\omega_0^2\left\{\frac{\mathrm{d}^2a_{12}}{\mathrm{d}\tau^2}+a_{12}\right\}-\omega_0^2a_{11}+2\omega_0\omega_1\frac{\mathrm{d}^2a_{10}}{\mathrm{d}\tau^2}+f_1a_{11}+3f_2a_{10}^2a_{11}\\ \qquad -f_3a_{21}=0,\quad a_{12}(0)=0,\quad \frac{\mathrm{d}a_{12}(0)}{\mathrm{d}\tau}=0,\\ \omega_0^2\left\{\frac{\mathrm{d}^2a_{22}}{\mathrm{d}\tau^2}+a_{22}\right\}-\omega_0^2a_{21}+2\omega_0\omega_1\frac{\mathrm{d}^2a_{20}}{\mathrm{d}\tau^2}+g_1a_{21}+3g_2a_{20}^2a_{21}\\ \qquad -g_3a_{11}-g_4a_{31}=0,\quad a_{22}(0)=0,\quad \frac{\mathrm{d}a_{22}(0)}{\mathrm{d}\tau}=0,\\ \omega_0^2\left\{\frac{\mathrm{d}^2a_{32}}{\mathrm{d}\tau^2}+a_{12}\right\}-\omega_0^2a_{31}+2\omega_0\omega_1\frac{\mathrm{d}^2a_{30}}{\mathrm{d}\tau^2}+\hbar_1a_{31}+3\hbar_2a_{30}^2a_{31}\\ \qquad -\hbar_3a_{21}-\hbar_4a_{41}=0,\quad a_{32}(0)=0,\quad \frac{\mathrm{d}a_{32}(0)}{\mathrm{d}\tau}=0,\\ \omega_0^2\left\{\frac{\mathrm{d}^2a_{42}}{\mathrm{d}\tau^2}+a_{41}\right\}-\omega_0^2a_{41}+r_1a_{41}+r_2a_{41}^3-r_3a_{40}=0,\\ \qquad a_{42}(0)=0,\quad \frac{\mathrm{d}a_{42}(0)}{\mathrm{d}\tau}=0,\end{cases}$$

$$\vdots$$

In order to calculate the nonlinear natural frequencies for QWNT, we assumed initial zeroth approximations given as

(D.13)
$$a_{10}(\tau)=X_1\cos\tau,$$

(D.14)
$$a_{20}(\tau)=X_2\cos\tau,$$

(D.15)
$$a_{30}(\tau)=X_3\cos\tau,$$

(D.16)
$$a_{40}(\tau)=X_4\cos\tau.$$

Substituting Eqs (D.13)–(D.16) into Eqs (D.11), and setting the coefficient of $\cos\tau$ to zero in order to eliminate the secular terms, the following nonlinear systems of equations are established

(D.17)
$$-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3-f_3X_2=0,$$

(D.18)
$$-X_2\omega_0^2+g_1X_2+\frac{3}{4}g_2X_2^3-g_3X_1-g_4X_3=0,$$

(D.19)
$$-X_1\omega_0^2+\hbar_1X_3+\frac{3}{4}\hbar_2X_3^3-\hbar_3X_2-\hbar_4X_4=0,$$

(D.20)
$$-X_4\omega_0^2+r_1X_4+\frac{3}{4}r_2X_4^3-r_3X_3=0.$$

From Eq. (D.17)

$$X_2 = \frac{-X_1\omega_0^2 + f_1 X_1 + \frac{3}{4} f_2 X_1^3}{f_3}. \tag{D.21}$$

Substituting Eq. (D.21) into Eq. (D.18) and making X_3 the subject of the formula gives

$$X_3 = \frac{-\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)\omega_0^2 + g_1\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)}{g_4} + \frac{\frac{3}{4}g_2\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)^3 - g_3X_1}{g_4}. \tag{D.22}$$

Again, by substituting Eqs (D.21) and (D.22) into Eq. (D.19) and making X_4 the subject of the formula we have

$$X_4 = \frac{1}{\hbar_4}\left\{-X_1\omega_0^2 + \hbar_1\frac{N^*_{(\mathrm{D.23})}}{g_4} + \frac{3}{4}\hbar_2\left(\frac{N^*_{(\mathrm{D.23})}}{g_4}\right)^3 - \hbar_3\left(\frac{-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}\right)\right\}, \tag{D.23}$$

where

$$N^*_{(\mathrm{D.23})} = -\left(\frac{-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}\right)\omega_0^2 + g_1\left(\frac{-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}\right) + \frac{3}{4}g_2\left(\frac{-X_1\omega_0^2 + f_1X_1 + \frac{3}{4}f_2X_1^3}{f_3}\right)^3 - g_3X_1.$$

