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ON PLASTIC DESIGN OF ROTATING COMPLEX MACHINE ELEMENTS

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The differential relations (analoguous to Henecky equations) along slip-lines in the presence of body forces are formulated. These relations may be used in constructing stress fields for plastic design of various machine elements undergoing large body forces. A slip-line net is constructed for the case of a strip loaded by tension and centrifugal forces. Application of this solution to plastic design of a rotor is demonstrated.

1. Introduction

Elements subject to large body loads form an essential part of many contemporary machines and mechanisms, such as high speed turbines, engines, pumps, generators etc. Fast moving parts like rotors, turbine blades, crankshafts or connecting rods are usually of compact but complex shape and their design often causes difficulties.

The method which proved to be useful in design of such complex elements is the method of limit carrying capacity [3]. This method is based on the limit design theorems of the theory of plasticity. Consider a body made of a rigid-perfectly plastic material, obeying the associated flow rule and the Drucker postulate (i.e. its yield surface is convex). Then the lower (safe) estimate of the limit load that such body is able to carry can be obtained from any statically admissible stress field. The stress field is defined to be statically admissible if it satisfies the equilibrium equations and the stress boundary conditions and if at any point the yield condition is not violated. If the load that the body is supposed to carry is assumed in advance, the shape of the body beying not determined, then the contour of any statically admissible stress field gives a safe estimate of the required shape.

A number of examples show that, despite crude simplification of the material properties assumed in the theory, the method of limit carrying capacity gives good results in designing various structural elements of practical importance. The basic way of constructing statically admissible stress fields for problems of design of complex shape elements applied in monograph [3] was the piecewise homogeneous stress field technique. Stress fields constructed in this way consist of a number of homogeneous subfields separated by the lines of stress discontinuity. However, in the case of continuously distributed body forces, the stress fields are in general nonhomogeneous, so this technique is not adequate. The right tool in this case is the slip-line technique, known from numerous applications in the theory of plasticity and in the mechanics of granular media.

2. The slip-line method in the presence of body forces

The slip-line method (known also as the method of characteristics) is the standard technique of solving various plane problems in the mechanics of plastic flow. Its detailed description may be found in any monograph dealing with the theory of plasticity or its applications (e.g. [2]). Thus in the present paper only the basic formulae will be presented, for the case, however, that is more general than the usual one. It is namely assumed, that the body under consideration is subject to body forces whose components are known functions of coordinates x and y.

Consider a body in the state of plane strain and just at the limit of plastic collapse. Thus the stress components satisfy the yield condition which in the present case assumes the form

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2 , \tag{2.1}$$

where k is the yield stress in simple shear. This formula holds valid for both the Tresca and Huber–Mises yield conditions. Equilibrium equations are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho X = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho Y = 0 , \tag{2.2}$$

where ρ is density, and X, Y are components of body force per unit mass. The above three equations form the set from which three unknown components of the stress tensor σ_x, σ_y and τ_{xy} can be found. The problem is thus statically determinate (there is no need to consider strains or velocities).

For the sake of simplifying this set of equations, two new unknown functions χ and φ are introduced. The first one is proportional to the sum of principal stresses σ_1 and σ_2

$$2k\chi = \frac{1}{2}(\sigma_1 + \sigma_2), \tag{2.3}$$

while the other is defined as the value of the angle between the direction of the larger principal stress ($\sigma_1 > \sigma_2$) and the x-coordinate axis.

The non-vanishing components of the stress tensor are expressed in terms of these functions by the formulae

$$\begin{aligned} \sigma_x &= 2k\chi + k\cos 2\varphi\,, \\ \sigma_y &= 2k\chi - k\cos 2\varphi\,, \\ \tau_{xy} &= k\sin 2\varphi\,. \end{aligned} \tag{2.4}$$

The yield condition (2.1) is thus identically satisfied. Substituting formulae (2.4) into equations (2.2) one obtains

$$\begin{aligned} \frac{\partial\chi}{\partial x} - \sin 2\varphi\frac{\partial\varphi}{\partial x} + \cos 2\varphi\frac{\partial\varphi}{\partial y} &= -\frac{\rho}{2k}X\,, \\ \frac{\partial\chi}{\partial y} + \cos 2\varphi\frac{\partial\varphi}{\partial x} + \sin 2\varphi\frac{\partial\varphi}{\partial y} &= -\frac{\rho}{2k}Y\,. \end{aligned} \tag{2.5}$$

This set of equations is of the hyperbolic type, so it has two families of real characteristics. Their equations can be found by the standard procedure (see e.g. [2]):

$$\frac{dy}{dx} = \operatorname{tg}\left(\varphi + \frac{\pi}{4}\right), \quad d\chi + d\varphi = -\frac{\rho}{2k}(X\,dx + Y\,dy) \quad (\alpha - \text{family}), \tag{2.6}$$

$$\frac{dy}{dx} = \operatorname{tg}\left(\varphi - \frac{\pi}{4}\right), \quad d\chi - d\varphi = -\frac{\rho}{2k}(X\,dx + Y\,dy) \quad (\beta - \text{family}). \tag{2.7}$$

The characteristics form angles $\pm\pi/4$ with the principal directions – like in absence of body forces. Therefore also in this case they coincide with slip-lines. The only difference between these two cases consists in the additional term on the right-hand side of the equations $(2.6)_2$ and $(2.7)_2$. This term reduces to zero when body forces X and Y vanish.

