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# OPTIMIZATION OF THE FORM OF A BUILDING WITH AN ARBITRARY BASE

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The present paper is devoted to the formulation and solution of the problem of multicriterial optimization of the form of an energy-saving building with vertical walls and constant volume and height. The base of the building is described by two arbitrary curves. The criterion of minimum building cost and minimum annual heating cost are assumed for optimization. The decision variables of the problem are the curves describing the base of the building. An algorithm for the EUREKA software package has been elaborated. The considerations are illustrated by a numerical example.

## 1. Introduction

If we study the development process of towns and housing estates in Poland, considerable variety of building forms can be observed. Buildings constructed of large reinforced concrete slabs have, from the geometrical point of view, the form of cuboids, or, sometimes combinations of several cubicoids with various side ratios. Buildings constructed of small elements are, on the contrary, much more differentiated, but their form can usually be reduced, to more or less fanciful combinations of prisms. Such buildings, the bases of which constitute rather complicated polygons are usually characterized by an unfavourable geometrical compactness coefficients [4].

The forms of some buildings with polygonal bases were optimized in [1] and [5].

Solution of the optimization problem of the form of a building, the ground plan of which is described by regular lines appears to be promising, such a form being interesting from the architectural point of view and giving good compactness coefficients.

Below we present a method for determining, on the basis of two criteria, the optimum form of a building, for prescribed climatic data and some geometrical quantities such as the area and the projection diameter of the building.

The method of multicriterion optimization enables us to take into consideration many antagonistic partial criteria. It is usually possible to find a compromising solution, in which, usually, none of the partial criteria reaches its extreme value, but all the requirements are satisfied to the highest degree possible, according to a global criterion.

Among the partial criteria used in various optimization procedures of building there are those of

- minimum volume or minimum weight of the structure,
- maximum safety or maximum reliability,
- maximum compactness,
- maximum natural lighting of rooms,
- minimum energy required for heating or cooling purposes of the building.

The criteria which will be used in the present paper are those of

- minimum building cost, and
- minimum annual maintenance cost of the building, including the cost of heating.

## 2. Components of thermal balance of a building

The thermal balance of a building is composed of losses and gains of heat in the course of the heat exchange processes under conditions of stabilized comfort inside the building and variable atmospheric conditions. The heat losses are due to

- heat transmission through external walls, ceilings and floors,
- heat transmission through transparent partitions,
- heat radiation through external walls, ceilings and floors,
- heating of ventillation air and
- infiltration of air through external partitions.

The gains are those of

- sun radiation through transparent partitions,
- heat emitted by lighting installations, household equipment and human organisms and
- heat recovered from the ventillation air.

The differences between the losses and the gains constitute that part of the energy, which must be supplied by the heating system. As regards the present paper, we shall take into consideration only those heat losses and gains which have an essential influence on the solution. They are

• the heat losses through walls, ceilings, floors and transparent partitions and

• the gains in the form of sun radiation heat through transparent partitions.

The effect of the physical environment on a building was the subject of numerous works [2, 5]. In the present optimization problem of the form of a building it will be assumed that the physical environment of the latter is characterized by the following quantities: $SD$ – annual number of degree ·days [K – day], $\theta_1$, $\theta_2$ – average values of the total amount of sun radiation falling during the heating season on the south, east or west vertical plane [kWh/m$^2$], $\alpha_1$ – coefficient of heat penetration from the inside [W/m$^2$K], $\alpha_2$ – coefficient of heat penetration from the outside [W/(m$^2$K)].

### 2.1. *Annual heat losses through the walls, ceilings and floors*

The problem of determining the heat losses through building partitions has been discussed in [5].

Problems of this type are considered with various assumptions concerning spatial temperature distribution, time variability of heat conduction processes and the convection and radiation type of heat exchange with the environment. From [3] it follows that, in the case of sufficiently long time intervals and processes in which the initial and final state of the partition differ little from each other, we can consider the problem, with sufficient accuracy, to be stationary and reducible to the one-dimensional case. In the case of the problem of optimum form of a building, we consider the heat balance of the building throughout the entire heating season, therefore use can be made of simplified relations.

The annual heat losses through an element $h \cdot dl$ of a wall, a ceiling and a floor were found from the formulae [5]:

(2.1) $$E_s = \left(\frac{1-p}{R_s} + \frac{p}{R_o}\right) \cdot 24 \cdot SD\, h\, dl,$$

(2.2) $$E_d = \frac{A_d}{R_d} \cdot \varphi_d \cdot 24 \cdot SD,$$

(2.3) $$E_p = \frac{A_p}{R_p} \cdot \varphi_p \cdot 24 \cdot SD,$$

where

(2.4) $$R_s = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{d}{\lambda}.$$

$A_d, A_p$ area of roofs and floors [$\mathrm{m}^2$], respectively,
$h$ building height [m],
$dl$ length element of a wall [m],
$p$ ratio of the area of the windows to that of a wall element,
$R_s, R_d, R_p$ thermal resistance of a wall, ceiling and floor element [$\mathrm{m}^2\mathrm{K/W}$],
$R_o$ thermal resistance of a window [$\mathrm{m}^2\mathrm{K/W}$],
$d$ partition thickness [m],
$\lambda$ coefficient of heat conduction of the partition material [W/(mK)],
$\varphi_d = \dfrac{t_{dz} - t_{dw}}{t_{sz} - t_{sw}}$,
$\varphi_p = \dfrac{t_{pz} - t_{pw}}{t_{sz} - t_{sw}}$,
$t_{dz}, t_{sz}, t_{pz}$ mean temperature of the external side of a roof, wall and floor [K], respectively,
$t_{dw}, t_{sw}, t_{pw}$ mean temperature of the internal side of a roof, wall and floor [K], respectively.

