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PLASTIC DESIGN OF A PRESSURE VESSEL

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The limit loads are given for various collapse mechanisms of a pressure vessel consisting of a cylindrical shell and hemispherical drumheads. Boundaries of different collapse mechanisms are determined in a parameter space encompassing the applicable range of vessel geometries. Limit loads corresponding to the neighboring collapse mechanisms remain continuous across these boundaries. These results shed some light on the plastic design and failure assessment in pressure vessel technology.

1. Introduction

A cylindrical vessel closed by hemispherical caps represents one of the structural configurations frequently encountered in engineering. Nevertheless, an analytical solution covering all applicable parameter ranges has not been achieved as yet. Satisfaction of the continuity conditions between the spherical and cylindrical portions of the structure, as well as the complications introduced by four geometric parameters, make an exact analysis of the problem difficult. Such a situation gives a strong impetus to the development of a simplified plastic design methodology based on limit analysis. In plastic limit analysis, an appropriate complete solution (standing for the solution which is identically assessed from both the lower and upper bound approaches) is frequently needed to provide consistent data on the plastic design of composite vessel structures.

Certain progress has been achieved in the present paper on the basis of the results given by the authors in Ref. [1]. These recent results make possible a convenient application of plastic design methodology in the aforementioned type of pressure vessels. By means of a simplified version of the two-moment limited-interaction yield condition, complete solutions are found for the whole applicable range of cylinder-hemispherical caps geometries. Information on the boundaries between adjacent collapse mechanisms are included. All results are written in analytical form and illustrated by graphs; they shed some light on the plastic design and failure mode assessments in pressure vessel technology.

2. Collapse mechanisms and yield conditions

Four geometric parameters, i.e., thickness of hemispherical cap, T; common radius of cylindrical and hemispherical shells, R; cylinder thickness, t; and cylinder length, L, characterize the present vessel-drumhead geometry. They can be replaced by three independent dimensionless parameters, such as $K = t/T$, T/R and L/R. The vessel is subjected to internal pressure P_0. The dimensionless limit load is written as

$$p = P_0 R/N_0, \tag{2.1}$$

where ultimate tensile force N_0 represents $\sigma_0 T$ with σ_0 denoting the yield stress. According to the geometric and loading characteristics, the collapse modes can be classified into seven cases, as shown in Fig. 1 to Fig. 8.

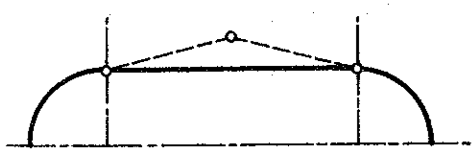

Fig. 1. Collapse mode of mechanism 1.

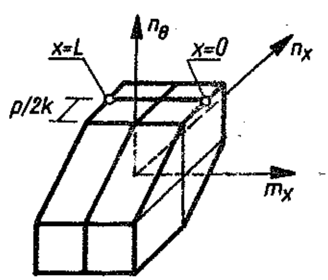

Fig. 2. Stress profile of mechanism 1.

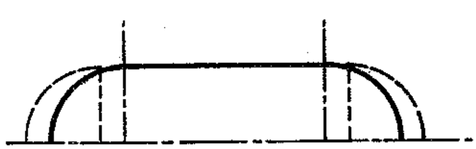

Fig. 3. Collapse mode of mechanism 2.

Only one complete solution for the above seven collapse mechanisms was obtained in Ref. [2] by using two-moment limited-interaction yield condition for the same configuration. However, it is rather difficult to obtain the remaining ones by assuming such a yield condition. In many engineering

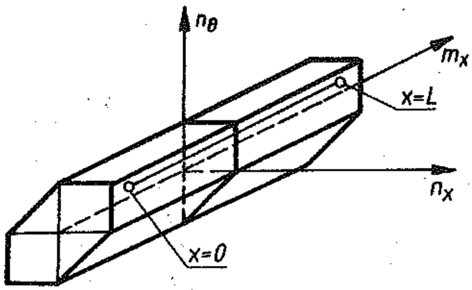

FIG. 4. Stress profile of mechanism 2.

