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RAIL ON LINEAR STOCHASTIC SUPPORTS (*)

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The field tests revealed that the vertical rail deflection under various service conditions is of a random character. The static linear rail response under vertical wheel load $P = 1\,\text{kN}$ the rail deflection and the bending moments of the rail are presented. The stiffnesses of fasteners, sleepers and subgrade are included by a set of discrete springs. The linear finite element procedure is applied. Parametric studies are carried out to examine the effect of randomly variable stiffnesses of the supports. Two types of reduction of the support stiffness are modelled: the nonstationary reduction in the stiffnesses of some supports, and the stationary random reduction in stiffnesses of the supports. The deterministic response results and the random ones are compared.

1. Introduction

During the lifetime of the track its components operate under conditions of repeated variable loading, variability of the foundation, irregularities of the rail etc., which introduce stochastic components to the problem, and the response of the track structure is of a random character. The quasi-static response of the track for these conditions can be described in terms of the deflection of the rail loaded by the concentrated force P, in the form of equation

$$L[v(x)] = \delta(x) \cdot P, \tag{1.1}$$

where $v(x)$ is the vertical deflection of the rail at point x, L is a linear differential operator, $\delta(x)$ is the Dirac delta function.

The formulation of the problem and the boundary conditions of Eq. (1.1) require the deflection, its slope, the bending moment and shear force at infinite distance to the right as well as to the left of the force P will be zero, see Fig. 1. Under these assumptions the beam is in a so-called quasi-stationary state, i.e., its behaviour depends on the distance from the moving point of

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application of the force, and the origin is moving together with the moving force P. In this paper the static case of the problem is analysed, i.e. without the dynamic effect. For more than one load, a solution may be obtained by superposition of the various wheel loads P. In a standard linear analysis, the deterministic quasi-static response was tackled by several authors [2], while the stochastic character was investigated in a few cases only.

In order to develop a rational method taking into account the variability of the input parameters, it is necessary to formulate the problem in a stochastic framework. Such stochastic formulation consists of two main elements:

a. Characterisation of the uncertainty of the input parameters.

b. Development of the relationship between the statistical characteristics of the output.

The aim of the paper is to present the quasi-static rail response for various support stiffness conditions. The rail response analysis is concerned with the determination of the vertical displacement $v(x)$, bending moments $M(x)$ and support reactions R_j. Stiffness of the foundation is a random variable and other input parameters of the problem are considered as deterministic data. In this analysis, the track components such as fasteners, sleepers, ballast and subgrade are modelled by a set of vertical springs. The finite element method is used to find the response. Using the IDA computer program [3], the analysis has been conducted to investigate the effect of variability of the support stiffness on the response. Practically, we could take into account other random factors of the problem, such as irregularities of the rail or the load of the rail. We suppose these sets are stochastically independent and we can investigate their effect in detail.

2. Linear track model

In a standard linear analysis the railway track structure consisting of rails, pads, sleepers, and subgrade is modelled as an infinitely long beam resting on a deterministic continuous Winkler foundation (Fig. 1a). The model just mentioned has been widely accepted for calculation of the rail response and it may be generally extended to a stochastic case (see Fig. 1b).

For a static case of the Bernouli - Euler beam on elastic foundation, the operator L takes a form