Consider now the special cases. Assume at first that the body forces are caused by gravity acting in the negative direction of y-axis. We have then $X = 0$, $Y = -g = -9.81\ \text{m/s}^2$, and the equations $(2.6)_2$ and $(2.7)_2$ become

$$\begin{aligned} d\chi + d\varphi &= \frac{\rho g}{2k}\,dy \qquad (\alpha - \text{family}), \\ d\chi - d\varphi &= \frac{\rho g}{2k}\,dy \qquad (\beta - \text{family}). \end{aligned} \tag{2.8}$$

They are identical (except for a possibly different notation) with the equations derived for the case of cohesive granular media in the mechanics of soils [1].

As the second special case, consider thin sheet of constant thickness whose center plane is Oxy. If the principal stresses σ_1 and σ_2 are of opposite signs, then for Tresca yield condition the formula (2.1) holds valid and therefore equations (2.6) and (2.7) may be used also for the state of plane stress. Suppose that the sheet rotates with the angular velocity ω about the y-coordinate axis. Neglecting the variation of the body force across the thickness of the sheet, we have in this case $X = \omega^2 x$, $Y = 0$ and then

$$(2.9)\qquad \begin{aligned} d\chi + d\varphi &= -\frac{\rho}{2k}\omega^2 x\, dx \qquad (\alpha-\text{family}),\\ d\chi - d\varphi &= -\frac{\rho}{2k}\omega^2 x\, dx \qquad (\beta-\text{family}). \end{aligned}$$

At the end, consider a plane strain case of a body rotating about the axis perpendicular to the plane Oxy intersecting this plane at the center of coordinate system O. The components of body force are now $X = \omega^2 x$ and $Y = \omega^2 y$. Equations $(2.6)_2$ and $(2.7)_2$ become

$$(2.10)\qquad \begin{aligned} d\chi + d\varphi &= -\frac{\rho}{2k}\omega^2 (x\, dx + y\, dy) \qquad (\alpha-\text{family}),\\ d\chi - d\varphi &= -\frac{\rho}{2k}\omega^2 (x\, dx + y\, dy) \qquad (\beta-\text{family}). \end{aligned}$$

Numerical integration of a boundary value problem based on equations (2.6) and (2.7) consists in replacing increments by finite differences, and in solving the resulting systems of algebraic equations step by step. This is the standard procedure described in textbooks and monographs dealing with the mechanics of plastic flow, so it will not be described here. The presence of body forces dependent on the position of a point causes that, in general, the system of finite difference equations for each point must be solved iteratively.

3. Example of plastic design - plane rotating element

Consider a strip subjected to tension under plane strain conditions, loaded additionally by body forces caused by rotation about the origin of coordinates O (Fig. 1). Thus the equations (2.10) hold. The strip may be, for example, a part of a rotor. Suppose that along the section AA' perpendicular to the x-coordinate axis, the uniformly distributed tensile stresses of the magnitude $2k$ are applied (Tresca yield condition is assumed). The safe profile of the strip under these loads is to be designed.

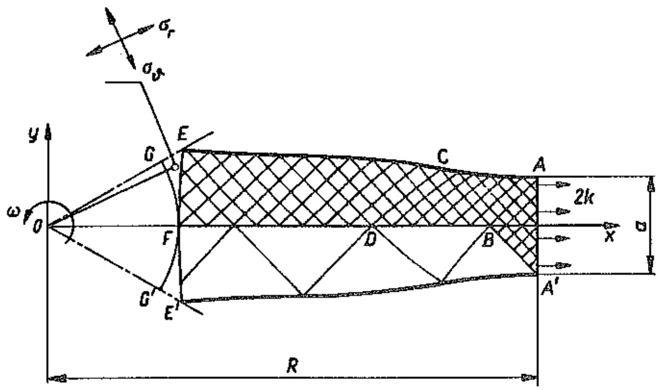

FIG. 1. Safe profile of a strip loaded by tension and body forces caused by high speed rotation.