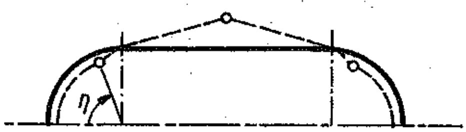

FIG. 5. Collapse mode of mechanism 3.

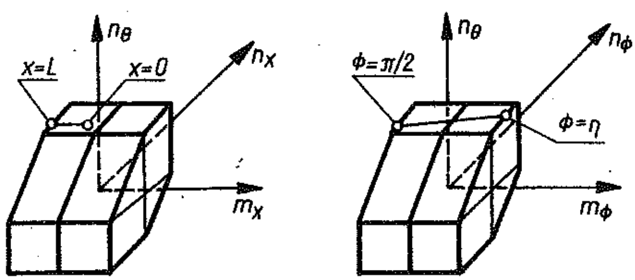

FIG. 6a. Stress profile of cylinder of mechanism 3. b. Stress profile of hemisphere of mechanism 3.

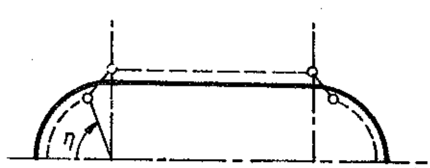

FIG. 7. Collapse mode of mechanism 4.

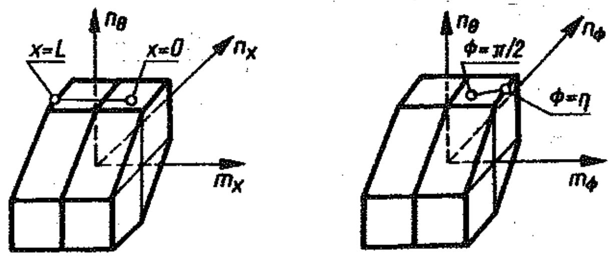

FIG. 8a. Stress profile of cylinder of mechanism 4. b. Stress profile of hemisphere of mechanism 4.

application, a simplified version of the two-moment limited-interaction yield condition (its bounding surfaces in stress resultant space are shown in Table 1) is sufficient to meet the requirements of the engineering design.

Table 1. Yield condition

Cylinder		Hemisphere	
stress equation	strain rate vector $(\dot{e}_\theta, \dot{e}_x, \dot{\varkappa}_x)$	stress equation	strain rate vector $(\dot{e}_\theta, \dot{e}_\phi, \dot{\varkappa}_\theta, \dot{\varkappa}_\phi)$
$n_\theta = 1$	$\mu(1, 0, 0)$	$n_\theta = 1$	$\mu(1, 0, 0, 0)$
$n_x = 1$	$\mu(0, 1, 0)$	$n_\phi = 1$	$\mu(0, 1, 0, 0)$
$-n_\theta + n_x = 1$	$\mu(-1, 1, 0)$	$-n_\theta + n_\phi = 1$	$\mu(-1, 1, 0, 0)$
$-n_\theta = 1$	$\mu(-1, 0, 0)$	$-n_\theta = 1$	$\mu(-1, 0, 0, 0)$
$-n_x = 1$	$\mu(0, -1, 0)$	$-n_\phi = 1$	$\mu(0, -1, 0, 0)$
$n_\theta - n_x = 1$	$\mu(1, -1, 0)$	$n_\theta - n_\phi = 1$	$\mu(1, -1, 0, 0)$
$m_x = 1$	$\mu(0, 0, 1)$	$\pm m_\theta = 1$	$\mu(0, 0, \pm 1, 0)$
$-m_x = 1$	$\mu(0, 0, -1)$	$\pm m_\phi = 1$	$\mu(0, 0, 0, \pm 1)$

