ENGINEERING TRANSACTIONS • Engng. Trans. • 42, 1–2, 145–155, 1994
Polish Academy of Sciences • Institute of Fundamental Technological Research

OPTIMAL DESIGN OF MISES TRUSS WITH RESPECT TO TIME TO CREEP RUPTURE(*)

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The paper deals with problem of optimal choice of slope angle for bars of Mises truss in creep conditions. Results of optimization with respect to brittle creep rupture coincide with elastic solution (45°), while for ductile creep rupture there is no optimum – the longest life-time is obtained for initially horizontal bars. Introduction of limitation for admissible strains makes it possible to find continuous transition from brittle to ductile rupture. The same possibility gives application of Kachonov's mixed rupture theory.

1. Introduction

The problems of optimal design in creep conditions have a short history. The first papers on this topic appeared in 1968 [12] and they are still scarce. Classification of such problems was given by Życzkowski [13], who pointed out many new possible criteria of optimization. One of the most important of them was connected with time to creep rupture.

There are several possible ways of formulation of such problems due to different theories of creep rupture. Till now, almost all papers on optimal design with respect to creep rupture time were based on Kachanov's theory [4] of brittle rupture. Such an approach was applied e.g. by Życzkowski and Rysz [14] for cylindrical shells, Rysz [7], [8] for pipelines, Ganczarski and Skrzypek [1], [2] for disks, Życzkowski and Świsterski [11], [15] for beams, and so on.

The application of the theory of ductile rupture, proposed by Hoff [3], according to which time of rupture is reached when transversal dimensions are reduced to zero, is more difficult. It is caused by the necessity of application of the finite strain theory. Therefore, there are only few papers using this theory [9], [10], dealing with problems of bars under nonuniform tension.

(*) Presented at the 6th Polish-German Symposium "Mechanics of Inelastic Solids and Structures", Poznań, Poland, September 1993.

In the present paper we shall discuss the optimal choice of the initial angle Φ in the Mises truss, consisting of two identical bars, coupled by a hinge, and in the same way connected with the base. The truss is loaded by a vertical constant force P, causing uniform tension in both bars. The optimal solution in the elastic range is well known, and is equal to 45°. The same value is optimal under the creep conditions for the Kachanov's theory of brittle rupture. It results from the condition of minimization of the initial stress in bars.

ŻYCZKOWSKI [12] stated that angle 45° for bars of Mises truss is optimal for various criteria of optimization, also in creep conditions, regardless of the constitutive law.

2. Ductile rupture theory

In Fig. 1, the solid lines show the current configuration of the truss for the given moment t, while the dashed lines present its initial configuration for $t = 0$.

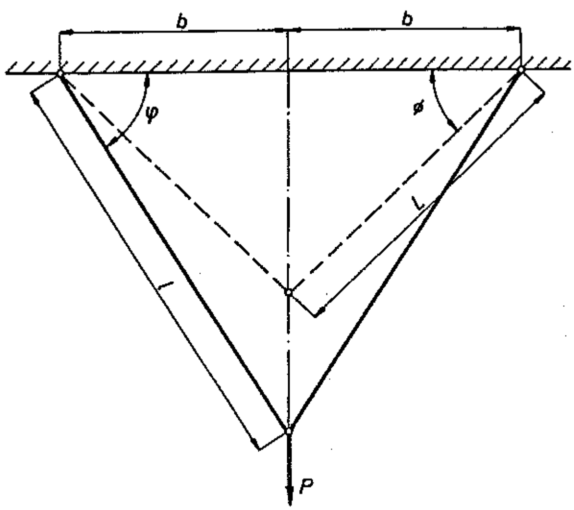

FIG. 1. Mises truss before and after deformation.

For the given volume of bars V and distance b, we look for such initial angle Φ, which leads to the longest life-time to ductile creep rupture. All quantities connected with the initial configuration are denoted by capital letters: L – length of bars, A – their cross-sectional area, Φ – angle of slope, R – reactive force. Corresponding parameters for the current configuration are denoted by the same small letters.