The problem is defined by two parameters: relative width of the strip a/R (a – length of AA', R – distance between AA' and the point of rotation), which determines the value of external load, and the parameter $\eta = \rho\omega^2R^2/2k$ determining the intensity of body forces. The solution presented in Fig.1 has been obtained for particular values of these parameters $a/R = 0.2$ and $\eta = 1$. This value of η corresponds e.g. to the situation when an element made of steel ($\rho = 7.85$ Mg/m^3) of the yield point $2k =$ 345 MPa loaded by tension at the distance of $R = 1$ m from the axis rotates at the speed of 2000 rpm.

The solution begins at the loaded boundary AA'. The stress field in the region $AA'B$ is found by solving the Cauchy boundary value problem based on AA'. Next, on the basis of the distribution of stresses along the slip-line AB just found, the so-called inverse Cauchy problem is solved, determining the stress field in the region ABC and generating the a priori unknown stress-free boundary AC. The stress-free boundary is defined by equations

$$\frac{dy}{dx} = \mathrm{tg}\varphi, \qquad \chi = 0.5 \quad (\text{i.e. } \sigma_1 = 2k). \tag{3.1}$$

Ox is the axis of symmetry of the strip, so the angle φ must be equal to 0 at every point of Ox. The stress field in the region BCD is therefore found as a solution of the mixed boundary value problem based on BC and BD.

Then the sequence of solving of these two (inverse Cauchy and the mixed one) boundary value problems is repeated as many times as required. In the present example three times were enough for the free profile AE to reach the axis OE.

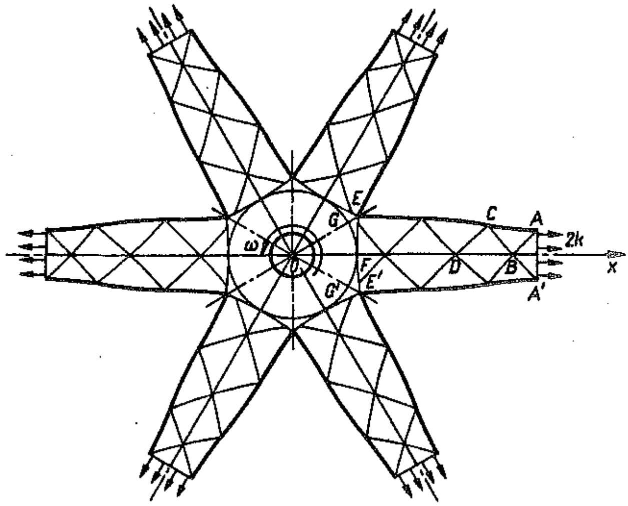

FIG. 2. Design of a six-arm rotor.

Figure 2 shows the possibility of application of the above solution to the plastic design of a rotor. Assume that the rotor consists of six identical elements whose axes form angles of $\pi/3$. Thus the axis OE that confines the single strip has the inclination of $\pi/6$ with respect to Ox. Equilibrium between the strips requires the principal directions along OE to be respectively parallel and perpendicular to its direction. The simple way to assure this is to assume axially symmetric state of stress in the region $EFOG$ (Fig.1), i.e. that principal directions at every point coincide with the directions of the polar coordinate system with the centre at the point O and principal stresses σ_r and σ_ϑ depend on the radius r only. The equilibrium equation is

$$\frac{d\sigma_r}{dr} + \frac{1}{r}(\sigma_r - \sigma_\vartheta) + \rho\omega^2 r = 0. \tag{3.2}$$

Curve EF separating this region from the slip-line field is the curve of stress discontinuity. Two equilibrium conditions at every point of EF together with Eq.(3.2) determine the actual orientation of the curve and both unknown stresses σ_r and σ_ϑ at such a point. The process of numerical integration has been performed beginning at the point E. The starting value of σ_r at this point must be assumed. In the present problem the assumption of $\sigma_r = 2.018\,k$ at E results in statically admissible stress field within the region EFG. While moving along EF, the radial stresses increase gradually up to $2.046\,k$ at the point G. The circumferential stresses rise at the beginning from $1.964\,k$ to $2.063\,k$ and then decrease to $0.046\,k$. At any point of the curve the magnitude of the difference $\sigma_r - \sigma_\vartheta$ does not exceed $2k$. Curve EF intersects x-axis at the distance of $0.261\,R$ to the point O.

The stress field within the circular section GOG' can be assumed in several ways. The most economical solution is obtained by assuming $\sigma_\vartheta - \sigma_r = 2k$. Integrating equation (3.2) with the initial condition $\sigma_r = 2.046\,k$ for $r = 0.261\,R$, one obtains

$$\sigma_r = 2k[\ln(r/R) - (r/R)^2\eta/2 + 2.397)].$$

For $r/R = 0.091$ the radial stress vanish. Therefore, this can be the relative radius of a circular hole around the center.

References

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