$$L[v(x)] = \frac{d^4}{dx^4}v(x) + c(x) \cdot v(x), \tag{2.1}$$

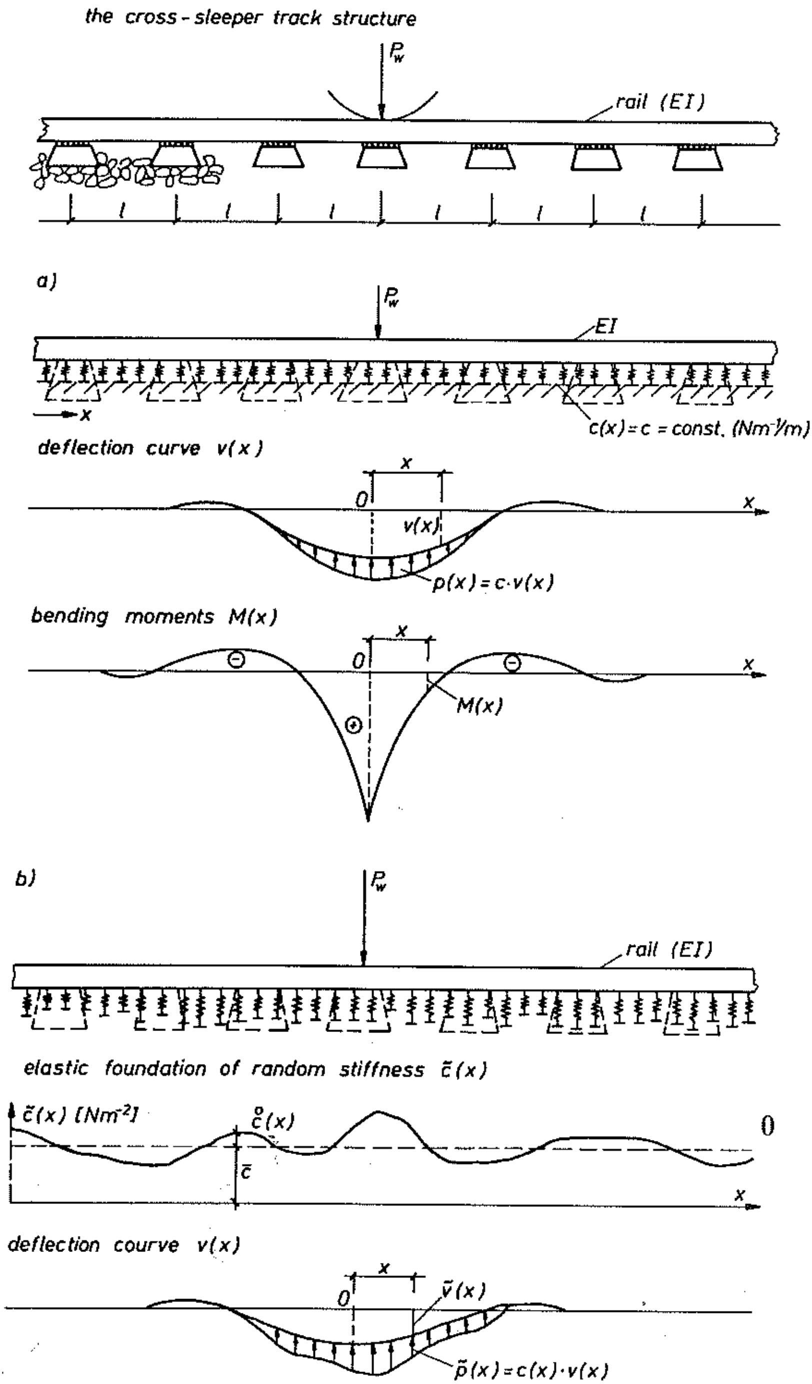

FIG. 1. Linear track models: a) Beam on deterministic Winkler foundation; b) Beam on a randomly variable Winkler foundation.

and the model in Fig. 1b is described by the stochastic differential equation

$$EI\frac{d^4v(x)}{dx^4}+\widetilde{c}(x)\cdot v(x)=P, \tag{2.2}$$

where $v(x)$ is the vertical deflection of point x on the rail axis, EI is the constant bending stiffness of the rail, P is vertical wheel load and $\widetilde{c}(x)$ is the stiffness of the foundation that varies randomly along its length coordinate x.

In Eq. (2.2) the coefficient is of a random character and the equation belongs to the class of differential equations with random coefficients. The solution of Eq. (2.2) is cumbersome, and one of possible approximation techniques is the perturbation approach. L. Fryba in [5] has applied this method to a similar dynamic problem. Because of the difficulties connected with the solution of the stochastic differential equation (2.2), a finite element model of the rail, a finite beam resting on discrete elastic supports with constant spacing (Fig. 2), was introduced and solved. The spring stiffnesses of discrete rail supports include the stiffnesses of fasteners, sleepers and of the subgrade. The spring supports are attached to the rail at the sleepers, see Fig. 2.

The stochastic finite element method [7], the direct Monte Carlo simulation or direct finite element method (FEM) simulation are suitable methods of solution of the problem. Because of the cost of the Monte Carlo simulation, the simple direct FEM was applied to estimate the track response. In the FEM, the structure is approximated by a set of discrete elements interconnected at nodal points, see Fig. 2. By using the virtual displacement theorem, the equilibrium equation of the set has the form

$$\mathbf{K}\cdot\mathbf{v}=\mathbf{P}, \tag{2.3}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{P}$ is the force vector, $\mathbf{v}$ is the vector of nodal displacements.

The elements of $\mathbf{K}$ are given in terms of the geometric variables $\mathbf{L}$, elasticity variables $\mathbf{E}$ and spring stiffness $\mathbf{k}$. Nodal displacements $\mathbf{v}$ are found from Eq. (2.3) directly

$$\mathbf{v}=\mathbf{K}^{-1}\cdot\mathbf{P}. \tag{2.4}$$

The set of load effects $\mathbf{S}=f(\mathbf{L},\mathbf{E},\mathbf{P},\mathbf{k})$ is correspondingly related to the nodal displacements $\mathbf{v}$ by

$$\mathbf{S}=\mathbf{D}\cdot\mathbf{v}, \tag{2.5}$$

where the elements of $\mathbf{D}$ are given in terms of $\mathbf{L}$ and $\mathbf{E}$ as well.

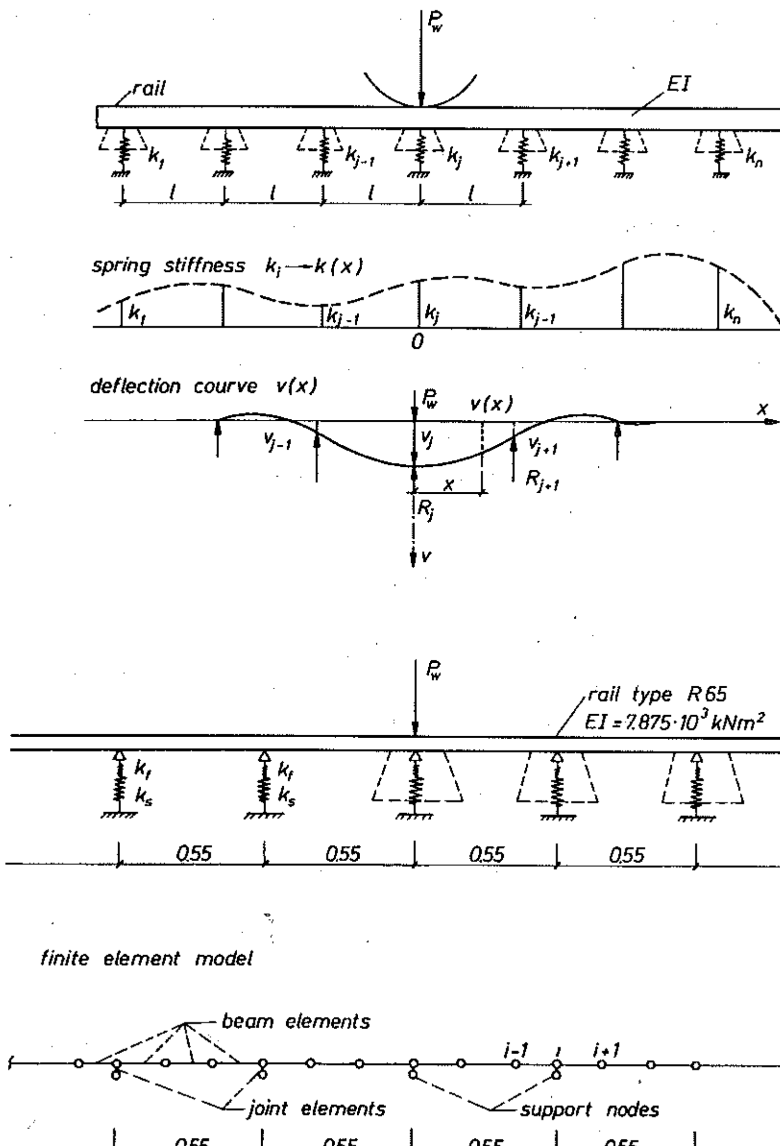

FIG. 2. Linear track model of a finite long beam on discrete elastic supports.

