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HARMONIC WAVE IN DISTURBED SYSTEM OF PERIODIC ELASTIC LAYERS

Z. WESOŁOWSKI (KIELCE)

The system of elastic layers periodic in space is considered. One extra cell situated between the cells of numbers k and $k+1$ disturbs the system. Harmonic wave of frequency ω propagates across the layers. The transparency of the system is defined as the ratio of the transmitted energy flux to the incident energy flux. Transparency depends on the position of the extra layer. The analytic expression for the transparency is given. Transparency is a periodic function of the position of the extra cell, and in general, a non-periodic function of ω. Assuming that the probability of finding the extra layer at the position k is given, the average transparency and its standard deviation has been calculated.

1. Homogeneous layers

The systems of layers were dealt with in many papers, e.g. in the already classical papers [1 – 7]. System of two or more layers constitute the elementary cell. If equal cells are repeated in space then the system is periodic. Properties of such a system are defined by the number of cells and the properties of elementary cell. In the present paper we consider the periodic system, disturbed by one extra cell of other properties.

The investigations are motivated by the dynamics of a disturbed mechanical system. In the fundamental papers [8 – 10], the cristal lattice was approximated by a chain of different interacting masses. Here we consider the chain of different elastic layers. The situation is more involved, since the frequency enters the equations via the trigonometric functions. Introduction of a new parameter φ and representation of the transition matrix M in the form satisfying the identity $M(\varphi)^n = M(n\varphi)$ simplifies the equations.

Consider the system of homogeneous elastic layers, Fig. 1. The layer situated between x_k and x_{k+1} is identified by the natural number k, $k = 1, 2, 3, \ldots$. Propagation speed, density and thickness of the layer k are denoted by c_k, ϱ_k and h_k, respectively. In direction x perpendicular to the layers, two sinusoidal waves of frequency ω propagate, one of them of amplitude A_k in the $+x$ direction, and the other of amplitude B_k in the $-x$ direction. The displacement u_k in the

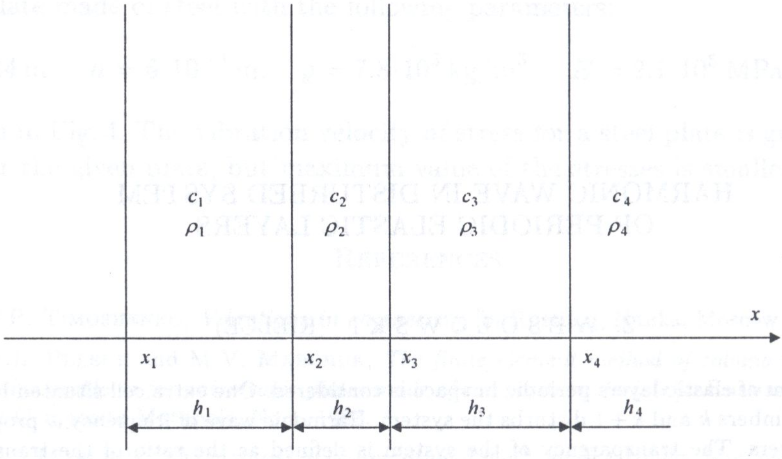

FIG. 1.

layer k is

$$(1.1)\qquad u_k = A_k \exp i\omega[t-(x-x_k)/c_k] + B_k \exp i\omega[t+(x-x_k)/c_k],$$

where t is time and $x_k \le x \le x_{k+1}$.

The displacement u_k satisfies the equation of motion

$$(1.2)\qquad c_k^2 u_{k,xx} = u_{k,tt}\,.$$

Physical displacement is the real part of the complex-valued function $u_k(x,t)$.

On both sides of the boundary between layers, the displacement and the stress vector must have the same values. This fact leads to the following relation between the amplitudes in layer k and layer $k+1$:

$$(1.3)\qquad \begin{bmatrix} A_{k+1} \\ B_{k+1} \end{bmatrix} = M_k \begin{bmatrix} A_k \\ B_k \end{bmatrix},$$

where

$$(1.4)\qquad M_k = \frac{1}{2}\begin{bmatrix} (1+\kappa_k)\exp(-i\alpha_k) & (1-\kappa_k)\exp(i\alpha_k) \\ (1-\kappa_k)\exp(-i\alpha_k) & (1+\kappa_k)\exp(i\alpha_k) \end{bmatrix},$$

$$(1.5)\qquad a_k = \omega h_k / c_k\,,$$

$$(1.6)\qquad \kappa_k = \varrho_k c_k / \varrho_{k+1} c_{k+1}\,,$$

$$(1.7)\qquad h_k = x_{k+1} - x_k\,.$$

The transfer matrix M_k allows us to express the amplitudes A_{k+1}, B_{k+1} by the amplitudes A_k, B_k. In further calculations the above results will be chained to express the amplitudes in the last layer in terms of the amplitudes in the first layer. The frequency ω influences the transfer matrix M via the functions α_k. The determinant of M does not depend on the frequency ω

(1.8) $$\det M_k = \kappa_k\,.$$

The energy flux S_k in layer k is the elastic energy transported in a unit of time. The following formula holds,

(1.9) $$S_k = \varrho_k c_k (A_k \overline{A}_k - B_k \overline{B}_k).$$

The bar over a complex quantity denotes complex conjugate value. The first part corresponds to the wave running to the right, and the second – to the wave running to the left. Since the material is elastic, there occurs no energy loss and the energy flux in each layer is the same,

(1.10) $$S_{k+1} = S_k\,.$$

Easy calculations prove that an elastically supported, sectionally homogeneous string is governed by the same system of equations.