Material of the bars is characterized by NORTON's creep law [6]:

(2.1) $$\dot{\varepsilon} = k\sigma^n ,$$

with material constants k and n, where $\dot{\varepsilon}$ stands for the velocity of logarithmic strains (dot over the symbol means its partial derivative with respect to time)

(2.2) $$\dot{\varepsilon} = \frac{d}{dt}\left(\ln\frac{l}{L}\right) = \frac{\dot{l}}{l} ,$$

while σ stands for the true stress:

(2.3) $$\sigma = \frac{P}{2a\sin\varphi} .$$

Assuming incompressibility of the material

(2.4) $$2AL = 2al = V = \text{const} ,$$

we can express the current cross-sectional area

(2.5) $$a = \frac{V}{2l} = \frac{V}{2L}e^{-\varepsilon} ,$$

and the current angle

(2.6) $$\cos\varphi = \frac{b}{l} = \frac{b}{L}e^{-\varepsilon} = e^{-\varepsilon}\cos\Phi ,$$

in terms of the logarithmic strain ε, and substituting them to Eq. (2.4), we finally obtain

(2.7) $$\sigma = \frac{Pb}{V\cos\Phi}\,\frac{e^{\varepsilon}}{\sqrt{1 - e^{-2\varepsilon}\cos^2\Phi}} .$$

Substitution of this formula into the Norton's creep law (2.1), leads to the differential equation with respect to the function $\varepsilon(t)$, in which the variables can be separated:

(2.8) $$e^{-n\varepsilon}\left(1 - e^{-2\varepsilon}\cos^2\Phi\right)^{n/2} d\varepsilon = k\left(\frac{Pb}{V\cos\Phi}\right)^n dt .$$

Integrating this equation at the initial condition

(2.9) $$\varepsilon(0) = 0,$$

and making use of the condition of ductile rupture in Hoff's sense:

(2.10) $$\varepsilon(t_*^{(d)}) \to \infty$$

we finally come to the expression for the time of ductile creep rupture:

$$t_*^{(d)} = \frac{1}{k}\left(\frac{V\cos\Phi}{Pb}\right)^n \int_0^\infty e^{-n\varepsilon}\left(1 - e^{-2\varepsilon}\cos^2\Phi\right)^{n/2} d\varepsilon \,. \tag{2.11}$$

The value of the improper integral in this formula can be easily found for even-numbered exponents $n = 2$ and $n = 4$. However, any attempts of finding the optimal value of the angle Φ fail. There is no extremum of function (2.11) with respect to Φ.

In fact, the best possible solution is horizontal position of bars ($\Phi = 0$) in the initial configuration. The compressive stresses in bars, leading to the possibility of loss of stability (snap-through problem), are here excluded. Any angle Φ different from zero is equivalent to the loss of time necessary for reaching this position by truss with initially horizontal bars. The optimal solution does not exist.

3. Limited admissible strains

The lack of optimal solution in Hoff's formulation was caused by the definition of rupture, connected with infinitely large strains. The result will be quite different if the admissible strains are limited to certain finite value:

$$\varepsilon(t_*) = \varepsilon_{\text{adm}} = \eta \,. \tag{3.1}$$

In the expression determining the time of ductile rupture (2.11), now we have a proper integral with finite upper limit, equal to η. Integration can be done by the substitution:

$$e^{-\varepsilon} = u. \tag{3.2}$$

At the beginning of the creep process, for $t = 0$, we have $u = 1$, while at the moment of rupture

$$u(t = t_*) = e^{-\eta} = w \,. \tag{3.3}$$

For example, taking for the exponent n in Norton's law the value of 2, we finally obtain

$$t_* = \frac{V^2}{4kP^2b^2}\left[(w^4 - 1)\cos^4\Phi + 2(1 - w^2)\cos^2\Phi\right] . \tag{3.4}$$

Comparing the first derivative of this function, with respect to Φ, to zero, we can easily find the optimal value of the initial angle Φ:

$$\Phi_{\text{opt}} = \arccos\sqrt{\frac{1}{w^2+1}}\,, \tag{3.5}$$

ensuring the longest possible life-time, for a given admissible logarithmic strain η. In the table, the values of optimal angles for various η are presented.

ε_{adm}	Φ_{opt}
0.01	44.71°
0.02	44.42°
0.05	43.57°
0.1	42.14°
0.2	39.31°
0.5	31.23°
1.0	20.20°
2.0	7.71°
5.0	0.39°

In this way, the continuous transition has been obtained, from 45° for very small ε_{adm} (almost brittle rupture), to 0° for ε_{adm} larger than 5 (Hoff's approach). For values of n different from 2, the solution can be found numerically. The results are similar, and for larger exponents n, Φ_{opt} tends to zero even faster.