Inserting Eq. (2.4) in Eq. (2.5) yields

$$\mathbf{S} = \mathbf{D} \cdot \mathbf{K}^{-1} \cdot \mathbf{P} = \mathbf{C} \cdot \mathbf{P}, \tag{2.6}$$

where $\mathbf{C} = \mathbf{P} \cdot \mathbf{K}^{-1}$ is a matrix connecting the load effects and external loads.

This linear finite element procedure was applied using the IDA computer program [3].

3. Modelling of the input parameters

The track stiffness, k (Nm^{-1}), is defined as

$$k = \frac{P}{v}, \tag{3.1}$$

where P is the concentrated wheel force applied to the rail, v is the rail deflection under the force.

The track foundation modulus, c (Nm^{-2}), is a widely used parameter to represent the vertical stiffness of the rail foundation, and it is defined as a force per length squared

$$c = \frac{p}{v}, \tag{3.2}$$

where p is the vertical rail foundation supporting force per unit length, and v is the vertical rail deflection. The relationship between the track modulus c and the track stiffness k is as follows

$$c = \frac{k^{4/3}}{(64EI)^{1/3}}. \tag{3.3}$$

The spring stiffness of discrete rail supports k_r should include the stiffness of fasteners, sleepers and of the subgrade. The conditions of the track stiffness are modelled by the four characteristic levels of the spring stiffness of the discrete supports with the mean value $\overline{k^{(i)}} = \mu_k^{(i)}$; $i = 1 \div 4$, see Table 1.

Idealised linear spring characteristics $k_r^{(i)}$ of discrete rail supports are expressed by the spring constant of the elastic joint k_f, and the spring constant of the subgrade $k_s^{(i)}$, Fig. 3.

$$k_r^{(i)} = \frac{k_f \cdot k_s^{(i)}}{k_f + k_s^{(i)}}, \qquad i = 1 \div 4. \tag{3.4}$$

Table 1. Characteristic stiffness levels of rail supports.

level of stiffness i	spring const. $\mu_k^{(i)} = \overline{k}^{(i)}$ [Nm^{-1}]	spring charact. of subgrade $k_s^{(i)}$ [Nm^{-1}]	modulus of compressibility of subgrade $K_s^{(i)}$ [Nm^{-3}]
1	$0.9 \cdot 10^7$	$1.2 \cdot 10^7$	$5 \cdot 10^7$
2	$1.5 \cdot 10^7$	$2.4 \cdot 10^7$	$10 \cdot 10^7$
3	$2.5 \cdot 10^7$	$6.7 \cdot 10^7$	$30 \cdot 10^7$
4	$3.2 \cdot 10^7$	$16.0 \cdot 10^7$	$70 \cdot 10^7$

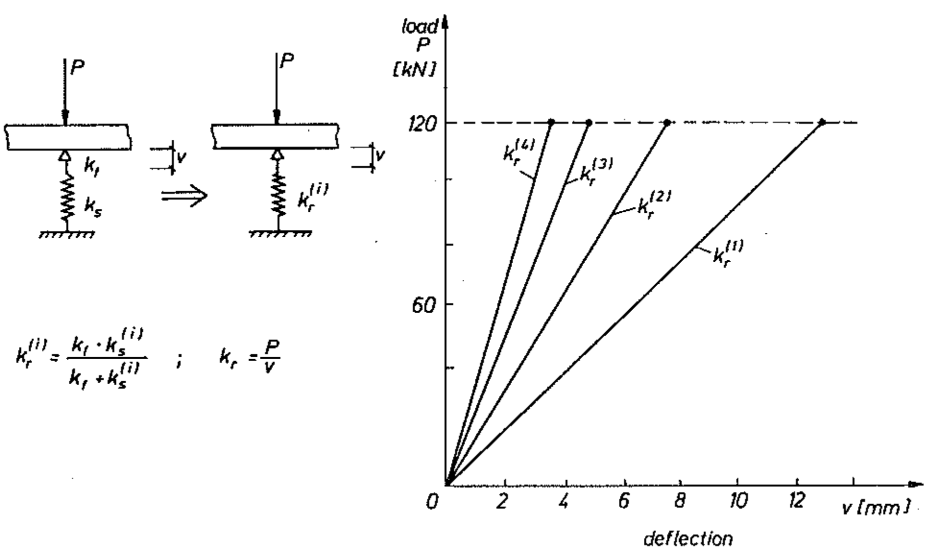

FIG. 3. Idealised linear spring characteristics k_r of discrete rail supports. Replaced linear spring characteristics.

The spring constant k_f was taken as the mean value $k_f = 4 \cdot 10^7\,\mathrm{Nm}^{-1}$. Four characteristic levels for the mean value of $k_s^{(i)}$ were chosen to model the foundation of the track, see Table 1.

Two basic models in the stiffness reduction of support were investigated:

a. Nonstationary reduction in the stiffness of some supports, see Fig. 4.

b. Stationary random stiffness of the discrete supports, see Fig. 5.