2. Periodic layers

Consider now the case, when a set of layers is repeated periodically in space. Such a set of layers constitutes the elementary cell. The elementary cell may consist of arbitrary number of layers; the simplest cell consists of two layers only. Confine the calculations to this case only, although the generalisation is immediate. The first layer in each cell will be identified by the subscript a, and the second by the subscript b. Denote

(2.1) $$\kappa = (\varrho_a c_a)/(\varrho_b c_b), \qquad \alpha_a = \omega d_a/c_a\,, \qquad \alpha_b = \omega d_b/c_b\,;$$

(2.2)
$$M_a = \frac{1}{2}\begin{bmatrix} (1+\kappa)\exp(-i\alpha_a) & (1-\kappa)\exp(i\alpha_a) \\ (1-\kappa)\exp(-i\alpha_a) & (1+\kappa)\exp(i\alpha_a)\end{bmatrix},$$
$$M_b = \frac{1}{2}\begin{bmatrix} (1+1/\kappa)\exp(-i\alpha_b) & (1-1/\kappa)\exp(i\alpha_b) \\ (1-1/\kappa)\exp(-i\alpha_b) & (1+1/\kappa)\exp(i\alpha_b)\end{bmatrix}.$$

Therefore

$$\kappa_k = \kappa, \qquad c_k = c_a, \qquad \alpha_k = \alpha_a, \qquad M_k = M_a \qquad \text{for} \quad k = 0, 2, 4, \ldots\,,$$
$$\kappa_k = 1/\kappa, \qquad c_k = c_b, \qquad \alpha_k = \alpha_b, \qquad M_k = M_b \qquad \text{for} \quad k = 1, 3, 5, \ldots\,.$$

In the above relations M_a is the transfer matrix from material a to material b, and M_b the transfer matrix from b to a.

Instead of identyfying the layers by subsequent numbers, in further investigations we shall identify them by the number of the cell and position in the cell. Concentrate first on the displacements in the layers of material a. The amplitudes in the layer a of the k-th elementary cell are $A_{k,a}$, $B_{k,a}$. In accord with (1.6), for each natural k there is

$$\begin{bmatrix} A_{k,a} \\ B_{k,a} \end{bmatrix} = M^k \begin{bmatrix} A_{0,a} \\ B_{0,a} \end{bmatrix}, \qquad M = M_b M_a, \quad k = 1,2,3,\ldots . \tag{2.3}$$

Here M is the transfer matrix for the elementary cell as a whole. From (2.2) it follows that this transfer matrix M of one elementary cell has the following components:

$$\begin{aligned} 4M_{11} &= (2+\kappa+1/\kappa)\exp i(-\alpha_a-\alpha_b) + (2-\kappa-1/\kappa)\exp i(-\alpha_a+\alpha_b), \\ 4M_{21} &= (\kappa-1/\kappa)\exp i(-\alpha_a-\alpha_b) - (\kappa-1/\kappa)\exp i(-\alpha_a+\alpha_b), \end{aligned} \tag{2.4}$$

$$M_{22} = \overline{M_{11}}, \qquad M_{12} = \overline{M_{21}}. \tag{2.5}$$

The above components prove, that

$$\det M = 1. \tag{2.6}$$

In general, the transfer matrix M is not a periodic function of ω. Note that the matrix M is non-Hermitean, since M_{11} is not real. The matrix with the symmetry (2.5) will be further called W-symmetric. The product of two W-symmetric matrices is W-symmetric.

The displacement (or stress) at one end of the system of layers is a given in advance, sinusoidal function of time. The displacement at the other end may be calculated taking into account the properties of the system. Therefore the frequency ω is a known parameter, in contrast to the situation, when the ends are subjected to homogeneous boundary conditions. In the latter case the frequency must be calculated as the proper frequency of the system. We do not consider here this case.