3. BASIC EQUATIONS

The sign convention and the junction conditions are illustrated in Fig. 9. At the hemisphere-cylinder junction, the following relations must be sarisfied:

$$N_\phi|_{\phi=\pi/2} = N_x|_{x=0} = N_1, \quad M_\phi|_{\phi=\pi/2} = M_x|_{x=0} = M_1,$$
$$Q_\phi|_{\phi=\pi/2} = Q_x|_{x=0} = Q_1, \quad W|_{\phi=\pi/2} = W|_{x=0}, \tag{3.1}$$
$$\psi|_{\phi=\pi/2} = \psi|_{x=0} \quad \text{(in absence of plastic hinge).}$$

The equilibrium equations can be expressed in the form

$$dN_x/dx = 0, \quad dM_x/dx = Q_x, \quad dQ_x/dx = P_0 - N_\theta/R \quad \text{for} \quad 0 \leqslant x \leqslant 1,$$
$$dQ_\phi/d\phi = \frac{1}{2} P_0 R - N_\theta, \quad N_\phi = \frac{1}{2} P_0 R - Q_\phi \operatorname{ctg} \phi, \quad \text{for} \quad 0 \leqslant \phi \leqslant \frac{\pi}{2}. \tag{3.2}$$
$$d(M_\phi \sin \phi) = M_\theta \cos \phi + Q_\phi R \sin \phi.$$

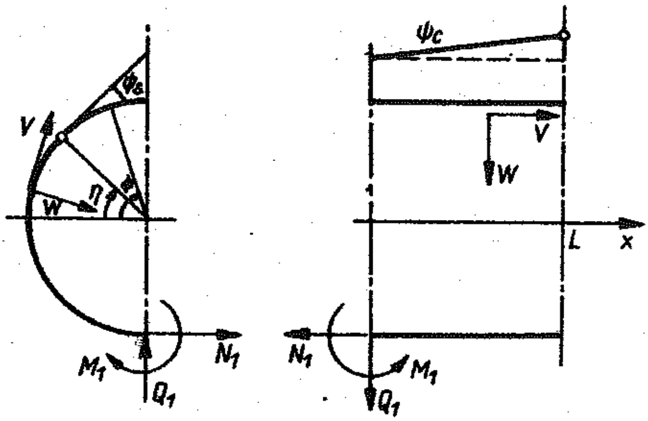

FIG. 9.

The geometrical relations are as follows:

$$
\begin{aligned}
&\dot{e}_\theta = -W/R, \qquad \dot{e}_x = V/R,\\
&\dot{\varkappa}_\theta = 0, \quad \dot{\varkappa}_x = -d^2 W/dx^2, \quad \psi = -dW/dx \quad \text{for} \quad 0 \leqslant x \leqslant 1,\\
(3.3)\qquad &\dot{e}_\theta = (V \operatorname{ctg}\phi - W)/R, \quad \dot{\varkappa}_\theta = -\operatorname{ctg}\phi\,(V + dW/d\phi)/R^2,\\
&\dot{e}_\phi = (dV/d\phi - W)/R, \quad \dot{\varkappa}_\phi = -\frac{d}{d\phi}\,(V + dW/d\phi)/R^2,\\
&\psi = -(V + dW/d\phi)/R \quad \text{for} \quad 0 \leqslant \phi \leqslant \frac{\pi}{2}.
\end{aligned}
$$

4. Limit loads and boundaries between various mechanisms

Based on the upper bound and lower bound theorems of limit analysis, the limit loads for various mechanisms can be derived [1]. They are complete solutions in the sense of satisfying both the statically and kinematically admissible conditions. A combination of different geometrical parameters corresponds to a single collapse mechanism. All the combinations corresponding to a specific collapse mechanism form a connected domain in the parameter space. These domains would fill the whole parameter space provided all the collapse mechanisms were exhausted. The inter-mechanism boundaries can be determined by statically admissible conditions. The limit loads and the boundaries for various mechanisms are discussed below.