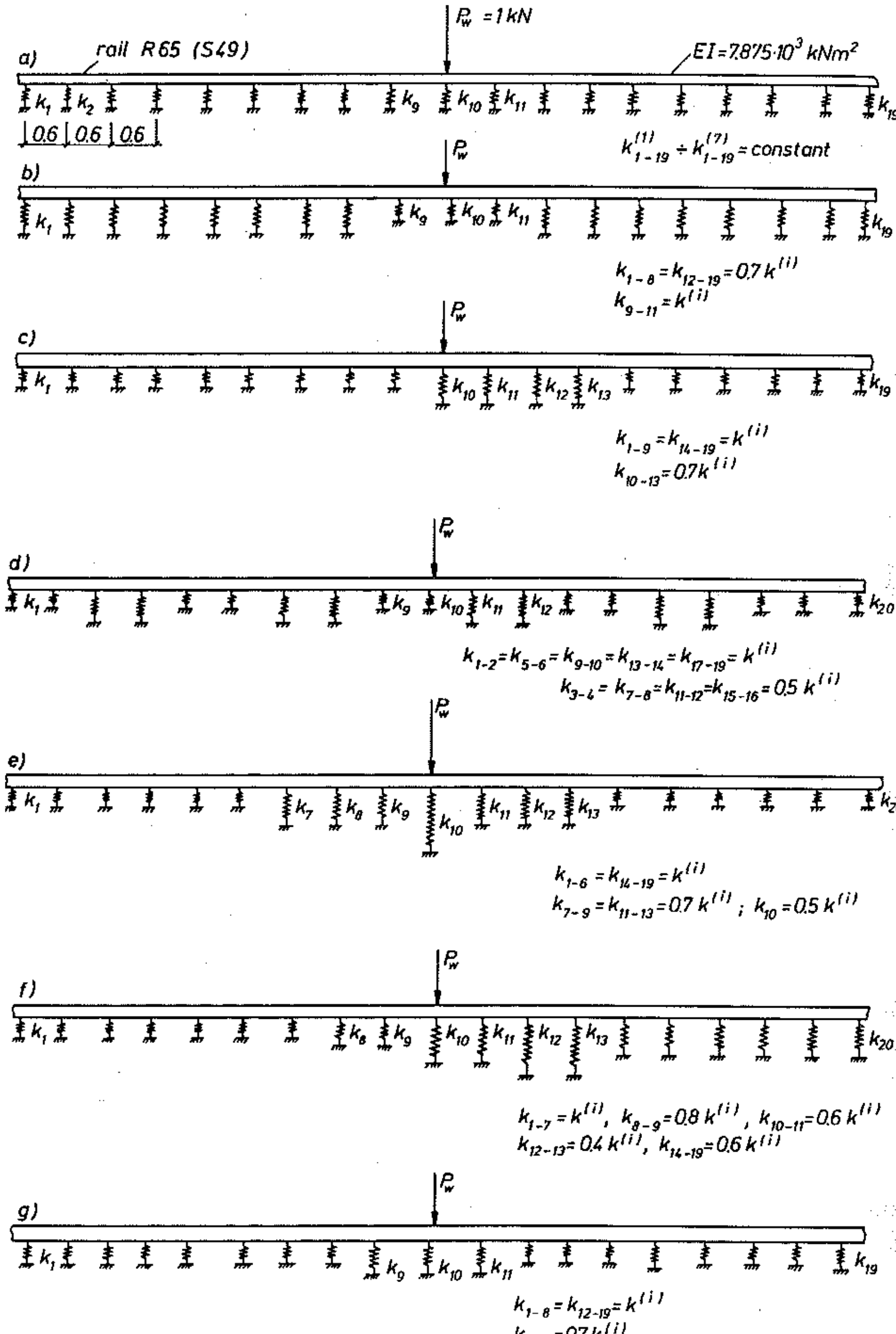

FIG. 4. Nonstationary modelled reduction in stiffness of some supports.

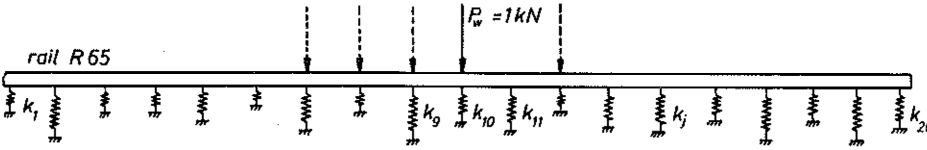

FIG. 5. Stationary randomly variable stiffness of the discrete supports. Characteristics of supports: k_j, μ_k, σ_k.

4. PARAMETRIC STUDY

Using the computer program IDA [3], the parametric studies were conducted to investigate the effect of reduction of the support stiffness. Two- dimensional finite element model with 57 beam elements supported by springs was used to represent the rail. The rail of type R65 was used for the studies, with the flexural stiffness $EI = 7.875 \cdot 10^6\,\mathrm{Nm}^2$, and spacing of the sleepers $l = 0.6\,\mathrm{m}$. The rail response analysis consists in the determination of the vertical displacements $v(x)$, bending moments $M(x)$, and support reaction R_j.

a. Results of the standard linear analysis for constant stiffnesses of supports, $k_j^{(i)} = \mathrm{const.}$, for the four characteristic deterministic stiffness levels of the rail supports $i = 1 \div 4$ (Tab. 4.1) are shown in Fig. 6. The vertical force $P = 1\,\mathrm{kN}$ is applied to the support.

b. Nonstationary reduction in the stiffness of some supports. Such modelling provides the basis for predicting the track performance with relatively poor ballast, or dipped rail joints. Typical result of the response analysis for the cases from Figs. 4e and 4g are shown in Figs. 7 and 8.

The elastic deflection curves illustrated in Figs. 6, 7 and 8 are, at the same time, the influence lines for deflection of the rail, because the unit force $P = 1\,\mathrm{kN}$ was applied. For the wheel force P acting on the rail at the point $x = 0$, the deflection $v(x = 0)$ can be found by multiplying the influence line ordinate by the magnitude of the force P. A comparison is made between the results of nonstationary response and the deterministic response for the characteristic levels of the support stiffness in Table 2.

c. Linear analysis of the stationary random stiffness of supports. For a chosen mean value $\mu_k^{(i)}$ and coefficient of variation V_k ($V_k = 0.1 \div 0.3$), the random variables stiffness $k_j^{(i)}$ were generated by means of the random number generator having a rectangular density function. Typical results of the response analysis for randomly variable stiffness of supports are shown in Figs. 9 and 10, and those for the characteristic stiffness levels $i = 2$ and 4 are displayed in Table 3.