Substituting Eqs (D.22) and (D.23) into Eq. (D.20) gives

(D.24)

$$\begin{aligned}
-\Bigg(\frac{-1}{h_4}\Bigg\{&X_1\omega_0^2 + \frac{\hbar_1}{g_4}\Bigg[-\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)\omega_0^2 + g_1\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)\\
&+\frac{3}{4}g_2\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)^3 - g_3X_1\Bigg] + \frac{3}{4}\frac{\hbar_2}{g_4^3}\Bigg[-\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)\omega_0^2\\
&+g_1\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right) + \frac{3}{4}g_2\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)^3 - g_3X_1\Bigg]^3\\
&-\hbar_3\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)\Bigg\}\Bigg)\left(\omega_0^2 + r_1\right) + \frac{3}{4}\frac{r_2}{\hbar_4^3}\Bigg(-\Bigg\{X_1\omega_0^2\\
&+\frac{h_1}{g_4}\Bigg[-\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)\omega_0^2 + g_1\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)\\
&+\frac{3}{4}g_2\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)^3 - g_3X_1\Bigg] + \frac{3}{4}\frac{h_2}{g_4^3}\Bigg[-\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)\omega_0^2\\
&+g_1\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right) + \frac{3}{4}g_2\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)^3 - g_3X_1\Bigg]^3\\
&-h_3\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)\Bigg\}\Bigg)^3 - \frac{r_3}{g_4}\Bigg(-\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)\omega_0^2\\
&+g_1\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right) + \frac{3}{4}g_2\left(\frac{-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3}{f_3}\right)^3 - g_3X_1\Bigg) = 0.
\end{aligned}$$

Equation (D.24) gives a nonlinear algebraic equation of degree 54. It is very difficult to develop exact analytical expressions or solutions of the roots finding. Therefore, the equation is solved numerically using the Newton-Raphson method. The smallest real value of ω_0 obtained from the solutions is the nonlinear natural frequency for the TWCNT.

To calculate the linear natural frequencies for QWNT, substitute:

$$a_{10}(\tau) = X_1 \cos\omega\tau, \tag{D.25}$$

$$a_{20}(\tau) = X_2 \cos\omega\tau, \tag{D.26}$$

$$a_{30}(\tau) = X_3 \cos\omega\tau, \tag{D.27}$$

$$a_{40}(\tau) = X_4 \cos\omega\tau, \tag{D.28}$$

into Eqs (D.17)–(D.20) and neglecting the nonlinear terms gives

$$-X_1\omega^2 + f_1X_1 - f_3X_2 = 0, \tag{D.29}$$

$$-X_2\omega^2 + g_1X_2 - g_3X_1 - g_4X_3 = 0, \tag{D.30}$$

$$-X_1\omega^2 + \hbar_1X_3 - \hbar_3X_2 - \hbar_4X_4 = 0, \tag{D.31}$$

$$-X_4\omega^2 + r_1X_4 - r_3X_3 = 0. \tag{D.32}$$

Equations (D.29)–(D.32) can be written in matrix form as

$$\begin{bmatrix} -\omega^2+f_1 & -f_3 & 0 & 0 \\ -g_3 & -\omega^2+g_1 & -g_4 & 0 \\ 0 & -h_3 & -\omega^2+\hbar_1 & -\hbar_4 \\ 0 & 0 & -r_3 & -\omega^2+r_1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{D.33}$$

Equating the determinant of the matrix of Eq. (D.33), the characteristic frequency equation is obtained as

$$\begin{aligned} \omega^8 &- (f_1+g_1+\hbar_1+r_1)\,\omega^6 + (f_1g_1+f_1r_1+f_1\hbar_1+g_1r_1+g_1\hbar_1+\hbar_1r_1+g_4\hbar_3+f_3g_3-\hbar_4r_3)\,\omega^4 \\ &- (f_1g_1\hbar_1+f_1g_1r_1+f_1\hbar_1r_1+g_1\hbar_1r_1+f_3g_3r_1+f_3g_3\hbar_1+f_1g_4\hbar_3-f_1\hbar_4r_3-g_4\hbar_3r_1-g_1\hbar_4r_3)\,\omega^2 \\ &+ (f_1g_1\hbar_1r_1+f_3g_3\hbar_1r_1-f_1g_1\hbar_4r_3-f_1g_4\hbar_3r_1-f_3g_4\hbar_4r_3) = 0. \end{aligned} \tag{D.34}$$

Equation (D.34) can be written as

$$\omega^8 + \beta_1\omega^6 + \beta_2\omega^4 + \beta_3\omega^2 + \beta_4 = 0, \tag{D.35}$$

where

$\beta_1 = -(f_1 + g_1 + \hbar_1 + r_1)$,

$\beta_2 = (f_1g_1 + f_1r_1 + f_1\hbar_1 + g_1r_1 + g_1\hbar_1 + \hbar_1r_1 + g_4\hbar_3 + f_3g_3 - \hbar_4r_3)$,

$\beta_3 = -(f_1g_1\hbar_1+f_1g_1r_1+f_1\hbar_1r_1+g_1\hbar_1r_1+f_3g_3r_1+f_3g_3\hbar_1+f_1g_4\hbar_3-f_1\hbar_4r_3-g_4\hbar_3r_1-g_1\hbar_4r_3)$,

$\beta_4 = (f_1g_1\hbar_1r_1 + f_3g_3\hbar_1r_1 - f_1g_1\hbar_4r_3 - f_1g_4h_3r_1 - f_3g_4\hbar_4r_3)$.

