ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSACTIONS • 35, 4, 721–727, 1987
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

ENERGY RELEASE RATES IN FRACTURE OF DISSIPATIVE MATERIALS(*)

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Expressions for the energy release rates in the presence of a propagating crack for a dissipative material are given. The relation between these energy release rates and J-type integrals is shown. This relation is used to calculate the global dissipation during the fracture process.

Notations

$\mathbf{x} = \chi(\mathbf{X}, t)$	the current position vector in the motion χ,
$\mathbf{S} = \mathbf{S}(\mathbf{X}, t)$	the nonsymmetric Piola–Kirchhoff stress tensor,
$\mathbf{u} = \chi(\mathbf{X}, t) - \mathbf{X}$	the displacement vector of $\mathbf{X}$ at the moment t,
$\mathbf{v} = \dot{\mathbf{u}}$	the velocity field,
$\mathbf{F}(\mathbf{X}, t) = [\mathrm{Grad}\,\chi]^T$	the deformation gradient,
$\theta = \theta(\mathbf{X}, t), \theta > 0$	the absolute temperature,
$\mathbf{Q} = \mathbf{Q}(\mathbf{X}, t)$	the heat flux vector,
$\varepsilon = \varepsilon(\mathbf{X}, t)$	the specific internal energy per unit mass,
$\eta = \eta(\mathbf{X}, t)$	the specific entropy per unit mass,
$p = p(\mathbf{X}, t)$	the heat supply per unit mass,
$\delta = \delta(\mathbf{X}, t)$	the entropy production per unit mass,
$\varrho_0 = \varrho_0(\mathbf{X}, t)$	the mass density,
$k = k(\mathbf{X}, t)$	the kinetic energy per unit mass.

1. Introduction

Analysis based on energy considerations plays an important role in fracture mechanics, both in static and dynamic situations. A remarkable result of this approach was the introduction of the energy release rate at the crack tip and the derivation of the relation between this energy release rate and the J-integral for a nonlinear elastic material [1].

(*) The paper has been presented at the Euromech 210 Colloquium on Postcritical Behaviour and Fracture of Dissipative Solids, Jabłonna, 19—21 June, 1986.

The purpose of this paper is to obtain some useful expressions for the energy release rates in a dissipative material in the presence of a propagating crack. These expressions have been obtained using the general balance laws of thermodynamics. We discuss also the relation between the J-type integrals and the energy release rates during the propagation of the crack and we give an expression for the global dissipation in terms of these J-type integrals.

2. Balance laws in dynamic crack propagation

We consider a body B identified with the region of R^2 occupied in a fixed reference configuration. Let us assume (Fig. 1) that B contains an edge and sharp crack described, for simplicity, by a nonintersecting smooth curve $\mathbf{c}(t)$, where $t \in [t_0, t_1]$.

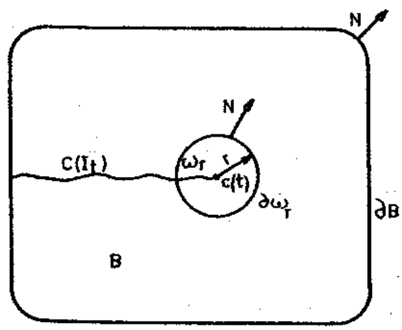

Fig. 1. The geometry of the body and the crack in the reference configuration.

Figure 1 also presents disc of radius r centered at the crack tip, denoted by ω_r, and $\partial\omega_r$, ∂B which are the boundaries of ω_r and B, respectively, in the absence of the crack, and the position of the crack tip $\mathbf{c}(t)$ at the current time t.

Let us denote by $\Sigma = \Sigma(t)$ the generalized boundary of B at time t, that is ∂B and the two faces of the crack. The boundary of the domain in which we may apply the usual continuum mechanics laws is defined by

$$\Sigma_r \cup \partial\omega_r, \quad \text{where} \quad \Sigma_r = \Sigma_r(t) = \Sigma - \omega_r.$$

The first law of thermodynamics for the material points which belong to $B-\omega_r$ has the form

$$(2.1) \qquad \int_{B-\omega_r} \varrho_0 (\dot{k}+\dot{\varepsilon})\, d\Omega = \int_{B-\omega_r} \varrho_0 (\mathbf{b}\cdot\mathbf{v}+p)\, d\Omega + \int_{\Sigma_r \cup \partial\omega_r} (\mathbf{SN}\cdot\mathbf{v}-\mathbf{Q}\cdot\mathbf{N})\, d\Sigma,$$

where $\mathbf{b}=\mathbf{b}(\mathbf{X},t)$ is the body force per unit mass and where $\varrho_0 k$, $\varrho_0 \varepsilon$, $\varrho_0 \mathbf{b}\cdot\mathbf{v}$, $\varrho_0 p$, $\mathbf{S}^T\mathbf{v}$ and $\mathbf{Q}$ satisfy the properties specified in the Appendix.