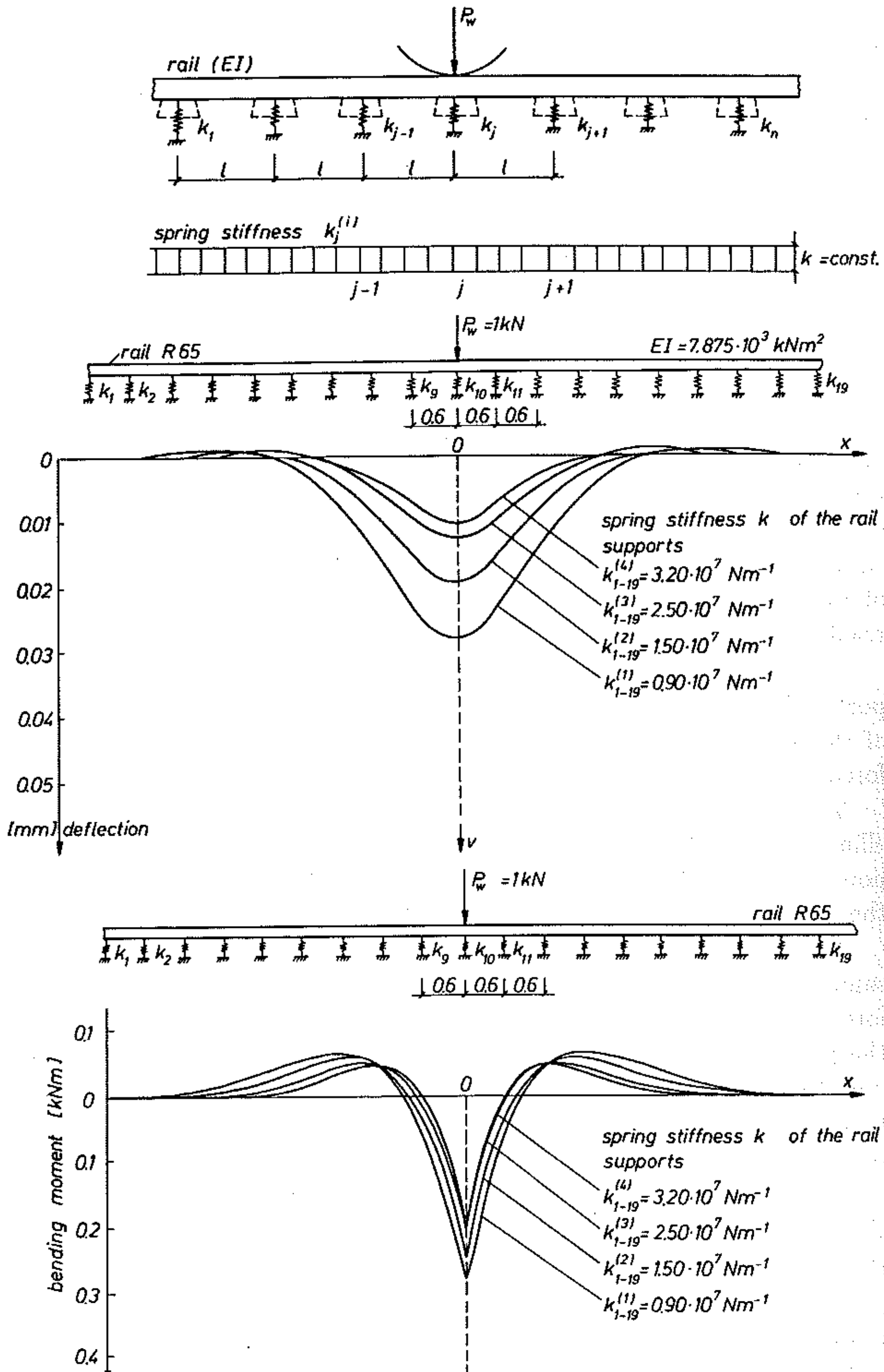

FIG. 6. Results of the standard deterministic analysis for the four stiffness levels.