The annual heat loss due to ventillation can be expressed by the formula.

$$E_w = 0.36 n_i \, 24 \, S D V, \tag{2.5}$$

where $n_i$ is a multiple of air exchange. For $n_i = 0.5$ we have

$$E_w = 0.18 \, 24 \, S D V. \tag{2.6}$$

### *2.2. The gains in energy by sun radiation*

The gains in heat by sun radiation through opaque walls will be disregarded. It is assumed that the daily gain due to sun radiation through a vertical window, the azimuth of which is $\alpha_w$ and the area $A$, is [5]

$$E_s = AJ \cos(\alpha_s - \alpha_w), \tag{2.7}$$

the symbols $J$ and $\alpha_s$ denoting the intensity and the azimuth of radiation. Assuming that the building is symmetric, the annual gain in heat due to sun radiation will be calculated from the formula

$$E_z = 2 \int_{B_1}^{A_1} \theta(\beta) p(x) h \, dl, \tag{2.8}$$

where $\beta$ is the orientation angle of the wall element (Fig. 1) and

$$\theta(\beta) = \theta_1 \cos\beta + \theta_2 \sin\beta. \tag{2.9}$$

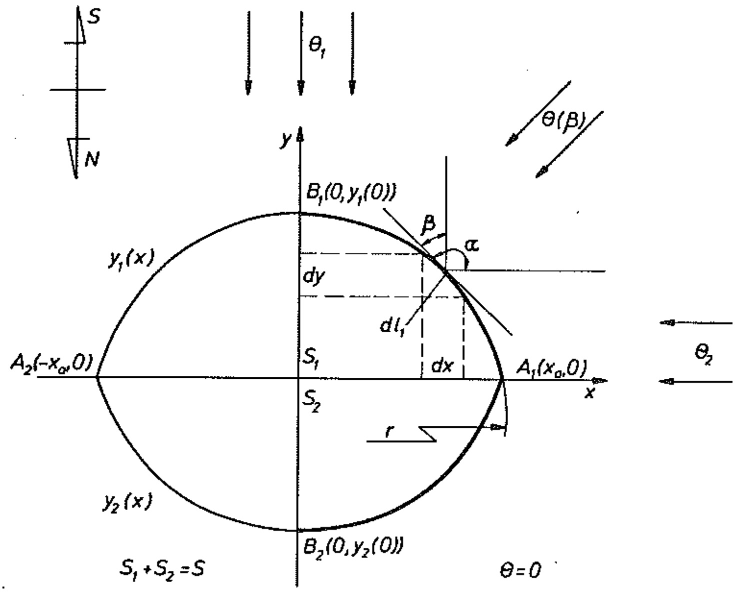


FIG. 1. The form of the building and the symbols used.

## 3. Formulation of the optimization problem

The subject of our considerations is a building with vertical walls, constant volume $V$ and height $h$. The base of the building is described by two arbitrary curves $y_1(x)$ and $y_2(x)$ (Fig. 1).

The gain in heat received through the north facing windows will be disregarded and the building will be assumed to be symmetric along the $N-S$ axis.

The aim of the present considerations is to determine the form of the curves $y_1(x)$ and $y_2(x)$ using two criteria;

- minimum building cost and
- minimum annual cost of heating.

The function expressing the construction cost is defined as follows

$$(3.1) \qquad F_1 = 2\int_{B_1}^{A_1} ((1-p(x))c_s + p(x)c_o)\, h\, dl_1 + 2\int_{B_2}^{A_1} hc_s\, dl_2 + \frac{V}{h}c_d + \frac{V}{h}c_p + D_1,$$

where $c_s, c_o, c_d$, and $c_p$ – are the construction costs of a wall, a window, a roof and a floor, [thousand zł/m$^2$], $V$ – volume of the building [m$^3$] and $D_1$ – other costs independent of the decision variables, [thousand zł].

The function expressing the annual heating cost is described in the following way

$$F_2 = 24SDc_e2\left\{\int_{B_1}^{A_1}\left(\frac{1}{R_s}(1-p(x))+\frac{1}{R_o}p(x)\right)h\,dl_1+\int_{B_2}^{A_1}h\,dl_2 + \left(\frac{1}{R_d}\varphi_d+\frac{1}{R_p}\varphi_p\right)\frac{V}{h}+f_wV\right\} - 2c_e\int_{B_1}^{A_1}\theta(\beta)p(x)h\,dl_1+D_2, \tag{3.2}$$

where $dl_1, dl_2$ are the lengths of elements of the curves $y_1$ and $y_2$, $D_2$ – other costs independent of the decision variables [thousand zł].