The bulk transfer matrix for a system of N equal cells equals M^N. Instead of such a perfect system, we consider here the system consisting of N equal elementary cells and one extra cell of other structure (size and/or properties). The extra cell is situated between the cells n and $n+1$. Total number of cells of the considered system is therefore $N+1$. The purpose of the further analysis is to calculate the bulk transfer matrix and the parameter defining the transparency of the set of layers. The transfer matrix for the extra cell will be denoted by M_e, and the wave amplitudes in the extra cell by A_e, B_e. In accord with the above formulae, the wave amplitudes in the last cell are connected with the amplitudes in the first cell by the relation

$$(2.7)\qquad \begin{bmatrix} A_{N+1,a} \\ B_{N+1,a} \end{bmatrix} = M^{N-n}\, M_e\, M^n \begin{bmatrix} A_{1,a} \\ B_{1,a} \end{bmatrix}, \qquad n = 1, ..., N,$$

$$(2.8)\qquad \begin{bmatrix} A_{N+1,a} \\ B_{N+1,a} \end{bmatrix} = M^N\, M_e \begin{bmatrix} A_e \\ B_e \end{bmatrix}.$$

The above formulae give the amplitudes of the waves at the end of the imperfect system in terms of the amplitudes at the beginning of the system. The extra cell may be situated at the beginning of the system, at its end, or inside. Equation (2.8) governs the first case, and Eq. (2.7) the two others. Note that the cell $N+1$ is used only for measuring the displacement at the end of the system consisting of N equal cells and one extra cell. Because of boundary conditions (1.4), the displacement at the beginning of the cell $N+1$ (in material a) equals the displacement at the end of the cell N (in material b). Assume that the amplitudes in the first cell are given. Then the amplitudes in the last cell depend on the number n defining the position of the extra layer.

Define the total transfer matrix

$$(2.9)\qquad M_T = M^{N-n}\, M_e\, M^n, \qquad n = 0, 1, 2, ..., N,$$

which governs the amplitudes of the whole (disturbed) system. The bulk matrix M_T depends on the position of the extra layer.

Consider $A_{1,a}$, $B_{1,a}$ and $A_{N+1,a}$ as the amplitudes of the incident, reflected and transmitted waves, respectively. In this situation it is necessary to take $B_{N+1,a} = 0$, since no incident wave is propagating from right to left. The other possible choice, easier to handle, is to take $A_{1,a} = 0$, and consider $B_{N+1,a}$, $A_{N+1,a}$ and $B_{1,a}$ as the amplitudes of the incident, reflected and transmitted waves, respectively. Obvious change in the amplitudes must be made for $n = 0$, when the first cell is the extra cell. The transparency τ may be defined as the ratio of the energy flux corresponding to $B_{1,a}$ and energy flux corresponding to $B_{N+1,a}$. Elementary calculations lead to the formula for transparency τ:

$$(2.10)\qquad \tau = \left[(M_T)_{11}\overline{(M_T)_{11}}\right]^{-1}.$$

Expression in the brackets equals the squared modulus of $(M_T)_{11}$ which in turn, in accord with (2.5), equals the squared modulus of $(M_T)_{22}$. The reflection coefficient λ equals the ratio of the reflected energy flux and the incident energy flux, and satisfies the relation $\lambda + \tau = 1$.

Further we confine the calculation to the case, when $\operatorname{Re} M_{11} < 1$. In accord with the derivation given in the Appendix, the transfer matrix M for one elementary cell may be written in the following trigonometric form [13]:

$$M = \begin{bmatrix} \cos\varphi - iE\sin\varphi & (C+iD)\sin\varphi \\ (C-iD)\sin\varphi & \cos\varphi + iE\sin\varphi \end{bmatrix}, \tag{2.11}$$

$$\varphi = \arccos(\operatorname{Re} M_{11}), \tag{2.12}$$

where φ, E, C, D are real parameters. There holds the identity, essential for further calculations,

$$M^k = \begin{bmatrix} \cos k\varphi - iE\sin k\varphi & (C+iD)\sin k\varphi \\ (C-iD)\sin k\varphi & \cos k\varphi + iE\sin k\varphi \end{bmatrix}, \qquad k = 1, 2, \ldots. \tag{2.13}$$

Note that M^k is given not by the reccursive formula $M^k = MM^{k-1}$ but is expressed as a function of $k\varphi$. This drastically simplifies the calculations and analysis.

3. Disturbed chain

In order to simplify the notation, denote the components of the transfer matrix M_e for the extra cell (cf. Appendix) by

$$M_e = \begin{bmatrix} P+iQ & R+iS \\ R-iS & P-iQ \end{bmatrix}, \tag{3.1}$$

where P, Q, R, S are real functions of ω and of the physical parameters of the extra cell (do not confuse S with the energy flux S_k). Calculate the bulk transfer matrix M_T for the whole system consisting of N cells and additionally, of the extra cell situated between the cells n and $n+1$. In accord with (2.13) and (A.8), there is

$$M_T = \begin{bmatrix} \cos m\varphi + iE\sin m\varphi & (C+iD)\sin m\varphi \\ (C-iD)\sin m\varphi & \cos m\varphi - iE\sin m\varphi \end{bmatrix} \times \begin{bmatrix} P+iQ & R+iS \\ R-iS & P-iQ \end{bmatrix} \begin{bmatrix} \cos n\varphi + iE\sin n\varphi & (C+iD)\sin n\varphi \\ (C-iD)\sin n\varphi & \cos n\varphi - iE\sin n\varphi \end{bmatrix}, \tag{3.2}$$