The roots of the octic equations are

$$\omega_1 = \sqrt{-\frac{\beta_1}{4} + \frac{1}{2}\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi} + \frac{1}{2}\sqrt{\frac{3\beta_1^2}{4} - \left(\frac{\beta_1^2}{4} - \beta_2 + \chi\right) - 2\beta_2 + \frac{4\beta_1\beta_2 - 8\beta_3 - \beta_1^3}{4\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi}}}},$$

$$\omega_2 = -\sqrt{-\frac{\beta_1}{4} + \frac{1}{2}\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi} + \frac{1}{2}\sqrt{\frac{3\beta_1^2}{4} - \left(\frac{\beta_1^2}{4} - \beta_2 + \chi\right) - 2\beta_2 + \frac{4\beta_1\beta_2 - 8\beta_3 - \beta_1^3}{4\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi}}}},$$

$$\omega_3 = \sqrt{-\frac{\beta_1}{4} + \frac{1}{2}\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi} - \frac{1}{2}\sqrt{\frac{3\beta_1^2}{4} - \left(\frac{\beta_1^2}{4} - \beta_2 + \chi\right) - 2\beta_2 + \frac{4\beta_1\beta_2 - 8\beta_3 - \beta_1^3}{4\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi}}}},$$

$$\omega_4 = -\sqrt{-\frac{\beta_1}{4} + \frac{1}{2}\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi} - \frac{1}{2}\sqrt{\frac{3\beta_1^2}{4} - \left(\frac{\beta_1^2}{4} - \beta_2 + \chi\right) - 2\beta_2 + \frac{4\beta_1\beta_2 - 8\beta_3 - \beta_1^3}{4\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi}}}},$$

$$\omega_5 = \sqrt{-\frac{\beta_1}{4} - \frac{1}{2}\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi} + \frac{1}{2}\sqrt{\frac{3\beta_1^2}{4} - \left(\frac{\beta_1^2}{4} - \beta_2 + \chi\right) - 2\beta_2 + \frac{4\beta_1\beta_2 - 8\beta_3 - \beta_1^3}{4\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi}}}},$$

$$\omega_6 = -\sqrt{-\frac{\beta_1}{4} - \frac{1}{2}\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi} + \frac{1}{2}\sqrt{\frac{3\beta_1^2}{4} - \left(\frac{\beta_1^2}{4} - \beta_2 + \chi\right) - 2\beta_2 + \frac{4\beta_1\beta_2 - 8\beta_3 - \beta_1^3}{4\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi}}}},$$

$$\omega_7 = \sqrt{-\frac{\beta_1}{4} - \frac{1}{2}\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi} - \frac{1}{2}\sqrt{\frac{3\beta_1^2}{4} - \left(\frac{\beta_1^2}{4} - \beta_2 + \chi\right) - 2\beta_2 + \frac{4\beta_1\beta_2 - 8\beta_3 - \beta_1^3}{4\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi}}}},$$

$$\omega_8 = -\sqrt{-\frac{\beta_1}{4} - \frac{1}{2}\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi} - \frac{1}{2}\sqrt{\frac{3\beta_1^2}{4} - \left(\frac{\beta_1^2}{4} - \beta_2 + \chi\right) - 2\beta_2 + \frac{4\beta_1\beta_2 - 8\beta_3 - \beta_1^3}{4\sqrt{\frac{\beta_1^2}{4} - \beta_2 + \chi}}}},$$

where

$$\chi = \left[\frac{9\beta_2\left(4\beta_4 + \beta_1\beta_3\right) - 27\left(4\beta_2\beta_4 - \beta_3^2 - \beta_1^2\beta_4\right) + 2\beta_2^3}{54} + \sqrt{\left(\frac{3\left(\beta_1\beta_3 - 4\beta_4\right) - \beta_2^2}{9}\right)^3 + \left(\frac{9\beta_2\left(4\beta_4 + \beta_1\beta_3\right) - 27\left(4\beta_2\beta_4 - \beta_3^2 - \beta_1^2\beta_4\right) + 2\beta_2^3}{54}\right)^2}\right]^{1/3}$$
$$+ \left[\frac{9\beta_2\left(4\beta_4 + \beta_1\beta_3\right) - 27\left(4\beta_2\beta_4 - \beta_3^2 - \beta_1^2\beta_4\right) + 2\beta_2^3}{54} - \sqrt{\left(\frac{3\left(\beta_1\beta_3 - 4\beta_4\right) - \beta_2^2}{9}\right)^3 + \left(\frac{9\beta_2\left(4\beta_4 + \beta_1\beta_3\right) - 27\left(4\beta_2\beta_4 - \beta_3^2 - \beta_1^2\beta_4\right) + 2\beta_2^3}{54}\right)^2}\right]^{1/3} + \frac{\beta_2}{3}.$$