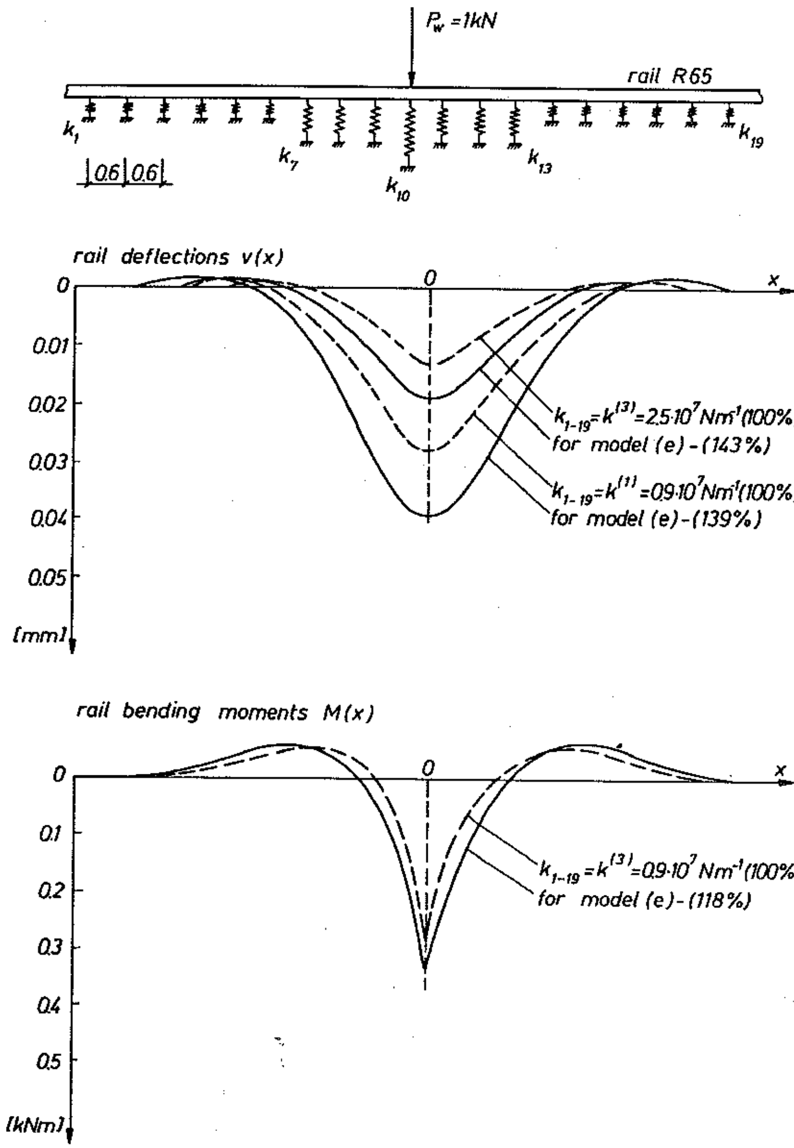

FIG. 7. Response analysis for the nonstationary modelled reduction in stiffness of some supports corresponding to Fig. 4e.

Table 2. Comparison of the rail response for modelled reduction of the support stiffness on Fig. 4e, g, b.

characteristic level (i)	support stiffness $k^{(i)}$ [Nm^{-1}]	Rail response for $P_w = 1\,kN$											
		constant stiffness of supports			modelled reduction of the support stiffness								
					Fig. e			Fig. g			Fig. b		
		$v(x=0)$ [mm]	$M(x=0)$ [kNm]	$R(x=0)$ [kN]	$v(x=0)$ [mm]	$M(x=0)$ [kNm]	$R_j(x=0)$ [kN]	$v(x=0)$ [mm]	$M(x=0)$ [kNm]	$R_j(x=0)$ [kN]	$v(x=0)$ [mm]	$M(x=0)$ [kNm]	$R_j(x=0)$ [kN]
1	$0.9 \cdot 10^7$	0.0284	0.285	0.251	0.0396 (139%)	0.338 (118%)	0.174 (69%)	0.0345 (121%)	0.321 (112%)	0.213 (84%)	0.03 (105%)	0.278 (97%)	0.265 (105%)
2	$1.5 \cdot 10^7$	0.0193	0.249	0.284	–	–	–	–	–	–	–	–	–
3	$2.5 \cdot 10^7$	0.0129	0.212	0.325	0.0185 (143%)	0.257 (121%)	0.232 (71%)	0.163 (126%)	0.241 (113%)	0.286 (88%)	0.013 (100%)	0.208 (98%)	0.334 (102%)
4	$3.2 \cdot 10^7$	0.0108	0.198	0.345	–	–	–	0.0138 (127%)	0.226 (114%)	0.304 (88%)	–	–	–

Table 3. The results of parametric study of the rail response for random stiffness of supports.

characteristic level (i)	input parameters		Response for $P_w = 1\,\text{kN}$					
			deflection $v(x=0)$			bending moment $M(x)$		
	support stiffness $\mu_k^{(i)}$ [Nm^{-1}]	coeff. of variation V_k	mean value μ_v [mm]	standard deviation σ_v [mm]	max. $v(x=0)$ [mm]	mean value μ_M [kNm]	standard deviation σ_M [kNm]	max. $M(x=0)$ [kNm]
2	$1.5 \cdot 10^{-7}$	0.1	$1.96 \cdot 10^{-2}$	$0.22 \cdot 10^{-3}$	0.0205 (106%)	0.248	$2.66 \cdot 10^{-3}$	0.256 (103%)
		0.2		$0.45 \cdot 10^{-3}$	0.0204 (105%)		$4.19 \cdot 10^{-3}$	0.255 (103%)
		0.3		$0.65 \cdot 10^{-3}$	0.0205 (106%)		$6.83 \cdot 10^{-3}$	0.258 (104%)
4	$3.2 \cdot 10^{-7}$	0.1	$1.1 \cdot 10^{-2}$	$0.13 \cdot 10^{-3}$	0.0113 (104%)	0.199	$2.51 \cdot 10^{-3}$	0.203 (102%)
		0.2		$0.27 \cdot 10^{-3}$	0.0114 (105%)		$3.62 \cdot 10^{-3}$	0.207 (104%)
		0.3		$0.42 \cdot 10^{-3}$	0.0118 (109%)		$5.01 \cdot 10^{-3}$	0.209 (103%)

