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ON AN INCORRECT CONSISTENT MASS MATRIX FORMULATION IN THE DYNAMICS OF TRUSSES

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In the recent bibliography of FEM applications in structural dynamics one can find an erroneous formulation of consistent mass matrix for truss-type element. The aim of the paper is to correct this formulation, to discuss the errors and to present the accurate values of eigenfrequencies computed for some selected types of plane trusses.

1. Introduction

The dynamics of trusses in FEM formulation seems to be the domain not completely verified. The legitimacy of such a statement may be confirmed by the fact that in the last decade's bibliography one can find entries [1, 2] containing erroneous numerical examples illustrating the application of incorrect consistent mass matrix formulation.

Such a state of affairs is caused undoubtedly by some kind of singularity in mechanical properties of the truss-type finite element with linear deformation state approximation. This singularity lies mainly in the fact that the stiffness matrix of such a finite element considered in its local coordinate system, may be represented by the 2×2 matrix. Its correct extension in dimensions to 4×4 [1] or 6×6 [2] contains only two non-zero rows and columns. Suggestion of the apparent analogy is here so strong that the procedure of extension is wrongly applied to the mass matrix. It causes, in consequence, complete neglecting of the inertia forces related to transverse motion of the finite element. This leads to very large errors in the computed eigenfrequencies of trusses. On the other hand, verification of numerical results even for the simplest kinds of trusses encounters the basic difficulty: in attainable bibliography there is an absolute lack of adequate and correct numerical examples.

The aim of the paper is mainly the correction of consistent mass matrix formulation for truss-type finite element. The results of the correction and the errors arising in consequence of using incorrect formulation are shown in Table 1. The correct eigenfrequencies computed for some selected types of trusses are listed in Table 2 to fill the gap observed in bibliography.

2. Correct consistent mass matrix

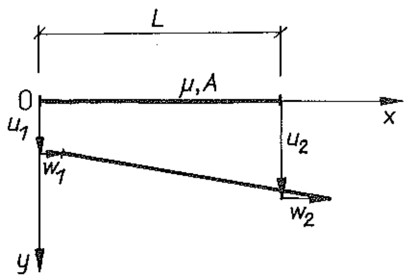

Fig. 1.

Linear displacement state approximation (Fig. 1) for truss-type finite element is represented in its local coordinate system $0xy$ by the formula

$$\mathbf{v}(x) = \begin{bmatrix} u(x) \\ w(x) \end{bmatrix} = \begin{bmatrix} N_1(x) & 0 & N_2(x) & 0 \\ 0 & N_1(x) & 0 & N_2(x) \end{bmatrix} \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \end{bmatrix} = \mathbf{N}(x)\mathbf{q} \,,$$

where

$$N_1(x) = 1 - x/l = 1 - \xi \,, \qquad N_2(x) = x/l = \xi \,.$$

Hence, using the general formula [3] for consistent mass matrix, we obtain in the local coordinate system

$$\mathbf{M} = \int_0^l \mu \mathbf{N}^T(x) \cdot \mathbf{N}(x) dx = \frac{\mu l}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} . \tag{2.1}$$

This form of the mass matrix does not change in the effect of transformation to the global coordinate system

$$\bar{\mathbf{M}} = \mathbf{T}^T \cdot \mathbf{M} \cdot \mathbf{T} = \mathbf{M} \,,$$

where

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_0 \end{bmatrix}, \qquad \mathbf{T}_0 = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

Matrix $\mathbf{M}$ has therefore the invariant property on account of the transformation $\mathbf{T}$. This result agrees with the data having mechanical nature.