The fundamental linear vibration frequency of QWNT is the lowest root. Substituting Eqs (D.13)–(D.16) into Eqs (D.11) and (D.12) gives

$$\text{(D.36)}\quad p^1:\begin{cases}\omega_0^2\left\{\frac{\mathrm{d}^2 a_{11}}{\mathrm{d}\tau^2}+a_{11}\right\}-\omega_0^2X_{10}\cos\tau+f_1X_{10}\cos\tau+f_2\left(X_{10}\cos\tau\right)^3\\ \qquad -f_3X_{20}\cos\tau=0,\quad a_{11}(0)=0,\quad \frac{\mathrm{d}a_{11}(0)}{\mathrm{d}\tau}=0,\\ \omega_0^2\left\{\frac{\mathrm{d}^2 a_{21}}{\mathrm{d}\tau^2}+a_{21}\right\}-\omega_0^2X_{20}\cos\tau+g_1X_{20}\cos\tau+g_2\left(X_{20}\cos\tau\right)^3\\ \qquad -g_3X_{10}\cos\tau=0,\quad a_{21}(0)=0,\quad \frac{\mathrm{d}a_{21}(0)}{\mathrm{d}\tau}=0,\\ \omega_0^2\left\{\frac{\mathrm{d}^2 a_{31}}{\mathrm{d}\tau^2}+a_{31}\right\}-\omega_0^2X_{30}\cos\tau+\hbar_1X_{30}\cos\tau+\hbar_2\left(X_{30}\cos\tau\right)^3\\ \qquad -\hbar_3X_{20}\cos\tau=0,\quad a_{31}(0)=0,\quad \frac{\mathrm{d}a_{31}(0)}{\mathrm{d}\tau}=0,\\ \omega_0^2\left\{\frac{\mathrm{d}^2 a_{41}}{\mathrm{d}\tau^2}+a_{41}\right\}-\omega_0^2X_{40}\cos\tau+r_1X_{40}\cos\tau+r_2\left(X_{40}\cos\tau\right)^3\\ \qquad -r_3X_{30}\cos\tau=0,\quad a_{41}(0)=0,\quad \frac{\mathrm{d}a_{41}(0)}{\mathrm{d}\tau}=0.\end{cases}$$

After the application of trigonometry identities to the fourth-term on the left-hand side

$$\text{(D.37)}\quad p^1:\begin{cases}\omega_0^2\left\{\frac{\mathrm{d}^2 a_{11}}{\mathrm{d}\tau^2}+a_{11}\right\}-\omega_0^2X_{10}\cos\tau+f_1X_{10}\cos\tau+\frac{3}{4}f_2X_{10}^3\cos\tau+\frac{1}{4}f_2X_{10}^3\cos3\tau\\ \qquad -f_3X_{20}\cos\tau=0,\quad a_{11}(0)=0,\quad \frac{\mathrm{d}a_{11}(0)}{\mathrm{d}\tau}=0,\\ \omega_0^2\left\{\frac{\mathrm{d}^2 a_{21}}{\mathrm{d}\tau^2}+a_{21}\right\}-\omega_0^2X_{20}\cos\tau+g_1X_{20}\cos\tau+\frac{3}{4}g_2X_{20}^3\cos\tau+\frac{1}{4}g_2X_{20}^3\cos3\tau\\ \qquad -g_3X_{10}\cos\tau=0,\quad a_{21}(0)=0,\quad \frac{\mathrm{d}a_{21}(0)}{\mathrm{d}\tau}=0,\\ \omega_0^2\left\{\frac{\mathrm{d}^2 a_{31}}{\mathrm{d}\tau^2}+a_{31}\right\}-\omega_0^2X_{30}\cos\tau+h_1X_{30}\cos\tau+\frac{3}{4}\hbar_2X_{30}^3\cos\tau+\frac{1}{4}\hbar_2X_{30}^3\cos3\tau\\ \qquad -\hbar_3X_{20}\cos\tau=0,\quad a_{31}(0)=0,\quad \frac{\mathrm{d}a_{31}(0)}{\mathrm{d}\tau}=0,\\ \omega_0^2\left\{\frac{\mathrm{d}^2 a_{41}}{\mathrm{d}\tau^2}+a_{41}\right\}-\omega_0^2X_{40}\cos\tau+r_1X_{40}\cos\tau+\frac{3}{4}r_2X_{30}^3\cos\tau+\frac{1}{4}r_2X_{30}^3\cos3\tau\\ \qquad -r_3X_{30}\cos\tau=0,\quad a_{41}(0)=0,\quad \frac{\mathrm{d}a_{41}(0)}{\mathrm{d}\tau}=0.\end{cases}$$