The function (2.9) can be expressed in the form $\theta(x)$, assuming as a first approximation that the sought-for curve is a segment of a circle. Then

$$\theta(x)=\theta_1\sqrt{1-\xi^2}+\theta_2\xi,$$

where $\xi=\sin\beta=x/r$.

The function $\theta(x)$ has been approximated by a polynomial of the following type

$$\theta(x)=a_0+a_1\xi+a_2\xi^2+\ldots+a_n\xi^n\,.$$

If the curve $y_1(x)$ obtained as a solution differs considerably from a circular arc, the analysis must be repeated assuming another function $y_1(x)$, with a form differing from that of a circular arc.

Because $dl=\sqrt{1+y'^2}\,dx$, $dl\cos\beta=dx$, $dl\sin\beta=dy$, we obtain

$$F_1 = 2\int_0^{x_a}\Big((1-p(x))c_s+p(x)c_oh\Big)\sqrt{1+{y_1'}^2}\,dx + 2\int_0^{x_a}hc_s\sqrt{1+{y_2'}^2}\,dx+\frac{V}{h}c_d+\frac{V}{h}c_p+D_1, \tag{3.3}$$

$$F_2 = 24SDc_e2\left\{\int_0^{x_a}\left(\frac{1}{R_s}(1-p(x))+\frac{1}{R_o}p(x)\right)h\sqrt{1+{y_1'}^2}\,dx + \int_0^{x_a}\frac{1}{R_s}h\sqrt{1+{y_2'}^2}dx+\left(\frac{1}{R_d}\varphi_d+\frac{1}{R_p}\varphi_p\right)\frac{V}{h}+f_wV\right\} -2c_e\int_0^{x_a}\theta(x)p(x)h\sqrt{1+{y_1'}^2}\,dx+D_2. \tag{3.4}$$

The decision variables of the problem are the functions $y_1(x)$, $y_2(x)$.
It is assumed that
1) $y_1(x)$ and $y_2(x)$ are continuos functions of $C^2$ class within range $[0, x_a]$,
2) the form of the building is symmetric in relation the $0Y$ axis:

$$y_1'(0) = 0, \tag{3.5}$$

$$y_2'(0) = 0, \tag{3.6}$$

3) the functions $y_1(x)$, $y_2(x)$ bound a region of an area $V/h$, that is

$$2\int_0^{x_a} (y_1(x) - y_2(x))\,dx = V/h, \tag{3.7}$$

4) the functions $y_1(x)$ and $y_2(x)$ are zero at the point, the abscissa of which is $x_a$:

$$y_1(x_a) = y_2(x_a) = 0. \tag{3.8}$$

A set of compromises can be determined by the method of weight coefficients. We seek for a minimum of the objective function

$$F = \lambda F_1 + (1 - \lambda)F_2, \tag{3.9}$$

where $\lambda \in <0, 1>$, the condition (3.7) being satisfied.

In the present work the weight coefficient $\lambda$ can be subjected to the modified number $N$ of utilization years of the building as follows:

$$\lambda = \frac{1}{N+1}.$$

The modified number of utilization years is the number of years multiplied by a coefficient expressing the rate of interest and inflation.

The value of the coefficient $\lambda$ being zero is a result of disregarding the costs of building materials and construction. The same effect is produced by assuming the time of utilization of the building to tend to infinity ($N \to \infty$).

The assumption of $\lambda = 1.0$ corresponds to the utilization costs being disregarded, that is to the assumption that $N = 0$. Both cases are not interesting, therefore it is not the entire set of compromises corresponding to $0 \le \lambda \le 1$ that will be determined, but its part

$$\frac{1}{2} \ge \lambda \ge \frac{1}{101},$$

corresponding a utilization period of 1 to 100 years:

$$1 \le N \le 100.$$

## 4. Solution of the optimization problem

From the objective function (3.9) we find

$$(4.1)\qquad F = \left(\lambda c_s + (1-\lambda)\frac{1}{R_s}\,24SDc_e\right)2h\int_0^{x_a}(1-p(x))\sqrt{1+y_1'^2}\,dx$$
$$+\left(\lambda c_o + (1-\lambda)\frac{1}{R_o}\,24SDc_e\right)2h\int_0^{x_a}p(x)\sqrt{1+y_1'^2}\,dx$$
$$+\left(\lambda c_s + (1-\lambda)\frac{1}{R_s}\,24SDc_e\right)2h\int_0^{x_a}\sqrt{1+y_2'^2}\,dx$$
$$+\left((c_d+c_p)\lambda + \left(\frac{1}{R_d}\varphi_d + \frac{1}{R_p}\varphi_p\right)(1-\lambda)\,24SDc_e\right)\frac{V}{h} + D_1\lambda$$
$$+(1-\lambda)f_wV\,24SDc_e - 2(1-\lambda)c_eh\int_0^{x_a}\theta(x)p(x)\sqrt{1+y_1'^2}\,dx$$
$$+(1-\lambda)D_2.$$