3. Numerical examples

In order to show and compare the results calculated with the use of mass matrices having two different forms let us consider first the two-bar truss (Fig. 2) adopted from [2]. In this paper the mass matrix was applied in its incorrect form

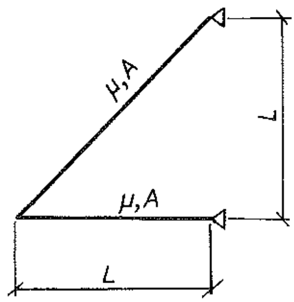

FIG. 2.

$$\mathbf{M} = \frac{\mu l}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the frequency equation was

$$\det\left\{ \begin{bmatrix} 1+2\sqrt{2} & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 2+\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{bmatrix} \right\} = 0, \qquad \lambda = \frac{\sqrt{2}}{3}\frac{\mu\omega^2 l^2}{EA},$$

having the roots

$$\omega_1 = 1.2247\sqrt{EA/(\mu l^2)}, \qquad \omega_2 = 1.7321\sqrt{EA/(\mu l^2)}.$$

Application of the correct mass matrix (2.1) leads to equation

$$\det\left\{ \begin{bmatrix} 1+2\sqrt{2} & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1+\sqrt{2} \end{bmatrix} \right\} = 0, \qquad \lambda = \frac{2\sqrt{2}}{3}\frac{\mu\omega^2 l^2}{EA},$$

with the roots

$$\omega_1 = 0.54745\sqrt{EA/(\mu l^2)}\,, \qquad \omega_2 = 1.3497\sqrt{EA/(\mu l^2)}\,.$$

Eigenfrequencies obtained when incorrect mass matrix is applied have considerably higher values than those in the correct case. The errors have the corresponding values: $\Delta\omega_1 = 124\%$, $\Delta\omega_2 = 28.3\%$.

Further analysis of errors calculated according to the formula

$$\Delta p_i = 100 \times (p_i^e - p_i)/p_i$$

was performed for An-type trusses shown in Fig.3. The results of analysis are listed in Table 1.

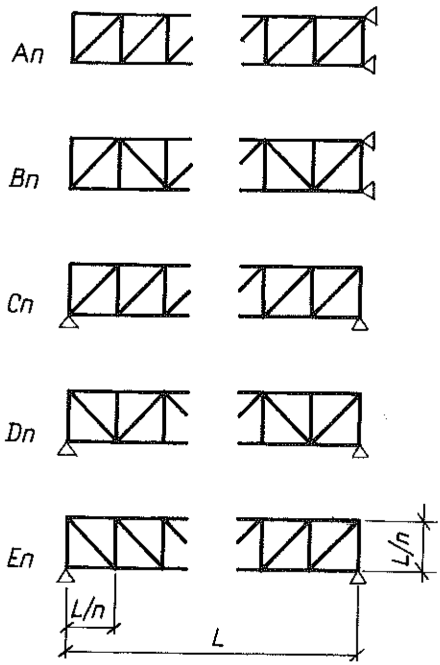

FIG. 3.

Table 2 contains specification of the exact eigenfrequencies computed for trusses shown in Fig. 3 having identical tension stiffnesses EA and mass intensities μ of all their bars. Trusses are vibrating around the static position of equilibrium with zero axial forces.

The results obtained for such "theoretical" trusses are of a mainly theoretical meaning, but may be essentially useful since they set up a kind of exact "reference point" for testing and verification the computer codes for dynamics of trusses.

Table 1.

Truss type	i	Correct model $p_i(An)$	Erroneous model $p_i(An)$	Error $\Delta p_i\%$
	1	0.33633	0.46446	38.1
	2	1.0488	1.5449	47.3
$n = 1$	3	1.3050	1.7321	32.7
	4	1.7672	3.0543	72.8
	1	0.29918	0.44285	48.0
	2	0.95830	1.3250	38.3
	3	1.0923	1.4812	35.6
	4	1.7333	2.1906	26.4
$n = 2$	5	2.7746	3.8903	40.2
	6	3.2011	5.4565	70.5
	7	3.4850	5.9845	71.7
	8	3.6868	6.3866	73.2
	1	0.22126	0.33642	52.1
	2	0.81736	1.2274	50.2
	3	1.0868	1.4520	33.6
	4	1.7375	2.5106	44.5
	5	2.4596	3.2350	31.5
	6	2.7956	3.6544	30.7
	7	3.4137	4.6138	35.2
$n = 4$	8	4.2080	5.4379	29.2
	9	5.1608	6.8731	33.2
	10	5.3828	7.6241	41.6
	11	5.7336	9.2004	60.5
	12	6.1491	11.131	81.0
	13	6.3151	11.709	85.4
	14	7.0583	12.317	74.5
	15	7.4654	12.743	70.7
	16	8.0367	13.406	66.8
$\omega_i = p_i\sqrt{EA/(\mu l^2)}$				

Table 2.