It can be easily shown that the solutions of Eqs (D.36) and (D.37) are

$$\text{(D.38)}\qquad a_{11}(\tau)=\frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos3\tau-\cos\tau)}{D^*_{(\mathrm{D}.38)}}$$

where

$$D^*_{(\mathrm{D}.38)}=32\left(\frac{N^*_{(\mathrm{D}.38)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)\,EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(\mathrm{D}.38)}=\alpha_1EI_1+(\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T+\left[h_{11}(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1H_x^2+k_p\Big),$$

$$\text{(D.39)}\qquad a_{21}(\tau)=\frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos3\tau-\cos\tau)}{D^*_{(\mathrm{D}.39)}},$$

where

$$D^*_{(\mathrm{D}.39)} = 32\left(\frac{N^*_{(\mathrm{D}.39)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}+\frac{c_1+c_2}{\rho A_2}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_2}\right),$$

$$N^*_{(\mathrm{D}.39)} = \alpha_1 EI_2 + (\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}\,(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2\,(\alpha_x)^2\,(\Delta T)^2+\eta A_2H_x^2+k_p\Big),$$

(D.40)
$$a_{31}\,(\tau) = \frac{X_3^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_3}{2L(\alpha_2-\alpha_5\mu)\rho A_3}\right)(\cos 3\tau-\cos\tau)}{D^*_{(\mathrm{D}.40)}},$$

where

$$D^*_{(\mathrm{D}.40)} = 32\left(\frac{N^*_{(\mathrm{D}.40)}}{(\alpha_2-\alpha_5\mu)\,\rho A_3}+\frac{c_2+c_3}{\rho A_3}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)\,EI_3}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_3}\right),$$

$$N^*_{(\mathrm{D}.40)} = \alpha_1 EI_3 + (\alpha_1\mu-\alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}\,(1-2v)-2h_{23}\left(v^2-1\right)+h_{33}v^2\right]A_3\,(\alpha_x)^2\,(\Delta T)^2+\eta A_3H_x^2+k_p\Big),$$

(D.41)
$$a_{41}\,(\tau) = \frac{X_3^3\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_4}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_4}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_4}\right)\left(\frac{I_1}{A_1}\right)(\cos 3\tau-\cos\tau)}{D^*_{(\mathrm{D}.41)}},$$

where

$$D^*_{(\mathrm{D}.41)} = 32\left(\frac{N^*_{(\mathrm{D}.41)}}{(\alpha_2-\alpha_5\mu)\,\rho A_4}\right)+\frac{c_3}{\rho A_4} + 24\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_4}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_4}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_4}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(\mathrm{D}.41)} = \alpha_1 EI_4+\alpha_2k_1-\alpha_5\mu k_1 + (\alpha_1\mu-\alpha_5)\Big(EA_4\alpha_x\Delta T + \left[h_{14}\,(1-2v)-2h_{24}\left(v^2-1\right)+h_{34}v^2\right]A_4\,(\alpha_x)^2\,(\Delta T)^2+\eta A_4H_x^2+k_p\Big).$$

Therefore, the first approximate solutions of Eqs (3.57)–(3.60) are

(D.42)
$$a_{11}\,(X,\tau) = X_1\cos\tau + \frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\tau-\cos\tau)}{D^*_{(\mathrm{D}.42)}},$$

where

$$D^*_{(\mathrm{D}.42)} = 32\left(\frac{N^*_{(\mathrm{D}.42)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)\,EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(\mathrm{D}.42)} = \alpha_1 EI_1 + (\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}\,(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1\,(\alpha_x)^2\,(\Delta T)^2+\eta A_1H_x^2+k_p\Big),$$

$$a_{21}(X,\tau) = X_2\cos\tau + \frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\tau - \cos\tau)}{D^*_{(\mathrm{D.43})}}, \tag{D.43}$$

where

$$D^*_{(\mathrm{D.43})} = 32\left(\frac{N^*_{(\mathrm{D.43})}}{(\alpha_2-\alpha_5\mu)\rho A_2} + \frac{c_1+c_2}{\rho A_2}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right),$$

$$N^*_{(\mathrm{D.43})} = \alpha_1 EI_2 + (\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v) - 2h_{22}\left(v^2-1\right) + h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2 + \eta A_2 H_x^2 + k_p\Big),$$