Taking into consideration (3.7) we obtain the functional

$$(4.2)\qquad F^* = A_s\int_0^{x_a}(1-p(x))\sqrt{1+y_1'^2}\,dx + A_o\int_0^{x_a}p(x)\sqrt{1+y_1'^2}\,dx$$
$$+A_s\int_0^{x_a}\sqrt{1+y_2'^2}\,dx + C + D - E\int_0^{x_a}\theta(x)p(x)\sqrt{1+y_1'^2}\,dx$$
$$+2\lambda_1\int_0^{x_a}(y_1(x)-y_2(x))\,dx - 2\lambda_1 V/h,$$

where

$$A_s = \left(\lambda c_s + (1-\lambda)\frac{1}{R_s}\,24SDc_e\right)2h,$$
$$A_o = \left(\lambda c_o + (1-\lambda)\frac{1}{R_o}\,24SDc_e\right)2h,$$
$$C = \lambda D_1 + (1-\lambda)f_wV\,24SDc_e,$$
$$D = \left((c_d+c_p)\lambda + \left(\frac{1}{R_d}\varphi_d + \frac{1}{R_p}\varphi_p\right)(1-\lambda)\,24SDc_e\right)\frac{V}{h} + (1-\lambda)D_2,$$
$$E = 2(1-\lambda)c_eh.$$

This is an isoperimetric problem of the variational calculus [6]. The conditions (3.5)–(3.8) enable us to determine the integration constants and the constant $\lambda_1$.

The functional (4.2) reaches its extreme value, if Euler's equations

$$f_{y_1' y_1'} y_1'' + f_{y_1 y_1'} y_1' + f_{x y_1'} - f_{y_1} = 0, \tag{4.3}$$

$$f_{y_2' y_2'} y_2'' + f_{y_2 y_2'} y_2' + f_{x y_2'} - f_{y_2} = 0, \tag{4.4}$$

are satisfied, the symbol $f$ denoting the integrand, that is

$$\begin{aligned}\Big(A_s(1-p(x)) + A_o p(x) - Ep(x)\theta(x)\Big)\frac{1}{\sqrt{1+y_1'^2}^{\,3}}y_1'' \\ + \left((-A_s + A_o - E\theta(x))\frac{dp(x)}{dx} - Ep(x)\frac{d\theta(x)}{dx}\right)\frac{y_1'}{\sqrt{1+y_1'^2}} - 2\lambda_1 = 0,\end{aligned} \tag{4.5}$$

$$A_s\frac{1}{\sqrt{1+y_1'^2}^{\,3}}y_2'' + 2\lambda_1 = 0. \tag{4.6}$$

The equation (4.5) can be reduced, by substituting

$$v(x) = \frac{y_1'}{\sqrt{1+y_1'^2}^{\,3}}, \qquad \frac{dv(x)}{dx} = \frac{y_1''}{\sqrt{1+y_1'^2}^{\,3}} \tag{4.7}$$

to the form

$$\begin{aligned}\frac{dv(x)}{dx} + v(x)\frac{\left((-A_s + A_o - E\theta(x))\frac{dp(x)}{dx} - Ep(x)\frac{d\theta(x)}{dx}\right)}{A_s(1-p(x)) + A_o p(x) - Ep(x)\theta(x)} \\ = \frac{2\lambda_1}{A_s(1-p(x)) + A_o p(x) - Ep(x)\theta(x)}.\end{aligned} \tag{4.8}$$

This is a linear differential equation of the first order. On integrating we obtain

$$v(x) = \frac{2\lambda_1 x + C}{A_s(1-p(x)) + A_o p(x) - Ep(x)\theta(x)}. \tag{4.9}$$

From the condition (3.5) we find the integration constant $C$. Because $y_1'(0) = 0$, $v(0) = 0$, therefore

$$0 = \frac{2\lambda_1 0 + C}{A_s(1-p(0)) + A_o p(0) - Ep(0)\theta(0)},$$

that is $C = 0$. Hence

$$v(x) = \frac{2\lambda_1 x}{A_s(1-p(x)) + A_o p(x) - Ep(x)\theta(x)} \tag{4.10}$$

or

$$\frac{y_1'(x)}{\sqrt{1+y_1'(x)^2}} = \frac{2\lambda_1 x}{A_s(1-p(x)) + A_o p(x) - Ep(x)\theta(x)}. \tag{4.11}$$

It is easy to show that $v(x)$ is the sine of the inclination angle of the tangent to the $0X$-axis at the point $(x, y_1(x))$. It follows that the value of $2\lambda_1$ is limited,

$$|2\lambda_1| \le \frac{A_s(1-p(x)) + A_o p(x) - Ep(x)\theta(x)}{x}, \tag{4.12}$$

for any $x \in (0, x_a]$.

From (4.7) we find

$$|y_1'(x)| = \frac{|v(x)|}{\sqrt{1-v^2(x)}}.$$

Because $y_1'(x) < 0$ and $v(x) < 0$ for $x \in (0, x_a)$, we have

$$y_1'(x) = \frac{v(x)}{\sqrt{1-v^2(x)}}. \tag{4.13}$$

Similarly, on integrating (4.6), we have

$$\frac{y_2'(x)}{\sqrt{1+y_2'(x)^2}} = \frac{2\lambda_1 x}{A_s}, \tag{4.14}$$

that is

$$|y_2'(x)| = \frac{x}{\sqrt{(A_s/2/\lambda_1)^2 - x^2}}. \tag{4.15}$$

Similarly to the function $v(x)$, the expression $(2\lambda_1 x/A_s)$ is the sine of the inclination angle of the tangent to the $0X$-axis at the point $(x, y_2(x))$. It follows that the value of $\lambda_1$ is limited:

$$|2\lambda_1| \le (A_s/x_a). \tag{4.16}$$

Because $y_2'(x) > 0$ for $x \in (0, x_a]$, we have

$$y_2'(x) = \frac{x}{\sqrt{(A_s/2/\lambda_1)^2 - x^2}}. \tag{4.17}$$

Hence, on integrating, we obtain

$$y_2(x) = -\sqrt{(A_s/2/\lambda_1)^2 - x^2} + C_{R_2}\,. \tag{4.18}$$

From the condition (3.8) $y_2(x) = 0$ we find the constant $C_{R_2}$:

$$0 = -\sqrt{(A_s/2/\lambda_1)^2 - x_a^2} + C_{R_2} \qquad \text{therefore} \qquad C_{R_2} = \sqrt{(A_s/2/\lambda_1)^2 - x_a^2}\,.$$

Hence

$$y_2(x) = -\sqrt{(A_s/2/\lambda_1)^2 - x^2} + \sqrt{(A_s/2/\lambda_1)^2 - x_a^2}\,. \tag{4.19}$$

It is seen that

$$\left(y_2(x) - \sqrt{(A_s/2/\lambda_1)^2 - x_a^2}\right)^2 + x^2 = (A_s/2/\lambda_1)^2.$$

This is an equation of a circle with its centre at the point $O_2(0, C_{R_2})$ and a radius $R_2 = A_s/2/\lambda_1$.

The area of the segment of that circle bounded by the $0X$-axis, is, for $y_2(x) < 0$,

$$\begin{aligned} S_2 &= 2\int_0^{x_a} |y_2(x)|\,dx \\ &= 2\int_0^{x_a} \left(\sqrt{(A_s/2/\lambda_1)^2 - x^2} - \sqrt{(A_s/2/\lambda_1)^2 - x_a^2}\right) dx \\ &= (A_s/2/\lambda_1)^2 \arcsin\left(\frac{x_a\, 2\lambda_1}{A_s}\right) - x_a\sqrt{(A_s/2/\lambda_1)^2 - x_a^2}\,. \end{aligned} \tag{4.20}$$

Integration of the Eq. (4.11) requires assumption of the form of the function $p(x)$. Then, the integration constant will be determined from the condition (3.8) $y_1(x_a) = 0$ and the parameter $\lambda_1$ – from the condition (3.7).

### *4.1. Solution of the optimization problem for assumed forms of the functions $\theta(x)$ and $p(x)$*

It is assumed that the function $\theta(x)$, the value of which is $\theta_1$ on the south facade and $\theta_2$ on the east or and west facade, can be approximated by a second order polynomial. We obtain

$$\theta(x) = \theta_1 - (\theta_1 - \theta_2)(x_a/r)^2(x/x_a)^2,$$

where $r$ is the abscissa, for which the tangent to the line $y_1(x)$ is parallel to the $0Y$-axis.

It is assumed that $p(x)$ is a trinomial square. Taking into consideration the symmetry condition we find that $p(x) = ax^2 + c$. On denoting $p(0) = p_0$ and $p(x_a) = p_a$, we find

$$p(x) = p_0 - (p_0 - p_a)(x/x_a)^2 . \tag{4.21}$$

On substituting the expressions for $p(x)$ and $\theta(x)$ into (4.13), we obtain the relation

$$y_1'(x) = \frac{2\lambda_1 x}{\sqrt{M^2 - 4\lambda_1^2 x^2}} , \tag{4.22}$$

where

$$M = A_s + \left(p_0 - (p_0 - p_a)(x/x_a)^2\right) \times \left(A_o - A_s - E\left[\theta_1 - (\theta_1 - \theta_2)(x_a/r)^2(x/x_a)^2\right]\right) .$$

Because the form of the function $y_1(x)$ cannot be determined from the Eq. (4.22) in an analytical manner, the function $y_1'(x)$ will be approximated within the interval $0 \le x \le x_a$ by the polynomial

$$y_1'(x) = B_2 x + B_4 x^3 + B_6 x^5 . \tag{4.23}$$

The coefficients $B_2$, $B_4$, $B_6$ can be determined by the method of smallest squares. On integrating we find

$$y_1(x) \simeq B_2 x^2/2 + B_4 x^4/4 + B_6 x^6/6 + C. \tag{4.24}$$

The condition (3.8) will be used to determine the integration constant

$$0 \simeq B_2 x_a^2/2 + B_4 x_a^4/4 + B_6 x_a^6/6 + C,$$

that is

$$y_1(x) \simeq B_2(x^2 - x_a^2)/2 + B_4(x^4 - x_a^4)/4 + B_6(x^6 - x_a^6)/6. \tag{4.25}$$

To determine the values of the coefficients $B_2$, $B_4$ and $B_6$, the knowledge of the constant $2\lambda_1$ is necessary. This quantity will be determined by iteration from the condition

$$\varepsilon_s = |S_1 + S_2 - S| \le \varepsilon_{so}, \tag{4.26}$$

where

$$(4.27)\qquad S_1 \simeq 2\int_0^{x_a} y_1(x)\,dx$$
$$\simeq 2\int_0^{x_a} \Big(B_2(x^2 - x_a^2)/2 + B_4(x^4 - x_a^4)/4 + B_6(x^6 - x_a^6)/6\Big)\,dx$$
$$= -2\left(B_2 x_a^3/3 + B_4 x_a^5/5 + B_6 x_a^7/7\right),$$

$S_2$ is determined by the Eq. (4.20) and $c_{so}$ denotes the tolerance of calculation of the area.