4. Mixed rupture theory

As the Hoff's ductile rupture theory did not give the optimal solution, now we shall investigate the possibility of such solution for the mixed rupture theory, proposed by Kachanov [5]. We shall apply his evolution equation:

$$\frac{d\psi}{dt} = -B\left(\frac{\sigma}{\psi}\right)^m, \tag{4.1}$$

where ψ is the ratio of the effective cross-sectional area a_{ef} to its initial value A; B and m are material constants. In contrast to the brittle rupture theory, σ denotes here the true stress – related to the current cross-section a (geometrical changes are taken into account).

When the parameter ψ is reduced from its initial value 1 to zero, the structure reaches its rupture time described by the formula

$$\int_0^{t_*^{(m)}} \sigma^m\,dt = \frac{1}{B(m+1)} = \text{const}\,. \tag{4.2}$$

The value of integral in this formula is expressed by the material constants in Kachanov's evolution law – B and m, and therefore, it is constant and independent of the physical law.

For the true stress in Eq. (4.2), we can substitute (2.4), by means of Eqs. (2.5) and (2.6) expressed in terms of the current angle φ:

$$\sigma = \frac{2Pb}{V \sin 2\varphi}\,. \tag{4.3}$$

From Eq. (4.3) it follows that true stress is minimal for $\varphi = 45°$. To determine its change in time we must apply the physical law, here adopted in form of Norton's law (2.1). From Eq. (2.6) we can find

$$\dot{l} = \frac{b \sin\varphi}{\cos^2\varphi}\dot{\varphi} \tag{4.4}$$

and putting it into Eq. (2.3), we obtain

$$\dot{\varepsilon} = \mathrm{tg}\,(\varphi)\dot{\varphi}\,. \tag{4.5}$$

Finally, the Norton's law takes form:

$$\frac{d\varphi}{dt} = k\left(\frac{2Pb}{V}\right)^n \frac{1}{\sin^n 2\varphi}\,\frac{1}{\mathrm{tg}\,\varphi}\,. \tag{4.6}$$

This equation can be solved only numerically, and therefore we shall introduce dimensionless quantities, denoted further by an overbar.

Stresses will be related to the stress in bars of the truss with the angle $\varphi = 45°$:

$$\sigma_0 = \frac{2Pb}{V}\,, \tag{4.7}$$

hence

$$\overline{\sigma} = \frac{\sigma}{\sigma_0} = \frac{1}{\sin 2\varphi}\,. \tag{4.8}$$

Time will be compared with the time of ductile rupture [3] of the bar extended by the initial stress σ_0:

$$\bar{t} = \frac{t}{t_0} = nk\sigma_0^n t = nk\left(\frac{2Pb}{V}\right)^n t\,. \tag{4.9}$$

Now we can rewrite the Norton's law (4.6) in the dimensionless form:

$$\frac{d\varphi}{d\bar{t}} = \frac{1}{n\,\mathrm{tg}\,\varphi \sin^n 2\varphi}\,. \tag{4.10}$$

Dimensionless time of the mixed rupture (4.2) will be now described by the equation

$$\int_0^{\bar{t}_*^{(m)}} \bar{\sigma}^m d\bar{t} = \int_0^{\bar{t}_*^{(m)}} \frac{d\bar{t}}{\sin^m 2\varphi} = \Theta . \tag{4.11}$$

On the right-hand side we have a constant parameter Θ, equal to the ratio of the brittle and ductile rupture times, for bars of the truss with the angle $\varphi = 45°$,

$$\Theta = \frac{t_{*0}^{(b)}}{t_{*0}^{(d)}} = \frac{nk\sigma_0^n}{(m+1)B\sigma_0^m} . \tag{4.12}$$

The value of this parameter depends on the kind of bar material (constants B, k, m and n), and the force P, volume of the material V, and distance b.