$$a_{31}(X,\tau) = X_3\cos\tau + \frac{X_3^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_3}{2L(\alpha_2-\alpha_5\mu)\rho A_3}\right)(\cos 3\tau - \cos\tau)}{D^*_{(\mathrm{D.44})}}, \tag{D.44}$$

where

$$D^*_{(\mathrm{D.44})} = 32\left(\frac{N^*_{(\mathrm{D.44})}}{(\alpha_2-\alpha_5\mu)\rho A_3} + \frac{c_2+c_3}{\rho A_3}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_3}{2L(\alpha_2-\alpha_5\mu)\rho A_3}\right),$$

$$N^*_{(\mathrm{D.44})} = \alpha_1 EI_3 + (\alpha_1\mu-\alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}(1-2v) - 2h_{23}\left(v^2-1\right) + h_{33}v^2\right]A_3(\alpha_x)^2(\Delta T)^2 + \eta A_3 H_x^2 + k_p\Big),$$

$$a_{41}(X,\tau) = X_4\cos\tau + \frac{X_3^3\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_4}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_4}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_4}\right)\left(\frac{I_1}{A_1}\right)(\cos 3\tau - \cos\tau)}{D^*_{(\mathrm{D.45})}}, \tag{D.45}$$

where

$$D^*_{(\mathrm{D.45})} = 32\left(\frac{N^*_{(\mathrm{D.45})}}{(\alpha_2-\alpha_5\mu)\rho A_4}\right) + \frac{c_3}{\rho A_4} + 24\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_4}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_4}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_4}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(\mathrm{D.45})} = \alpha_1 EI_4 + \alpha_2k_1 - \alpha_5\mu k_1 + (\alpha_1\mu-\alpha_5)\Big(EA_4\alpha_x\Delta T + \left[h_{14}(1-2v) - 2h_{24}\left(v^2-1\right) + h_{34}v^2\right]A_4(\alpha_x)^2(\Delta T)^2 + \eta A_4 H_x^2 + k_p\Big).$$

The displacements of the QWCNTs are expressed as

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$$w_1(x,t) = \left(X_1\cos\tau + \frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\tau - \cos\tau)}{D^*_{(\mathrm{D.46})}}\right)\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right), \tag{D.46}$$

where

$$D^*_{(\mathrm{D}.46)} = 32\left(\frac{N^*_{(\mathrm{D}.46)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(\mathrm{D}.46)} = \alpha_1 EI_1+(\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}\,(1-2v)-2h_{21}\left(v^2-1\right)+h_{31}v^2\right]A_1\,(\alpha_x)^2\,(\Delta T)^2+\eta A_1 H_x^2+k_p\Big),$$

$$\text{(D.47)}\qquad w_2\,(x,t) = \left(X_2\cos\tau+\frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\tau-\cos\tau)}{D^*_{(\mathrm{D}.47)}}\right)\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right),$$

where

$$D^*_{(\mathrm{D}.47)} = 32\left(\frac{N^*_{(\mathrm{D}.47)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}+\frac{c_1+c_2}{\rho A_2}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)\,EI_2}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_2}\right),$$

$$N^*_{(\mathrm{D}.47)} = \alpha_1 EI_2+(\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}\,(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2\,(\alpha_x)^2\,(\Delta T)^2+\eta A_2 H_x^2+k_p\Big),$$

$$\text{(D.48)}\qquad w_3\,(x,t) = \left(X_3\cos\tau+\frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_3}{2L(\alpha_2-\alpha_5\mu)\rho A_3}\right)(\cos 3\tau-\cos\tau)}{D^*_{(\mathrm{D}.48)}}\right)\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right),$$

where

$$D^*_{(\mathrm{D}.48)} = 32\left(\frac{N^*_{(\mathrm{D}.48)}}{(\alpha_2-\alpha_5\mu)\,\rho A_3}+\frac{c_2+c_3}{\rho A_3}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_3}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_3}\right),$$

$$N^*_{(\mathrm{D}.48)} = \alpha_1 EI_3+(\alpha_1\mu-\alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}\,(1-2v)-2h_{23}\left(v^2-1\right)+h_{33}v^2\right]A_3\,(\alpha_x)^2\,(\Delta T)^2+\eta A_3 H_x^2+k_p\Big),$$

$$\text{(D.49)}\qquad w_4\,(x,t) = \left(X_4\cos\tau+\frac{X_3^3\left(\frac{\alpha_3 k_3-\alpha_4\frac{EA_4}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_4}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_4}\right)}{D^*_{(\mathrm{D}.49)}} \cdot\left(\frac{I_1}{A_1}\right)(\cos 3\tau-\cos\tau)\right)\sqrt{\frac{I_1}{A_1}}\sin\left(\frac{n\pi x}{l}\right),$$