Truss type	i	$p_i(An)$	$p_i(Bn)$	$p_i(Cn)$	$p_i(Dn)$	$p_i(En)$
	1	0.29918	0.31290	0.62157	0.60549	0.60549
	2	0.95830	0.84942	0.82622	0.63950	0.63950
	3	1.0923	1.2069	2.0288	1.9040	1.9040
	4	1.7333	1.8376	2.1368	1.9466	1.9466
$n=2$	5	2.7746	2.7701	2.3732	2.4799	2.4799
	6	3.2011	3.2536	2.6779	2.7065	2.7065
	7	3.4850	3.5345	3.1896	2.9633	2.9633
	8	3.6868	3.6621	3.7727	3.9669	3.9669
	1	0.22126	0.22920	0.60751	0.59085	0.55455
	2	0.81736	0.85161	1.1697	1.0876	1.0043
	3	1.0868	1.0323	1.4865	1.3973	1.4202
	4	1.7375	1.7618	2.3519	2.1321	2.1201
	5	2.4596	2.4973	2.4812	2.6096	2.6800
$n=4$	6	2.7956	3.0547	3.1703	3.1186	3.2361
	7	3.4137	3.5002	4.3155	4.2288	4.1640
	8	4.2080	4.4367	4.3565	4.2330	4.2417
	9	5.1608	5.4804	4.7907	4.8382	4.6449
	10	5.3828	5.5456	5.0252	5.2447	5.0158
	1	0.13364	0.13552	0.43252	0.43444	0.39911
	2	0.63188	0.66576	1.0593	1.1003	1.0683
	3	1.0668	1.0397	1.6472	1.5456	1.4132
	4	1.4266	1.5059	1.9873	2.0199	1.9591
	5	2.2852	2.3800	2.8687	2.8967	2.8894
$n=8$	6	3.0827	3.0106	3.2093	2.9359	2.9911
	7	3.2829	3.2729	3.8583	3.7514	3.7415
	8	4.0774	3.9496	4.3808	4.0420	4.7097
	9	4.8566	4.9320	4.7873	5.1334	4.8149
	10	5.1968	5.4992	5.4958	5.4216	5.2981

4. Conclusion

Consistent mass matrix formulation proposed in papers [1, 2] is completely unverified. It leads to inadmissibly large errors in eigenfrequencies which result from neglecting the inertia forces acting transversely to the bars. Verification of the computed eigenfrequencies, even for the simplest trusses, is still now practically nearly impossible. This is due to the lack of adequate and verified examples in the bibliography.

References

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Streszczenie

O BŁĘDNEJ POSTACI KONSYSTENTNEJ MACIERZY MAS W DYNAMICE KRATOWNIC

Opisywane w literaturze zastosowania MES w dynamice układów kratowych okazują się nie zawsze merytorycznie poprawne. W niniejszej pracy zajęto się sprostowaniem błędu polegającego na wykorzystaniu w obliczeniach częstości drgań własnych niepoprawnej postaci konsystentnej macierzy mas. Dla wybranych typów kratownic płaskich podano zestawienia błędów powstających przy takich obliczeniach. Podano również wyniki poprawne uzyskane dzięki zastosowaniu macierzy mas o właściwej postaci.

Резюме

ОБ ОШИБОЧНОЙ ФОРМЕ КОНСИСТЕНТНОЙ МАТРИЦЫ МАСС В ДИНАМИКЕ ФЕРМ

В литературе последнего десятилетя из области приложений МКЭ в динамике сооружений можно обнаружить некоторую ошибочную формулировку касаюшыюся формы консистентной матрицы масс фермового элемента. Целю настоящей статии есть корректировка ошибочного положения, дискуссия ошибок ипредставление правильных результатов вычислений для некоторых типов плоских ферм.

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