4.1. Limit loads

Mechanism 1 (cylinder collapse mechanism)

$$
(4.1)\qquad p_1 = K\,(1 + 2/w^2), \qquad w^2 = 2L^2/KTR.
$$

Mechanism 2 (cylinder collapse mechanism)

$$p_2 = 2K. \tag{4.2}$$

Mechanism 3 (combined collapse mechanism)

$$p_3 = 2\,(\pi/2-\eta+KL/R)/(\pi/2-\eta+2L/R), \tag{4.3}$$

where the angular location of plastic hinge circle η is determined by

$$(2-K)\,[(1-\sin\eta+(\pi/2-\eta)\,(L/2R)]\,(L/R)- \\ -(1+K^2)\,(\pi/2-\eta+2L/R)\,(T/4R)=0. \tag{4.4}$$

Mechanism 4 (combined collapse mechanism)

The expression for p_4 is identical with that in Eq. (4.3), where η is determined by

$$(2-K)\,[1-\sin\eta)\,(L/R)-(\pi/2-\eta+2L/R)\,(T/2R)=0. \tag{4.5}$$

Mechanism 5 (combined collapse mechanism)

$$p_5 = (\pi/2+KL/R)/(\pi/2+2L/R). \tag{4.6}$$

Mechanism 6 (combined collapse mechanism)

$$p_6 = 2-T/R. \tag{4.7}$$

Mechanism 7 (spherical mechanism)

$$p_7 = 2. \tag{4.8}$$

4.2. Boundaries between various mechanisms

The complete solutions of the preceding seven mechanisms are valid only in certain regions of the geometrical parameter space. The boundaries between various mechanisms are obtained by examination of the statically admissible conditions, especially by satisfying the corresponding yield conditions.

By analyzing the analytical expressions and the computational results, several conclusions may be drawn.

Boundary between mechanisms 1 and 2

By equating p_1 to p_2, one arrives at

$$w^2 = 2, \tag{4.9}$$

what implies that mechanism 1 and mechanism 2 are connected along a boundary determined by continuous transition of the complete solutions from mechanism 1 to mechanism 2. When $w^2 > 2$, the complete solution for mechanism 1 is relevant, otherwise the complete solution of mechanism 2 is appropriate.

Boundary between mechanism 1 and 3

From $p_1 = p_3$ it can be obtained that

$$(4.10)\quad (2-K)(L/R)\left[1-\cos\frac{4KL/R}{2w^2-K(2+w^2)}\right]+[(2-K)(L/R)^2- -(1+K^2)(T/2R)]\frac{2KL/R}{2w^2-K(2+w^2)}-(1+K^2)(T/2R)=0.$$

The above formula can be abbreviated as $f_{13}(L/R, T/R, K) = 0$. The parameters $(L/R, T/R, K)$ satisfying Eq. (4.10) form a surface of the parameter space which describes the boundary between mechanism 1 and mechanism 3. Complete solutions of mechanism 1 and mechanism 3 correspond to geometric parameters located at the right and left-hand sides of this boundary, respectively.

Boundary between mechanisms 2 and 3

From $p_2 = p_3$ one obtains the condition

$$(4.11)\quad (2-K)\left[1-\cos\frac{KL/R}{1-K}\right](L/R)+[(2-K)(L/R)^2- -(1+K^2)(T/2R)]\frac{KL/R}{2(1-K)}-(T/2R)(1+K^2)=0$$

which may be written in a simplified form as $f_{23}(L/R, T/R, K) = 0$. On the surface $f_{23}(L/R, T/R, K) = 0$, mechanism 2 and mechanism 3 are linked up. The complete solutions of mechanism 2 and mechanism 3 can be found at the right-hand and the left-hand sides of this surface, respectively.

Boundary between mechanism 3 and mechanism 4

$$(4.12)\quad 2(2-K)(L/R)\left[1-\cos\frac{(2L/R)(K^2-1)}{(2-K)Kw^2-(K^2-1)}\right]-\left[2L/R+ +\frac{(2L/R)(K^2-1)}{(2-K)K-(K-1)}\right](T/R)=0.$$

The above formula can be written in the form $f_{34}(L/R, T/R, K) = 0$, in which the two mechanisms are linked up. The right-hand side and the left-hand side of the surface are dominated by the complete solutions relevant to mechanism 3 and mechanism 4, respectively.