where, in order to shorten the formulae, we have denoted $m = N - n$. Evidently

$M_T = M_T(n)$. Performing the multiplications, we obtain

$$
\begin{aligned}
\mathrm{Re}(M_T)_{11} &= P\cos N\varphi + (-EQ + CR + DS)\sin N\varphi,\\
\mathrm{Im}(M_T)_{11} &= Q\cos m\varphi\cos n\varphi + (EP + CS - DR)\cos m\varphi \sin n\varphi\\
&\quad + (EP - CS + DR)\sin m\varphi\cos n\varphi\\
&\quad + (-E^2Q - C^2Q - D^2Q + 2ECR + 2EDS)\sin m\varphi\sin n\varphi,
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
\mathrm{Re}(M_T)_{12} &= R\cos m\varphi\cos n\varphi + (ES + CP - DQ)\cos m\varphi\sin n\varphi\\
&\quad + (-ES + CP + DQ)\sin m\varphi\cos n\varphi\\
&\quad + (E^2R + C^2R - D^2R - 2ECQ + 2CDS)\sin m\varphi\sin n\varphi,\\
\mathrm{Im}(M_T)_{12} &= S\cos m\varphi\cos n\varphi + (-ER + CQ + DP)\cos m\varphi\sin n\varphi\\
&\quad + (ER - CQ + DP)\sin m\varphi\cos n\varphi\\
&\quad + (E^2S - C^2S + D^2S - 2EDQ + 2CDR)\sin m\varphi\sin n\varphi.
\end{aligned}
\tag{3.4}
$$

The remaining components are determined by the W-symmetry of the bulk transfer matrix M_T.

The trigonometric identities expressing the product of two trigonometric functions by the sums of trigonometric functions

$$2\cos m\varphi\cos(N-m)\varphi = \cos N\varphi + \cos(2m-N)\varphi, \quad \text{etc.}$$

allow to express the functions (3.3), (3.4) in a simpler form, namely

$$
\begin{aligned}
\mathrm{Re}(M_T)_{11} &= P\cos N\varphi + (-EQ + CR + DS)\sin N\varphi,\\
\mathrm{Im}(M_T)_{11} &= [(1 + C^2 + D^2)Q - ECR + EDS]\cos N\varphi EP\sin N\varphi\\
&\quad + (-C^2Q - D^2Q + 2ECR - EDS)\cos(N-2n)\varphi + (CS - DR)\sin(N-2n)\varphi,
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
\mathrm{Re}(M_T)_{12} &= [(1 - E^2 + D^2)R + ECQ - CDS]\cos N\varphi + CP\sin N\varphi\\
&\quad + (E^2R - D^2R - ECQ + CDS)\cos(N-2n)\varphi + (ES - DQ)\sin(N-2n)\varphi,\\
\mathrm{Im}(M_T)_{12} &= [(1 - E^2 + C^2)S + EDQ - CDR]\cos N\varphi + DP\sin N\varphi\\
&\quad + (E^2S - C^2S - EDQ + CDR)\cos(N-2n)\varphi + (-ER + CQ)\sin(N-2n)\varphi.
\end{aligned}
\tag{3.6}
$$

For the extra layer situated at the left end of the chain there is $(N - 2n) = N$, and for the extra layer situated at the right end there is $(N - 2n) = -N$. Note that the real part of $(M_T)_{11}$ does not depend on the positon of the extra layer. In the important special case $CS - DR = 0$ (cf. Appendix), n enters the expressions for $(M_T)_{11}$ only via the cosine function. Consequently $(M_T)_{11}$ and the modulus of $(M_T)_{11}$ are even functions of n. In accord with (2.10), in this special case the transparency is a symmetric function of $(N - 2n)$. Therefore for $CS - DR = 0$, the transparency is a symmetric function of the distance of the extra layer from the centre of the system.

The components of M_T depend on n via the trigonometric functions sine and cosine only. Therefore the bulk transfer matrix M_T is a periodic function of n. The spatial period L is determined by the formula

$$L = \pi/\varphi. \tag{3.7}$$

It should be remembered that n is integer. If L as given by (3.7) or (3.8) is an integer, then L is the period of the functions $(M_T)_{11}$, $(M_T)_{12}$, as given by (3.5), (3.6). In the opposite case the period in the exact sense does not exist.

The bulk transfer matrix M_T is a periodic function not only of n, but of the total number N of elementary cells, too. From (3.5), (3.6) it follows that the corresponding period equals $2T$. Concluding: for arbitrary integers k_1, k_2, we face the identities

$$M_T(N + 2k_1T, n + k_2T) = M_T(N, n). \tag{3.8}$$

The periodicity of M_T has an important physical consequence: in accord with (2.10), the transparency τ, which depends only on the first component of the bulk transfer matrix, is a periodic function of n and N.