If the inequality (4.26) is satisfied for the values of $S_1$ and $S_2$ thus determined, the calculation work may be finished. If this inequality is not satisfied and the expression under the modulus sign is positive, then, for the new calculation, the parameter $2\lambda_1$ should be reduced.

The value of the parameter $2\lambda_1$ thus obtained determines, in an unambiguous manner, the lines $y_1(x)$ and $y_2(x)$, thus making us able to determine the functions $F_1$, $F_2$ and $F$. To do this we must calculate the integrals:

$$\int_0^{x_a}\sqrt{1+y_1'^2}\,dx,\qquad \int_0^{x_a} p(x)\sqrt{1+y_1'^2}\,dx,$$
$$\int_0^{x_a}\sqrt{1+y_2'^2}\,dx,\qquad \int_0^{x_a} \theta(x)p(x)\sqrt{1+y_1'^2}\,dx.$$

Making use of the Eqs. (4.10), (4.13) and (4.23), we obtain

$$(4.28)\qquad \sqrt{1+y_1'^2} = \frac{1}{\sqrt{1-v^2(x)}} = \frac{y_1'(x)}{v(x)}$$
$$= \frac{B_2 + B_4x^2 + B_6x^4}{2\lambda_1}\left(A_s + p(x)(A_o - A_s - E\theta(x))\right),$$

and, for the functions $p(x)$, $\theta(x)$, which have already been assumed,

$$(4.29)\qquad \sqrt{1+y_1'^2} = \frac{B_2 + B_4x^2 + B_6x^4}{2\lambda_1}(\alpha_0 + \alpha_2x^2 + \alpha_4x^4)$$
$$\simeq 1/2/\lambda_1\Big(B_2\alpha_0 + x^2(B_2\alpha_2 + B_4\alpha_0) + x^4(B_2\alpha_4 + B_4\alpha_2 + B_6\alpha_0)$$
$$+x^6(B_4\alpha_4 + B_6\alpha_2) + x^8B_6\alpha_4\Big),$$

where

$$\alpha_0 = A_s + p_o(A_o - A_s - E\theta_1),$$
$$\alpha_2 = -\frac{1}{x_a^2}\left((p_o - p_a)(A_o - A_s - E\theta_1) - E(\theta_1 - \theta_2)(x_a/r)^2 p_o\right),$$
$$\alpha_4 = -\frac{1}{x_a^4}(p_o - p_a)E(\theta_1 - \theta_2)(x_a/r)^2.$$

Hence

$$\int\limits_0^{x_a} \sqrt{1 + {y'_1}^2}\, dx \simeq 1/2/\lambda_1\Big(B_2\alpha_0 x_a + (B_2\alpha_2 + B_4\alpha_0)x_a^3/3 + (B_2\alpha_4 + B_4\alpha_2 + B_6\alpha_0)x_a^5/5 + (B_4\alpha_4 + B_6\alpha_2)x_a^7/7 + B_6\alpha_4 x_a^9/9\Big), \tag{4.30}$$

$$\int\limits_0^{x_a} p(x)\sqrt{1 + {y'_1}^2}\, dx \simeq 1/2/\lambda_1\left(B_2\alpha_0 x_a\left(p_o - \frac{p_o - p_a}{3}\right) + (B_2\alpha_2 + B_4\alpha_0)x_a^3\left(p_o/3 - \frac{p_o - p_a}{5}\right) + (B_2\alpha_4 + B_4\alpha_2 + B_6\alpha_0)x_a^5\left(p_o/5 - \frac{p_o - p_a}{7}\right) + (B_4\alpha_4 + B_6\alpha_2)x_a^7\left(p_o/7 - \frac{p_o - p_a}{9}\right) + B_6\alpha_4 x_a^9\left(p_o/9 - \frac{p_o - p_a}{11}\right)\right), \tag{4.31}$$

$$\begin{aligned}\int\limits_0^{x_a} \theta(x)p(x)\sqrt{1 + {y'_1}^2}\, dx &\simeq 1/2/\lambda_1\Big(B_2\alpha_0\gamma_0 x_a + (B_2(\alpha_2\gamma_0 + \alpha_0\gamma_2) + B_4\alpha_0\gamma_0)x_a^3/3 \\ &+ (B_2(\alpha_4\gamma_0 + \alpha_2\gamma_2 + \alpha_0\gamma_4) + B_4(\alpha_2\gamma_0 + \alpha_0\gamma_2) + B_6\alpha_0\gamma_0)x_a^5/5 \\ &+ (B_2(\alpha_4\gamma_2 + \alpha_2\gamma_4) + B_4(\alpha_4\gamma_0 + \alpha_2\gamma_2 + \alpha_0\gamma_4) + B_6(\alpha_2\gamma_0 + \alpha_0\gamma_2))x_a^7/7 \\ &+ (B_2\alpha_4\gamma_4 + B_4(\alpha_4\gamma_2 + \alpha_2\gamma_4) + B_6(\alpha_4\gamma_0 + \alpha_2\gamma_2 + \alpha_0\gamma_4))x_a^9/9 \\ &+ (B_4\alpha_4\gamma_4 + B_6(\alpha_4\gamma_2 + \alpha_2\gamma_4))x_a^{11}/11 + B_6\alpha_4\gamma_4 x_a^{13}/13\Big),\end{aligned} \tag{4.32}$$