where

$$D^*_{(\mathrm{D}.49)} = 32\left(\frac{N^*_{(\mathrm{D}.49)}}{(\alpha_2-\alpha_5\mu)\,\rho A_4}\right)+\frac{c_3}{\rho A_4} + 24\left(\frac{\alpha_3 k_3-\alpha_4\frac{EA_4}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_4}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_4}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(D.49)} = \alpha_1 EI_4 + \alpha_2 k_1 - \alpha_5 \mu k_1 + (\alpha_1\mu - \alpha_5)\Big(EA_4\alpha_x\Delta T + \left[h_{14}(1-2v) - 2h_{24}\left(v^2-1\right) + h_{34}v^2\right] A_4(\alpha_x)^2(\Delta T)^2 + \eta A_4 H_x^2 + k_p\Big).$$

- **C-C support**

$$\text{(D.50)}\quad w_1(x,t) = X_1\cos\tau + \frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\tau - \cos\tau)}{D^*_{(D.50)}} \cdot \sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\sinh\beta + \sin\beta}{\cosh\beta - \cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(D.50)} = 32\left(\frac{N^*_{(D.50)}}{(\alpha_2-\alpha_5\mu)\rho A_1} + \frac{c_1}{\rho A_1}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right),$$

$$N^*_{(D.50)} = \alpha_1 EI_1 + (\alpha_1\mu - \alpha_5)\Big(EA_1\alpha_x\Delta T + \left[h_{11}(1-2v) - 2h_{21}\left(v^2-1\right) + h_{31}v^2\right] A_1(\alpha_x)^2(\Delta T)^2 + \eta A_1 H_x^2 + k_p\Big),$$

$$\text{(D.51)}\quad w_2(x,t) = X_2\cos\tau + \frac{X_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\tau - \cos\tau)}{D^*_{(D.51)}} \cdot \sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\sinh\beta + \sin\beta}{\cosh\beta - \cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(D.51)} = 32\left(\frac{N^*_{(D.51)}}{(\alpha_2-\alpha_5\mu)\rho A_2} + \frac{c_1+c_2}{\rho A_2}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right),$$

$$N^*_{(D.51)} = \alpha_1 EI_2 + (\alpha_1\mu - \alpha_5)\Big(EA_2\alpha_x\Delta T + \left[h_{12}(1-2v) - 2h_{22}\left(v^2-1\right) + h_{32}v^2\right] A_2(\alpha_x)^2(\Delta T)^2 + \eta A_2 H_x^2 + k_p\Big),$$

$$\text{(D.52)}\quad w_3(x,t) = X_3\cos\tau + \frac{X_3^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_3}{2L(\alpha_2-\alpha_5\mu)\rho A_3}\right)(\cos 3\tau - \cos\tau)}{D^*_{(D.52)}} \cdot \sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right) - \cos\left(\frac{\beta x}{L}\right)\right] - \left(\frac{\sinh\beta + \sin\beta}{\cosh\beta - \cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right) - \sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(D.52)} = 32\left(\frac{N^*_{(D.52)}}{(\alpha_2-\alpha_5\mu)\rho A_3} + \frac{c_2+c_3}{\rho A_3}\right) + 24\left(\frac{(\alpha_8\mu-\alpha_4)EI_3}{2L(\alpha_2-\alpha_5\mu)\rho A_3}\right),$$

$$N^*_{(D.52)} = \alpha_1 EI_3 + (\alpha_1\mu - \alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}(1-2v) - 2h_{23}\left(v^2-1\right) + h_{33}v^2\right] A_3(\alpha_x)^2(\Delta T)^2 + \eta A_3 H_x^2 + k_p\Big),$$

$$w_4(x,t)=X_4\cos\tau+\frac{X_3^3\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_4}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_4}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_4}\right)\left(\frac{I_1}{A_1}\right)(\cos 3\tau-\cos\tau)}{D^*_{(\mathrm{D}.53)}}$$
$$\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\}, \tag{D.53}$$

where

$$D^*_{(\mathrm{D}.53)}=32\left(\frac{N^*_{(\mathrm{D}.53)}}{(\alpha_2-\alpha_5\mu)\,\rho A_4}\right)+\frac{c_3}{\rho A_4}$$
$$+24\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_4}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_4}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_4}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(\mathrm{D}.53)}=\alpha_1EI_4+\alpha_2k_1-\alpha_5\mu k_1+(\alpha_1\mu-\alpha_5)\Big(EA_4\alpha_x\Delta T$$
$$+\left[h_{14}(1-2v)-2h_{24}\left(v^2-1\right)+h_{34}v^2\right]A_4(\alpha_x)^2(\Delta T)^2+\eta A_4H_x^2+k_p\Big).$$