4. Random position

Assume now that the probability of finding the extra layer between the cells n and $n+1$ is $p(n)$,

$$\sum_{n=0}^{N} p(n) = 1.$$

The average value $\langle M_T \rangle$ of the bulk transfer matrix M_T and the standard deviation σ may be calculated from the formulae

$$\langle M_T \rangle = \sum_{n=0}^{N} p(n)\, M_T(n), \tag{4.1}$$

$$\sigma = \frac{1}{N} \sum_{n=0}^{N} [M_T(n) - \langle M_T \rangle]^2. \tag{4.2}$$

The transparency τ is given by (2.10). The expression in the brackets is the modulus of $(M_T)_{11}$. It follows that the the average value $\langle \tau \rangle$ of τ and its standard deviation σ_τ are given by the formulae

$$(4.3)\qquad \langle\tau\rangle = \sum_{n=0}^{N} p(n)\left\{[\mathrm{Re}(M_T)_{11}]^2 + [\mathrm{Im}(M_T)_{11}]\right\}^{-1},$$

$$(4.4)\qquad \sigma_\tau = \frac{1}{N}\sum_{n=0}^{N}\left\{\{[\mathrm{Re}(M_T)_{11}]^2 + [\mathrm{Im}(M_T)_{11}]^2\}^{-1} - \langle\tau\rangle\right\}^2.$$

The case of equal probability for each position characterized by n is of special importance. In this situation there is

$$(4.5)\qquad p(n) = \frac{1}{N}, \qquad \langle M_T\rangle = \frac{1}{N+1}\sum_{n=0}^{N} M_T(n).$$

It is easy to calculate the average matrix $\langle M_T\rangle$ and its standard deviation for each (given in advance) transfer matrix for an elementary cell and the transfer matrix for the extra cell. In the above case the analytical summation in (4.1) may be performed. However, the corresponding formulae are relatively simple only for special values of N, in particular for $N = 2^p - 1$, where p is an arbitrary integer. Further simplification is obtained, if $\cos[(N+1)\varphi/2] = 0$. Since neither of the above simplifications posseses physical meaning, we do not explore them.

Here we present the numerical results for two limiting situations: first, when the calculated $\tau(n)$ fits a smooth curve, and the second, when the $\tau(n)$ is spread chaotically over some region.

We prefer not to consume space for introduction of dimensionless variables. The data given below are valid in one arbitrary fixed system of units. Consider the set of $N = 200$ elementary cells disturbed by the extra cell. Define the elementary cell and the extra cell by the relations

$$(4.6)\qquad \begin{aligned} &c_a = 1, && c_b = 2, && c_e = 1, && \\ &h_a = 1, && h_b = 1, && h_e = 2, && \varrho_a = \varrho_b = \varrho_e = 1. \end{aligned}$$

The elementary cell has the total length 2 and consists of two layers: one layer of propagation speed $c_a = 1$ and other layer of propagation speed $c_b = 2$. The extra cell has the total length 2, but consists of one layer only, which is characterized by the propagation speed the same as that of the layer a, namely $c_e = 1$. We consider therefore the special situation, when the extra cell could be obtained from the elementary cell by replacing material b with material a.

For frequency $\omega = 0.1$, the spatial period T given by (3.7) is $T = 19.86$. The corresponding values of transparency τ are shown at Fig. 2. We face a set of separate points situated on one curve. The maximum and minimum values of τ are

$$(4.7)\qquad \tau_{\max} = .9911, \qquad \tau_{\min} = .9154.$$

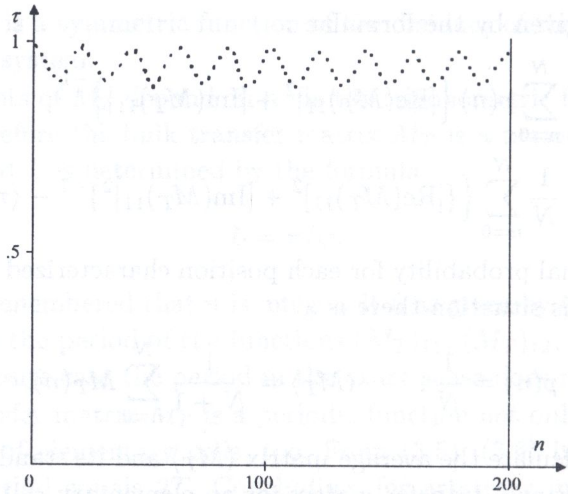

FIG. 2.

The mean value of τ, and its standard deviation are

(4.8) $$\langle\tau\rangle = .9489, \qquad \sigma = .0052.$$

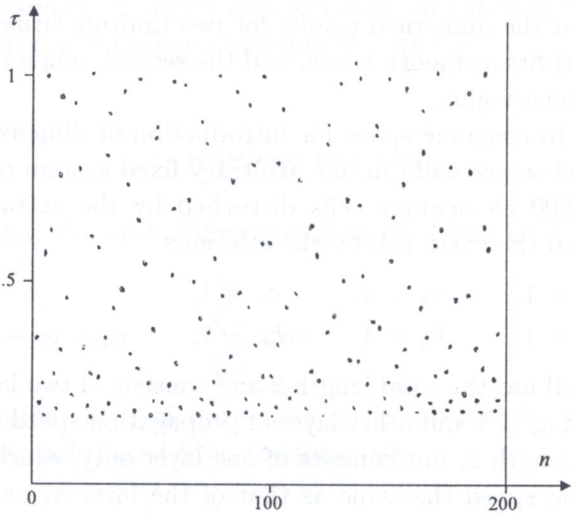

FIG. 3.