where

$$\gamma_0 = p_o\theta_1,$$
$$\gamma_2 = -\frac{1}{x_a^2}\left(p_o - p_a - (\theta_1 - \theta_2)(x_a/r)^2\right),$$
$$\gamma_4 = -\frac{1}{x_a^4}(p_o - p_a)(\theta_1 - \theta_2)(x_a/r)^2.$$

From (4.30) we find

$$(4.33)\qquad \int_0^{x_a} \sqrt{1+y_2'^2}\, dx = \int_0^{x_a} \frac{A_s/2/\lambda_1}{\sqrt{(A_s/2/\lambda_1)^2 - x^2}}\, dx = \frac{A_s}{2\lambda_1} \arcsin(2\lambda_1 x_a/A_s).$$

The above integrals having been determined, we find from (3.3), (3.4) and (3.9) the functions $F_1$, $F_2$ and $F$.

## 5. Numerical example

The present calculations have been performed with the IBM PC 12 MHz computer using the Borland International EUREKA software package. The data assumed for computation were as follows:

$$\begin{aligned}
c_s &= 400\,[\text{thousand zł/m}^2], & c_o &= 300\,[\text{thousand zł/m}^2],\\
c_d &= 320\,[\text{thousand zł/m}^2], & c_p &= 210\,[\text{thousand zł/m}^2],\\
c_e &= 0.6\,[\text{thousand zł/kWh}], & &\\
1/R_s &= 0.72\,[\text{W/m}^2\text{K}], & 1/R_o &= 2.6\,[\text{W/m}^2\text{K}],\\
1/R_d &= 0.44\,[\text{W/m}^2\text{K}], & 1/R_p &= 0.57\,[\text{W/m}^2\text{K}],\\
\varphi_d &= 1.1, & \varphi_p &= 0.9,\\
SD &= 4000\,\text{K day}, & h &= 1\,\text{m},\\
p_o &= 0.3, & p_a &= 0.2,\\
\theta_1 &= 350\,[\text{kWh/m}^2], & \theta_2 &= 120\,[\text{kWh/m}^2],\\
x_a &= 0.8\,\text{m}, & S &= 1.0\,\text{m}^2, \qquad \varepsilon_{so} = 0.01\,\text{m}^2.
\end{aligned}$$

The values of the parameter $\lambda$ used for computation were

$$\lambda = 1/2,\ 1/11,\ 1/26,\ 1/51 \ \text{ and } \ 1/101,$$

which corresponds to the modified utilization times of the building, expressed in years, $N = 1,\ 10,\ 25,\ 50$ and 100.

For such assumptions we find, from (4.12) and (4.16), the maximum value of the parameter $|2\lambda_1|$.

The points of the set of compromises which have been determined are shown in Fig. 2 and the corresponding forms of the bases of the buildings – in Fig. 3.

The final results of computation are quoted in Tab. 1.

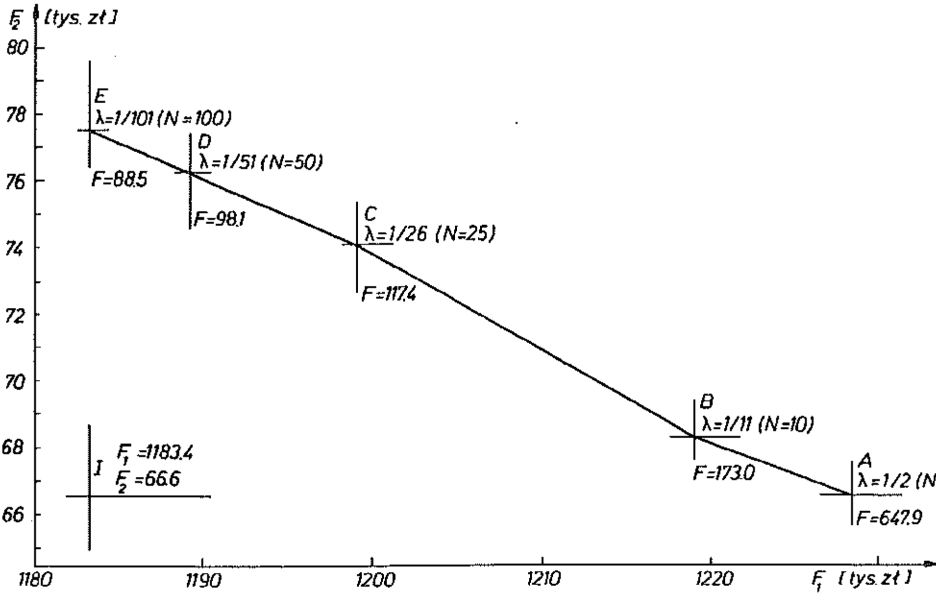


FIG. 2. The set of compromises and indication of the perfect solution.