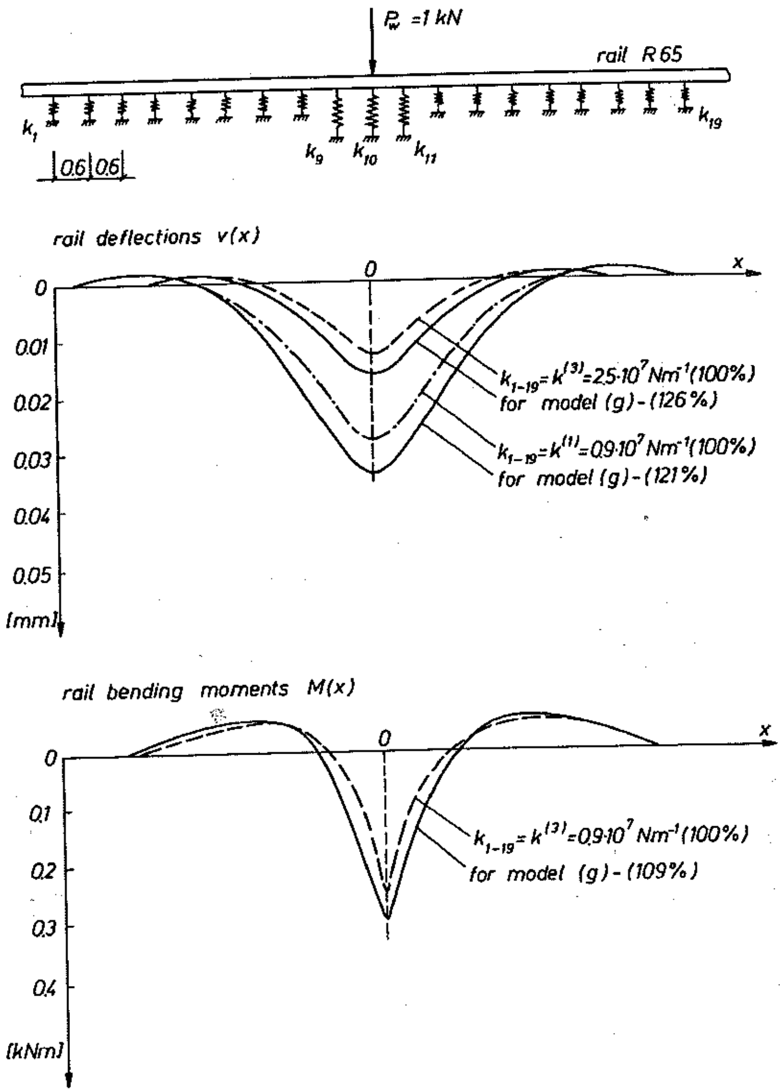

FIG. 8. Response analysis for the nonstationary modelled reduction in stiffness of some supports corresponding to Fig. 4g.

The rail with a stationary, randomly variable stiffnesses of discrete supports was successively statically loaded by the force $P = 1\,\mathrm{kN}$ in positions $j = 1 \div 19$, and the corresponding amplitudes v_j, M_j in positions j were computed. As an example, two elastic deflection curves of the rail for the input parameters $\mu_k = 1.5 \cdot 10^7\,\mathrm{Nm}^{-1}$ and $V_k = 0.1 \div 0.3$, are shown in Fig. 11. They represent the static deformation of the rail that is successively loaded by the force $P = 1\,\mathrm{kN}$ in positions $j = 1 \div 19$.

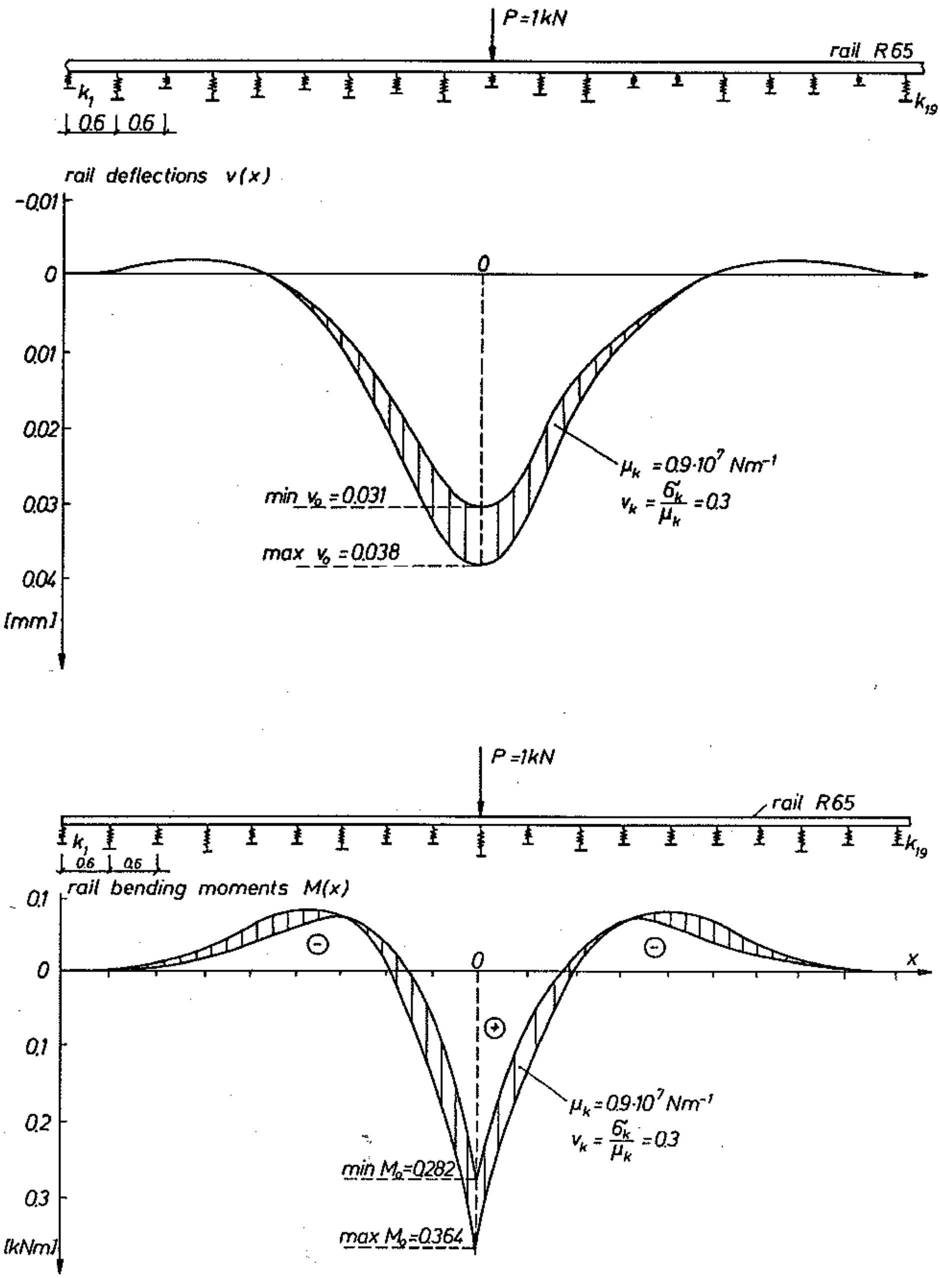

FIG. 9. Response analysis for stationary randomly variable stiffness of supports.