For frequency $\omega = 10.0$ the spatial period T is $T = 6.639$. The corresponding values of transparency τ are shown at Fig. 3. The calculated extrema, mean value and standard deviation are

(4.9) $$\tau_{\max} = .997, \qquad \tau_{\min} = .165,$$

(4.10) $$\langle\tau\rangle = .447, \qquad \sigma = .0804.$$

The set of separate points shown at Fig. 3 consists of points situated on one, rapidly oscillating curve. This fact becomes evident if τ is imagined to be calculated for each real n, not only for the physically allowed integer n. However also for integer n, the apparent chaos may be ordered, if only the values of τ for equally spaced $n > 1$ are plotted. For example, if only the values of τ for $n = 0, 13, 26, 39, \ldots$ are plotted, then the set shown at Fig. 4 is obtained. Similar regular curves are obtained for other equally spaced n, e.g. $n = 2, 12, 22, 32, \ldots$.

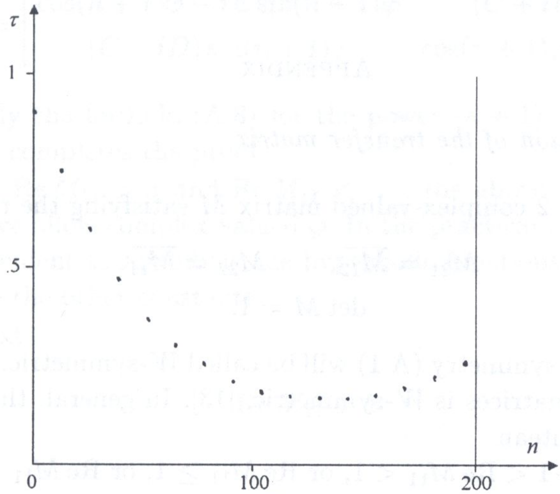

FIG. 4.

If the period T is close to an integer, then the values of τ are not so chaotic as in Fig. 3. For $\omega = 6$ there is $T = 7.011$, and the values of τ are given at Fig. 5. The points are situated on seven relatively flat sine curves.

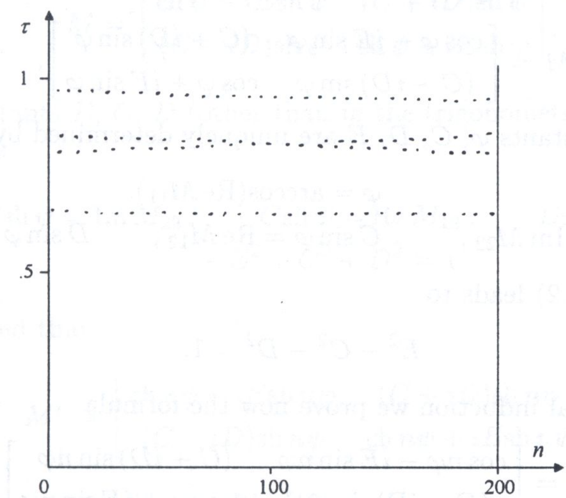

FIG. 5.

Essential for the the function $\tau(n)$ is the behaviour of the modulus of $(M_T)_{11}$ i.e. the value of $[\mathrm{Re}(M_T)_{11}]^2 + [\mathrm{Im}(M_T)_{11}]^2$ as a function of n. In accord with (3.5), it consists of constant part, independent of n, and a part depending on the trigonometric functions of $(N-2n)$. If the last part is small when compared with the first one, then $\tau_{\max}$ does not differ very much from $\tau_{\min}$ and the discrete set of points $\tau(n)$ fits one, relatively smooth curve. In the opposite case $\tau_{\max}$, $\tau_{\min}$ are far apart and the dependence $\tau(n)$ is chaotic, as shown in Fig. 3.

Appendix

A.1. Representation of the transfer matrix

Consider the 2×2 complex-valued matrix M satisfying the relations

$$M_{21} = \overline{M_{12}}, \qquad M_{22} = \overline{M_{11}}, \tag{A.1}$$

$$\det M = 1. \tag{A.2}$$

The matrix with symmetry (A.1) will be called W-symmetric. The product of two W-symmetric matrices is W-symmetric, [13]. In general, the W-symmetric matrix is non-Hermitean.