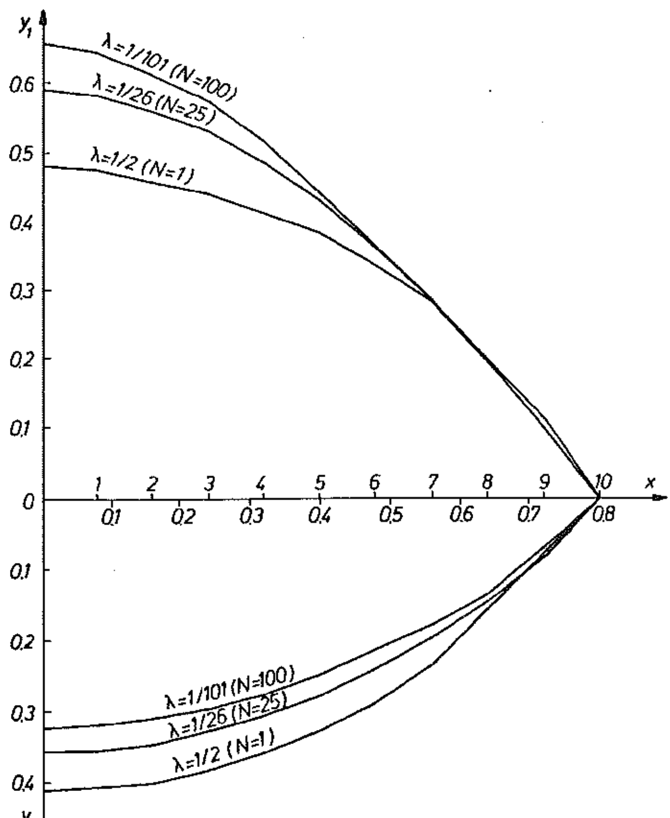


FIG. 3. The form of the optimum base of a building for various values of the parameter λ.

**Table 1. The results of computation.**

| $\lambda$ [–] | 1/2 | 1/11 | 1/26 | 1/51 | 1/101 |
|---|---|---|---|---|---|
| $\lvert 2\lambda_1 \rvert$ [–] | 450.3 | 147.4 | 102.5 | 86.4 | 78.2 |
| $x_a/r$ [–] | 0.855 | 0.830 | 0.794 | 0.777 | 0.767 |
| $x$ [m] | $y_1(x)$ [m] | | | | |
| 0.00 | 0.480 | 0.560 | 0.590 | 0.624 | 0.654 |
| 0.08 | 0.476 | 0.555 | 0.583 | 0.616 | 0.645 |
| 0.16 | 0.464 | 0.538 | 0.563 | 0.592 | 0.618 |
| 0.24 | 0.444 | 0.511 | 0.530 | 0.554 | 0.575 |
| 0.32 | 0.416 | 0.473 | 0.484 | 0.501 | 0.516 |
| 0.40 | 0.380 | 0.423 | 0.426 | 0.436 | 0.446 |
| 0.48 | 0.334 | 0.363 | 0.358 | 0.362 | 0.366 |
| 0.56 | 0.277 | 0.291 | 0.281 | 0.279 | 0.279 |
| 0.64 | 0.206 | 0.207 | 0.195 | 0.190 | 0.189 |
| 0.72 | 0.116 | 0.111 | 0.111 | 0.097 | 0.095 |
| 0.80 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| $B_2$ | −1.2471 | −1.7165 | −2.1284 | −2.5067 | −2.8517 |
| $B_4$ [–] | +0.1524 | +0.1144 | +1.0957 | +2.1993 | +3/3103 |
| $B_6$ | −2.1999 | −0.5205 | −0.4906 | −1.0693 | −1.8443 |
| $R_2$ [m] | 0.980 | 1.005 | 1.078 | 1.123 | 1.151 |
| $C_{R_2}$ | 0.567 | 0.608 | 0.723 | 0.788 | 0.827 |
| $S_1$ | 0.5375 | 0.5533 | 0.6123 | 0.6314 | 0.6500 |
| $S_2$ [m$^2$] | 0.4640 | 0.4434 | 0.3939 | 0.3699 | 0.3556 |
| $S$ | 1.0015 | 0.9966 | 1.0061 | 1.0013 | 1.0056 |
| $F_1$ | 1229.1 | 1219.1 | 1199.1 | 1189.2 | 1183.4 |
| $F_2$ [1000 zl] | 66.6 | 68.4 | 74.1 | 76.3 | 77.5 |
| $F$ | 647.9 | 173.0 | 117.4 | 98.1 | 88.5 |

## 6. Inferences

The optimization problem of the form of a building with an arbitrary base has been solved by variational methods. The solution obtained is composed of a circular segment bounding the northern part of the building and a curve described by a sixth degree polynomial bounding its southern part. The ratio of the area of the southern part to that of the northern part depends on the size of the windows, the density of sun radiation energy and the ratio of unit costs of the windows and walls. They increase along with the number $N$, which determines the modified utilization time of the building. In the present problem they are

$$S_{pd}/S_{pn} = 1.1584 \quad \text{for} \quad N = 1 \quad \text{and}$$
$$S_{pd}/S_{pn} = 1.8279 \quad \text{for} \quad N = 100.$$

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