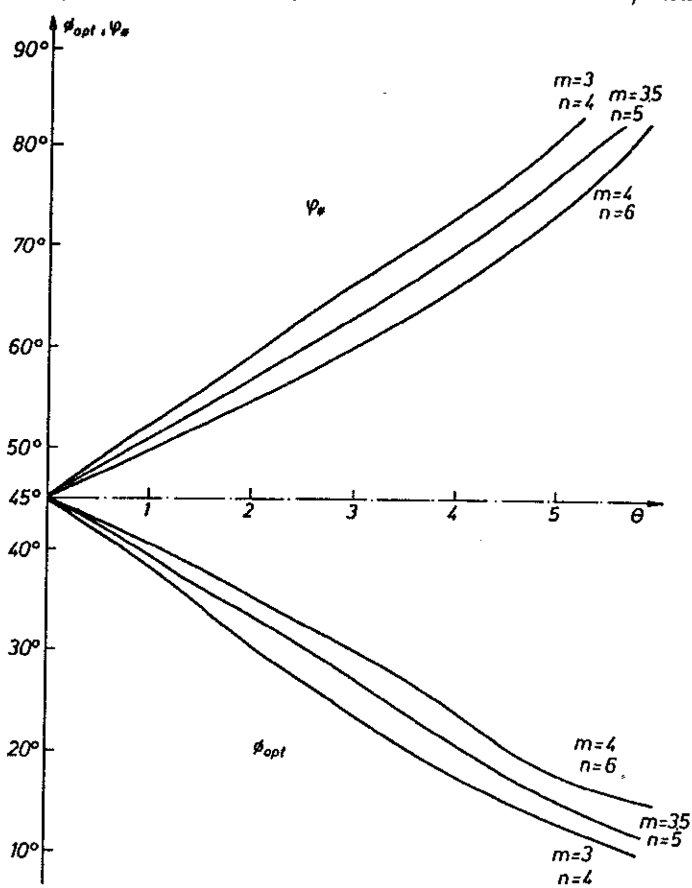

FIG. 2. Optimal initial angles Φ_{opt} and angles φ_* at the moment of mixed rupture.

In order to find the optimal solution, the equation (4.11) was integrated numerically and the upper limit, for which the value Θ is reached, was sought for. Calculations were carried out for various initial angles Φ and, among them, the one leading to the longest life-time (the greatest upper limit) were chosen.

As the free parameters in calculations, besides Θ, the following ones were used: in Norton's law – n, and in Kachanov's law – m. The results for three pairs of exponents taken as an example: $m = 3$ and $n = 4$; $m = 3.5$ and $n = 5$; $m = 4$ and $n = 6$, are presented in Fig. 2, as functions of parameter Θ. The lower curves present values of optimal initial angles Φ_{opt}. For $\Theta = 0$, corresponding to brittle rupture, this angle is equal 45°. As parameter Θ increases, Φ_{opt} tends to zero (horizontal bars), as in case of ductile rupture.

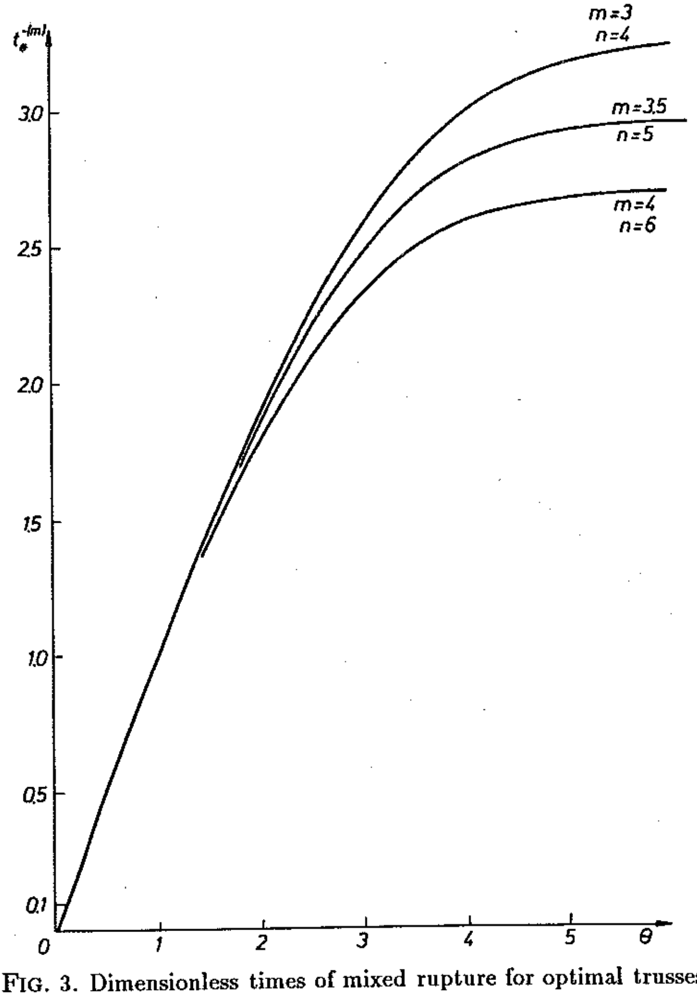

FIG. 3. Dimensionless times of mixed rupture for optimal trusses.