There is either $-1 < \mathrm{Re}\, M_{11} < 1$, or $\mathrm{Re}\, M_{11} \geq 1$, or $\mathrm{Re}\, M_{11} \leq -1$. Consider first the physically important case

$$-1 < \mathrm{Re}\, M_{11} < 1. \tag{A.3}$$

Without loosing the generality, assume the range $0 < \varphi < \pi$, and write the matrix M in the following form:

$$M = \begin{bmatrix} \cos\varphi - iE\sin\varphi & (C+iD)\sin\varphi \\ (C-iD)\sin\varphi & \cos\varphi + iE\sin\varphi \end{bmatrix}, \tag{A.4}$$

where the real constants φ, C, D, E are uniquely determined by the relations

$$\varphi = \arccos(\mathrm{Re}\, M_{11}), \tag{A.5}$$

$$E\sin\varphi = \mathrm{Im}\, M_{22}\,, \qquad C\sin\varphi = \mathrm{Re}\, M_{12}\,, \qquad D\sin\varphi = \mathrm{Im}\, M_{12}\,. \tag{A.6}$$

The relation (A.2) leads to

$$E^2 - C^2 - D^2 = 1. \tag{A.7}$$

By mathematical induction we prove now the formula

$$M^n = \begin{bmatrix} \cos n\varphi - iE\sin n\varphi & (C+iD)\sin n\varphi \\ (C-iD)\sin n\varphi & \cos n\varphi + iE\sin n\varphi \end{bmatrix}. \tag{A.8}$$

Multiplying by M we get

$$(M^{n+1})_{11} = \cos n\varphi \cos\varphi - (E^2 - C^2 - D^2)\sin n\varphi \sin\varphi - iE\sin(n+1)\varphi,$$

$$(M^{n+1})_{21} = (C - iD)\sin(n+1)\varphi, \qquad M_{22}^{n+1} = \overline{M_{11}^{n+1}}, \qquad M_{12}^{n+1} = \overline{M_{21}^{n+1}}.$$

Taking now into account (A.7) we get

$$\text{(A.9)} \quad M^{n+1} = \begin{bmatrix} \cos(n+1)\varphi - iE\sin(n+1)\varphi & (C+iD)\sin(n+1)\varphi \\ (C-iD)\sin(n+1)\varphi & \cos(n+1)\varphi + iE\sin(n+1)\varphi \end{bmatrix},$$

therefore exactly the formula (A.8) for the power $(n+1)$. The fact that (A.8) holds for $n = 1$ completes the proof.

In the cases $\operatorname{Re} M_{11} > 1$ and $\operatorname{Re} M_{11} < -1$, the above results may be used provided that we allow complex-valued φ. In the practical calculations however, it is more convenient to introduce the hyperbolic functions and real parameter ψ and re-define the other constants.

Consider next

$$\text{(A.10)} \qquad \operatorname{Re} M_{11} > 1.$$

Defining

$$\text{(A.11)} \qquad \psi = \operatorname{Arch}(\operatorname{Re} M_{11}),$$

we can represent M in the form

$$\text{(A.12)} \qquad M = \begin{bmatrix} \operatorname{ch}\psi - iE\operatorname{sh}\psi & (C+iD)\operatorname{sh}\psi \\ (C-iD)\operatorname{sh}\psi & \operatorname{ch}\psi + iE\operatorname{sh}\psi \end{bmatrix},$$

where the constants E, C, D (other than in the trigonometric case) are defined by the relations

$$\text{(A.13)} \qquad E\operatorname{sh}\psi = \operatorname{Im} M_{22}, \qquad C\operatorname{sh}\psi = \operatorname{Re} M_{12}, \qquad D\operatorname{sh}\psi = \operatorname{Im} M_{12},$$

$$\text{(A.14)} \qquad -E^2 + C^2 + D^2 = 1.$$

It may be proved that

$$\text{(A.15)} \qquad M^n = \begin{bmatrix} \operatorname{ch} n\psi - iE\operatorname{sh} n\psi & (C+iD)\operatorname{sh} n\psi \\ (C-iD)\operatorname{sh} n\psi & \operatorname{ch} n\psi + iE\operatorname{sh} n\psi \end{bmatrix}.$$

The remaining cases are discussed in [13].

A.2. Extra cell

The extra cell may consist of one or more layers. The formulae given in Sec. 1 allow to calculate the transfer matrix for arbitrary number of layers in the extra cell. We present here the expression for M_e for the important case when the extra cell consists of one layer only. Denote the elastic modulus, density and thickness of the extra layer by E_e, ϱ_e, and h_e, respectively. Its left-hand neighbour is the layer b of the cell n, and its right-hand neighbour is the layer a of the cell $n+1$.