Region of validity of mechanism 5 (boundary of mechanism 3)

Mechanism 5 is equivalent to the limiting case of mechanism 3 in which the angular location of plastic hinge circle η tends to zero. In the region of validity of mechanism 5, the geometric parameters $(L/R, T/R, K)$ must satisfy the following equation:

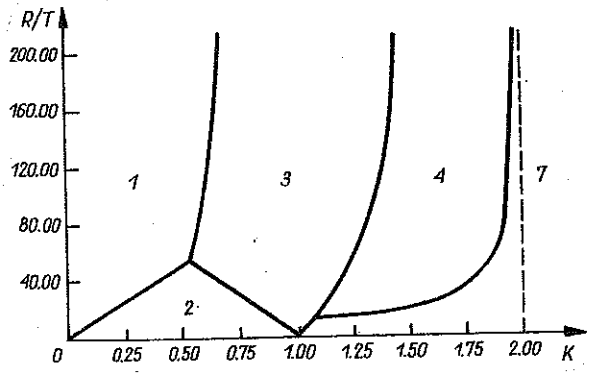

FIG. 10.

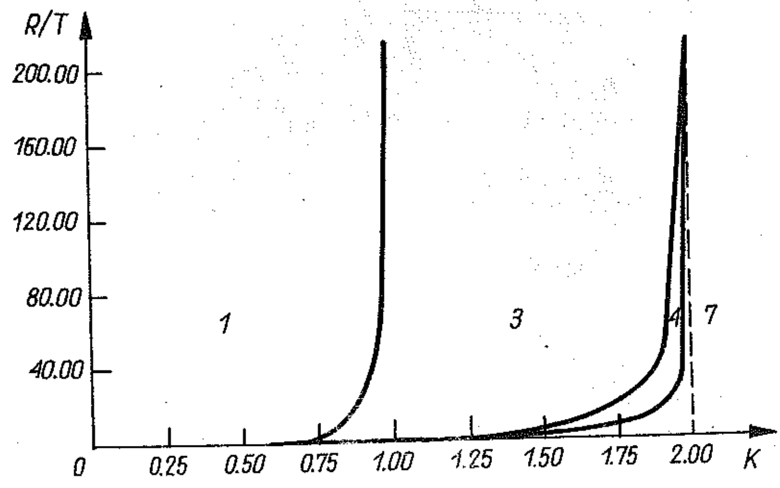

FIG. 11.

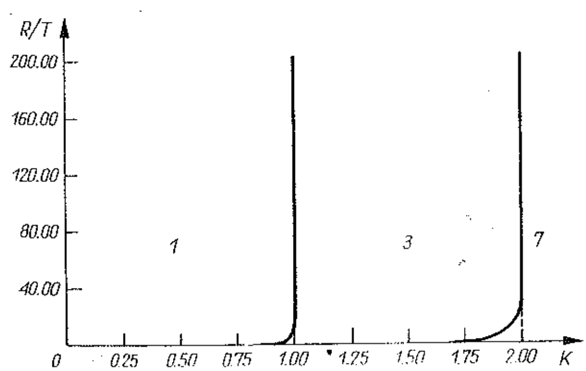

FIG. 12.

$$(2-K)\left(1+\frac{\pi L}{4R}\right)-(T/2R)(1+K^2)\left(1+\frac{\pi R}{4L}\right)=0. \tag{4.13}$$

Region of validity of mechanism 6 (boundary of mechanism 4)