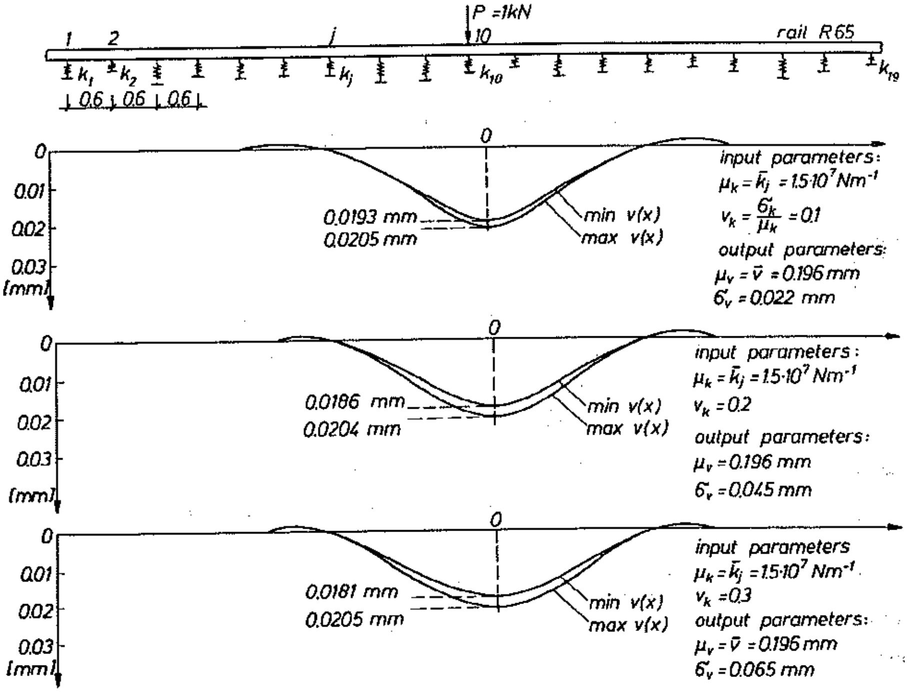

FIG. 10. Response analysis for stationary randomly variable stiffness of supports. Deflection curves for the uniform distribution of spring stiffness $k_j^{(i)}$ and $P_w = 1\,\mathrm{kN}$ in position $j = 10$.

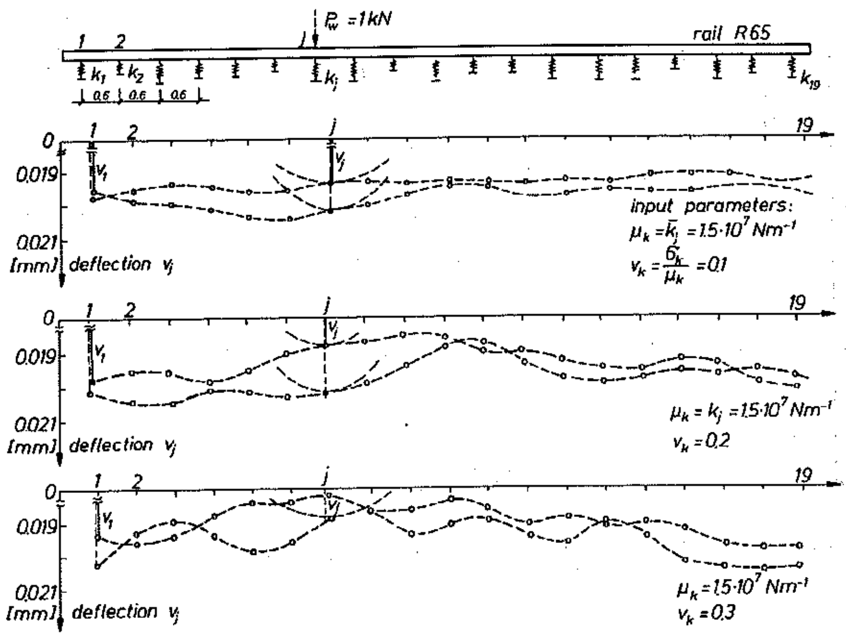

FIG. 11. Elastic rail deflection curves for quasi-stationary wheel force $P = 1\,\mathrm{kN}$ in positions $j = 1 \div 19$.

5. Conclusions

In the presented parametric studies of the quasi-static rail response, the effect of reduction of rail support stiffness is examined. The rail response analysis is concerned with the determination of the vertical displacement $v(x)$, bending moments $M(x)$ and discrete support reactions R_j. The stiffness of discrete supports is a variable quantity and other input parameters of the problem are considered as deterministic ones. The reduction in stiffness of the supports models some important practice cases. The finite element method was used to find the response. Two types of reduction of the support stiffness modelled: the nonstationary reduction in the stiffness of some supports and the stationary random reduction in stiffness of the supports. The first type enables the qualitative prediction of the rail performance with relatively poor ballast or dipped rail joints, while the second type of reduction enables us to assess the effect of random variation in sleeper stiffness. Numerical studies showed that the simulation is an effective method for response analysis of the track structure. The response results for the stationary random reduction in stiffness of the supports show that the response is not so unfavourable as in the case of a nonstationary reduction in support stiffness. Thus, the stiffness of vertical springs can considerably influence the response of the track. The knowledge of the rail variable response is important both for determination of the force transfer from the rail to the sleepers and for assessing the serviceability of the railway track.

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