In accord with the formulae (1.7), there is

$$\begin{bmatrix} A_{n,b} \\ B_{n,b} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \left(1+\dfrac{\varrho_a c_a}{\varrho_b c_b}\right)\exp\left(-i\omega\dfrac{h_a}{c_a}\right) & \left(1-\dfrac{\varrho_a c_a}{\varrho_b c_b}\right)\exp\left(i\omega\dfrac{h_a}{c_a}\right) \\ \left(1-\dfrac{\varrho_a c_a}{\varrho_b c_b}\right)\exp\left(-i\omega\dfrac{h_a}{c_a}\right) & \left(1+\dfrac{\varrho_a c_a}{\varrho_b c_b}\right)\exp\left(i\omega\dfrac{h_a}{c_a}\right) \end{bmatrix} \times \begin{bmatrix} A_{n,a} \\ B_{n,a} \end{bmatrix},$$

$$\text{(A.16)} \qquad \begin{bmatrix} A_e \\ B_e \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \left(1+\dfrac{\varrho_b c_b}{\varrho_e c_e}\right)\exp\left(-i\omega\dfrac{h_b}{c_b}\right) & \left(1-\dfrac{\varrho_b c_b}{\varrho_e c_e}\right)\exp\left(i\omega\dfrac{h_b}{c_b}\right) \\ \left(1-\dfrac{\varrho_b c_b}{\varrho_e c_e}\right)\exp\left(-i\omega\dfrac{h_b}{c_b}\right) & \left(1+\dfrac{\varrho_b c_b}{\varrho_e c_e}\right)\exp\left(i\omega\dfrac{h_b}{c_b}\right) \end{bmatrix} \times \begin{bmatrix} A_{n,b} \\ B_{n,b} \end{bmatrix},$$

$$\begin{bmatrix} A_{n+1,a} \\ B_{n+1,a} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \left(1+\dfrac{\varrho_e c_e}{\varrho_a c_a}\right)\exp\left(-i\omega\dfrac{h_e}{c_e}\right) & \left(1-\dfrac{\varrho_e c_e}{\varrho_a c_a}\right)\exp\left(i\omega\dfrac{h_e}{c_e}\right) \\ \left(1-\dfrac{\varrho_e c_e}{\varrho_a c_a}\right)\exp\left(-i\omega\dfrac{h_e}{c_e}\right) & \left(1+\dfrac{\varrho_e c_e}{\varrho_a c_a}\right)\exp\left(i\omega\dfrac{h_e}{c_e}\right) \end{bmatrix} \times \begin{bmatrix} A_e \\ B_e \end{bmatrix}.$$

Multiplication proves that

$$\text{(A.17)} \qquad \begin{bmatrix} A_{n+1,a} \\ B_{n+1,a} \end{bmatrix} = M\, M_e \begin{bmatrix} A_{n,a} \\ B_{n,a} \end{bmatrix},$$

where M is the transfer matrix for a single cell, and M_e is given by the following formulae:

$$\text{(A.18)} \qquad 4(M_e)_{11} = \left(2+\frac{\varrho_e c_e}{\varrho_a c_a}+\frac{\varrho_a c_a}{\varrho_e c_e}\right)\exp\left(-i\omega\frac{h_e}{c_e}\right) + \left(2-\frac{\varrho_e c_e}{\varrho_a c_a}-\frac{\varrho_a c_a}{\varrho_e c_e}\right)\exp\left(i\omega\frac{h_e}{c_e}\right),$$

(A.18) [cont.]

$$4(M_e)_{12} = \left(\frac{\varrho_e c_e}{\varrho_a c_a} - \frac{\varrho_a c_a}{\varrho_e c_e}\right) \exp\left(-i\omega\frac{h_e}{c_e}\right) + \left(-\frac{\varrho_e c_e}{\varrho_a c_a} + \frac{\varrho_a c_a}{\varrho_e c_e}\right) \exp\left(i\omega\frac{h_e}{c_e}\right),$$

$$4(M_e)_{22} = \overline{(M_e)_{11}}, \qquad (M_e)_{21} = \overline{(M_e)_{12}}.$$

In the special case, when density and propagation speed in the extra layer satisfy the relation $\varrho_e c_e = \varrho_a c_a$, the above expressions are considerably simplified to yield

$$M_e = \begin{bmatrix} \exp\left(-i\omega\frac{h_e}{c_e}\right) & 0 \\ 0 & \exp\left(i\omega\frac{h_e}{c_e}\right) \end{bmatrix}. \tag{A.19}$$

Note that this case corresponds to removal of the layer b from an elementary cell and appriopriate change of the thickness of the layer a in this cell. Similar result holds for the special case $\varrho_e c_e = \varrho_b c_b$.

If the extra cell consists of more than one layer, then decomposition and the formulae discussed in Sec. 1 allow to express M_e by the physical constants of separate layers. The calculations may be simplified, basing on the known fact, that the layer of arbitrary properties but zero thickness, does not influence the behaviour of a set of layers. Between the cell n and the extra cell, a virtual layer of $E = E_a$, $\varrho = \varrho_a$, and of thickness $h = 0$ must be added. Between the extra cell and the cell $n+1$ must be added a virtual layer of $E = E_b$, $\varrho = \varrho_b$, and of thickness $h = 0$. In each case the transfer matrix M_e possesses the W-symmetry.

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UNIVERSITY OF TECHNOLOGY, KIELCE.

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