The upper curves in this diagram show the values of φ_*, at which the mixed rupture occurs. These curves are almost symmetrical to curves of Φ_{opt}.

Dimensionless times of mixed rupture, for the same three pairs of exponents, are presented in Fig. 3. For Θ smaller than 2, the results do not differ, and the influence of values of the exponents m and n is distinct for larger Θ.

The gain, in comparison to the time of mixed rupture of the truss with the initial angle $\Phi = 45°$, is shown in Fig. 4. It increases quickly with the growth of Θ.

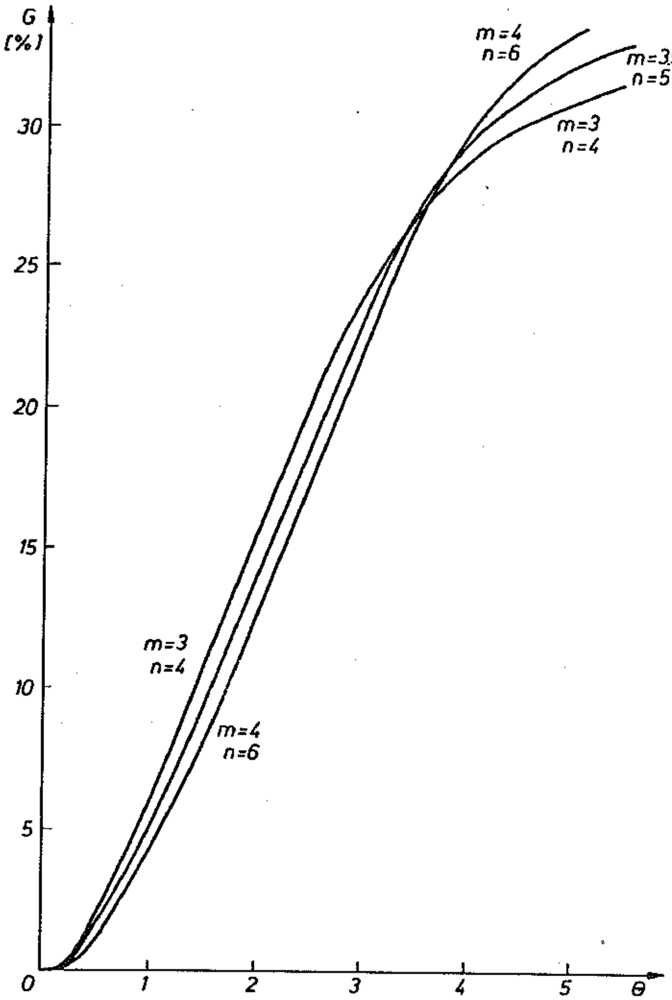

FIG. 4. Gain of time to rupture in comparison with truss of initial angle 45°.

5. Final remarks

The example of Mises truss shows that problems of optimal design with respect to ductile creep rupture not always have a solution. Such a solution exists for the brittle rupture theory and coincides with the one in the elastic range.

To avoid this complication, a limitation of admissible strains may be introduced. In this way all solutions: from brittle rupture (for very small admissible strains), to ductile rupture (for sufficiently large strains) may be obtained.

Problem may be also formulated by introduction of admissible vertical displacement of the hinge joining the truss bars f_{adm}:

$$\frac{f_{\mathrm{adm}}}{b}\cos\Phi + \sin\Phi = \sqrt{e^{2\varepsilon} - \cos^2\Phi}\,. \tag{5.1}$$

The strain corresponding to given f_{adm} may be found

$$\varepsilon = \frac{1}{2}\ln\left[\left(1 + \frac{f_{\mathrm{adm}}^2}{b^2}\right)\cos^2\Phi + \frac{f_{\mathrm{adm}}}{b}\sin 2\Phi + \sin^2\Phi\right], \tag{5.2}$$

and final remarks are obviously the same.

Interesting results are found by application of the mixed rupture theory. In this way the continuous transition from brittle to ductile rupture theory may be obtained, too.

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Received January 21, 1994.