- **C-S support**

$$w_1(x,t)=\left[X_1\cos\tau+\frac{X_1^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L(\alpha_2-\alpha_5\mu)\rho A_1}\right)(\cos 3\tau-\cos\tau)}{D^*_{(\mathrm{D}.54)}}\right]$$
$$\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\}, \tag{D.54}$$

where

$$D^*_{(\mathrm{D}.54)}=32\left(\frac{N^*_{(\mathrm{D}.54)}}{(\alpha_2-\alpha_5\mu)\,\rho A_1}+\frac{c_1}{\rho A_1}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)EI_1}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_1}\right),$$

$$N^*_{(\mathrm{D}.54)}=\alpha_1EI_1+(\alpha_1\mu-\alpha_5)\Big(EA_1\alpha_x\Delta T$$
$$+\left[h_{11}(1-2\nu)-2h_{21}\left(\nu^2-1\right)+h_{31}\nu^2\right]A_1(\alpha_x)^2(\Delta T)^2+\eta A_1H_x^2+k_p\Big),$$

$$w_2(x,t)=\left[X_2\cos\tau+\frac{\mathrm{X}_2^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_2}{2L(\alpha_2-\alpha_5\mu)\rho A_2}\right)(\cos 3\tau-\cos\tau)}{D^*_{(\mathrm{D}.55)}}\right]$$
$$\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\}, \tag{D.55}$$

where

$$D^*_{(\mathrm{D}.55)}=32\left(\frac{N^*_{(\mathrm{D}.55)}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}+\frac{c_1+c_2}{\rho A_2}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)\,EI_2}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_2}\right),$$

$$N^*_{(\mathrm{D}.55)}=\alpha_1EI_2+(\alpha_1\mu-\alpha_5)\Big(EA_2\alpha_x\Delta T$$
$$+\left[h_{12}(1-2v)-2h_{22}\left(v^2-1\right)+h_{32}v^2\right]A_2(\alpha_x)^2(\Delta T)^2+\eta A_2H_x^2+k_p\Big),$$

$$\text{(D.56)}\quad w_3(x,t) = \left[X_3\cos\tau + \frac{X_3^3\left(\frac{(\alpha_8\mu-\alpha_4)EI_3}{2L(\alpha_2-\alpha_5\mu)\rho A_3}\right)(\cos 3\tau - \cos\tau)}{D^*_{(\mathrm{D.56})}}\right]$$
$$\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(\mathrm{D.56})} = 32\left(\frac{N^*_{(\mathrm{D.56})}}{(\alpha_2-\alpha_5\mu)\,\rho A_2}+\frac{c_2+c_3}{\rho A_3}\right)+24\left(\frac{(\alpha_8\mu-\alpha_4)\,EI_3}{2L\,(\alpha_2-\alpha_5\mu)\,\rho A_3}\right),$$

$$N^*_{(\mathrm{D.56})} = \alpha_1 EI_3 + (\alpha_1\mu-\alpha_5)\Big(EA_3\alpha_x\Delta T + \left[h_{13}(1-2v)-2h_{23}\left(v^2-1\right)+h_{33}v^2\right]A_3(\alpha_x)^2(\Delta T)^2+\eta A_3H_x^2+k_p\Big),$$

$$\text{(D.57)}\quad w_4(x,t) = \left[X_4\cos\tau + \frac{X_3^3\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_4}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_4}{2L}}{(\alpha_2-\alpha_5\mu)\rho A_4}\right)\left(\frac{I_1}{A_1}\right)(\cos 3\tau-\cos\tau)}{D^*_{(\mathrm{D.57})}}\right]$$
$$\cdot\sqrt{\frac{I_1}{A_1}}\left\{\left[\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right]-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left[\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right]\right\},$$

where

$$D^*_{(\mathrm{D.57})} = 32\left(\frac{N^*_{(\mathrm{D.57})}}{(\alpha_2-\alpha_5\mu)\,\rho A_4}\right)+\frac{c_3}{\rho A_4} + 24\left(\frac{\alpha_3k_3-\alpha_4\frac{EA_4}{2L}-6\alpha_6\mu k_3-3\alpha_7\mu k_3+\alpha_8\mu\frac{EA_4}{2L}}{(\alpha_2-\alpha_5\mu)\,\rho A_4}\right)\left(\frac{I_1}{A_1}\right),$$

$$N^*_{(\mathrm{D.57})} = \alpha_1 EI_4+\alpha_2k_1-\alpha_5\mu k_1+(\alpha_1\mu-\alpha_5)\Big(EA_4\alpha_x\Delta T + \left[h_{14}(1-2v)-2h_{24}\left(v^2-1\right)+h_{34}v^2\right]A_4(\alpha_x)^2(\Delta T)^2+\eta A_4H_x^2+k_p\Big).$$

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