Mechanism 6 is equivalent to the limiting case of mechanism 4 in which η is taken to be zero. In the region of validity of mechanism 6, the geometric parameters $(L/R, T/R, K)$ must satisfy the following equation

$$(2-K)-(T/R)(1+\pi R/4L)=0. \tag{4.14}$$

Region of validity of mechanism 7

Mechanism 7 represents the pure membrane failure mode of the hemispheres, their ragion of validity being

$$K \geqslant 2. \tag{4.15}$$

Common boundary of mechanisms 1, 2 and 3

Common boundary of these three mechanisms is described by the following two relations

$$\begin{gathered} 1-[T/R+(L/R)^2]/4-\cos[(L/R)^3/(T/R-L^3/R^3)]=0, \\ K=(L/R)^2/(T/R). \end{gathered} \tag{4.16}$$

At this common boundary, one has

$$T/R=\frac{(4L/R)(1+\pi L/4R)}{\pi/2+L/R}, \quad K=(\pi/2)/(\pi/2+L/R). \tag{4.17}$$

Common boundary of mechanisms 4, 5 and 6

At this boundary, one has

$$T/R=(2-K)/(1+\pi R/4L), \quad K=(1+\pi L/2R)^{\frac{1}{2}}. \tag{4.18}$$

Boundaries in the parameter space between different neighbouring collapse mechanisms are shown in Figs. 10—12. They have been prepared for representative cross-sections in the parameter space of $L/R=0{,}1$, 1 and 10, respectively.

5. Discussion

In engineering practice, determination of the limit load is a crucial step for designing or testing a pressure vessel. According to the formulae given in this paper, the values of the corresponding limit loads as well as the structure failure modes can be found for a set of given geometric parameters. It can be shown that p increases monotonically as K increases when the values of L/R and T/R are fixed. As K approaches 2, p tends to the membrane solution of the hemispherical shell. A gap still remeines between mechanism 6 and mechanism 7 where we failed to identify the appropriate collapse mechanism. The limit load in this unidentified region is, however, bounded by the value $p_6 = 2 - T/R$ from the left side and by $p_7 = 2$ from the right side.

According to the analytical expressions given in the present paper, a computational program is prepared providing the value of limit load. An optimal design of the vessel geometry can also be achieved by this program according to the pressure applied to the structure.

The analysis and computations presented are carried out within the range of geometrical parameters used in engineering practice. The analytical solutions and computational data provide a rational, quantitative basis for vessel design and can be included in the standards of designing or testing of the combined pressure vessel structures. Effective and accurate determination of the limit loads and the associated failure modes will certainly have a significant effect on the design and construction of composite pressure vessels, resulting in a more reasonable safety margin, economic gains and better material utilization.

References

1. Cheng Li, Xu Bing-Ye, Hwang Kehchih, Acta Mech. Sinica, 2, 135—150, 1985.
2. Xu Bing-Ye, Cheng Li, Arch. Mech., 36, 5—6, 771—778, 1984.

Streszczenie

NOŚNOŚĆ GRANICZNA NACZYŃ CIŚNIENIOWYCH

Podano wartości obciążeń granicznych dla różnych mechanizmów zniszczenia zbiornika ciśnieniowego złożonego z powłoki walcowej zamkniętej powłokami półsferycznymi. Określono granice różnych mechanizmów zniszczenia w przestrzeni parametrów zawierającej zakres stosowalnych geometrii zbiornika. Wartości obciążeń granicznych na tych granicach zmieniają się w sposób ciągły. Wyniki analizy rzucają pewne światło na analizę plastyczną i określenie warunków zniszczenia w konstrukcji zbiorników ciśnieniowych.

Резюме

ПРЕДЕЛЬНАЯ НЕСУЩАЯ СПОСОБНОСТЬ НАПОРНЫХ РЕЗЕРВУАРОВ

В работе даются значения предельных нагрузок в случае различных механизмов разрушения цилиндрического, закрытого полусферическими оболочками, напорного резер-

вуара. Определены пределы для различных механизмов разрушения, в таком диапазоне параметров. что учтена геометрия всех практически используемых разновидностей резервуаров. Значения предельных нагрузок в этих предельных случаях изменяются непрерывным образом. Результаты анализа разъясняют проблемы, связанные с пластическим анализом и задачей определения условий разрушения, в строении напорных резервуаров.
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