By using the law of balance of linear momentum,

$$\operatorname{Div}\mathbf{S}^T+\varrho_0\,\mathbf{b}=\varrho_0\,\dot{\mathbf{v}}, \tag{2.2}$$

Eq. (2.1) becomes

$$\int_{B-\omega_r}[\varrho_0(\dot{\varepsilon}-p)-\mathbf{S}\cdot\dot{\mathbf{F}}+\operatorname{Div}\mathbf{Q}]\,d\Omega=0. \tag{2.3}$$

The second law of thermodynamics in the local form is

$$\varrho_0\,\delta=\varrho_0\,\dot{\eta}-\varrho_0\,p/\theta+\operatorname{Div}(\mathbf{Q}/\theta)\quad\text{in } B-\omega_r, \tag{2.4}$$

where $\varrho_0\,\delta$, $\varrho_0\,\eta$, $\mathbf{Q}/\theta$ are fields which satisfy the properties given in the Appendix. From Eqs. (2.3) and (2.4) we can obtain

$$\int_{B-\omega_r}\varrho_0\,\theta\delta\,d\Omega=\int_{B-\omega_r}(\varrho_0\,\theta\dot{\eta}-\varrho_0\,\dot{\varepsilon}+\mathbf{S}\cdot\dot{\mathbf{F}}-\mathbf{Q}\cdot\operatorname{Grad}\theta/\theta)\,d\Omega, \tag{2.5}$$

and from Eq. (2.2) it follows that

$$\mathbf{S}\cdot\dot{\mathbf{F}}=\operatorname{Div}(\mathbf{S}^T\mathbf{v})-\varrho_0\,\dot{\mathbf{v}}\cdot\mathbf{v}+\varrho_0\,\mathbf{b}\cdot\mathbf{v}. \tag{2.6}$$

Then Eq. (2.5) can be written, using the divergence theorem,

$$\int_{B-\omega_r}\varrho_0\,\theta\delta\,d\Omega=-\int_{B-\omega_r}\varrho_0(\dot{\varepsilon}+\dot{k})\,d\Omega+\int_{B-\omega_r}(\varrho_0\,\theta\dot{\eta}-\mathbf{Q}\cdot\operatorname{Grad}\theta/\theta)\,d\Omega+ \\ +\int_{B-\omega_r}\varrho_0\,\mathbf{b}\cdot\mathbf{v}\,d\Omega+\int_{\Sigma_r}\mathbf{S}^T\mathbf{v}\cdot\mathbf{N}\,d\Sigma-\int_{\partial\omega_r}\mathbf{S}^T\mathbf{v}\cdot\mathbf{N}\,d\Sigma. \tag{2.7}$$

In the sequel we shall need the following classical transport theorem [2]:

$$\frac{d}{dt}\int_{B-\omega_r}\Gamma\,d\Omega=\int_{B-\omega_r}\dot{\Gamma}\,d\Omega-\int_{\partial\omega_r}\Gamma\,\dot{\mathbf{c}}\cdot\mathbf{N}\,d\Sigma. \tag{2.8}$$

Applying the theorem (2.8) to Eqs. (2.1) and (2.7), we obtain

$$\frac{d}{dt}\int_{B-\omega_r}\varrho_0(k+\varepsilon)\,d\Omega+\int_{\partial\omega_r}\varrho_0(k+\varepsilon)\,\dot{\mathbf{c}}\cdot\mathbf{N}\,d\Sigma=\int_{B-\omega_r}\varrho_0(\mathbf{b}\cdot\mathbf{v}+p)\,d\Omega+ \\ +\int_{\Sigma_r}(\mathbf{SN}\cdot\mathbf{v}-\mathbf{Q}\cdot\mathbf{N})\,d\Sigma-\int_{\partial\omega_r}(\mathbf{SN}\cdot\mathbf{v}-\mathbf{Q}\cdot\mathbf{N})\,d\Sigma, \tag{2.9}$$

$$\int_{B-\omega_r}\varrho_0\,\theta\delta\,d\Omega=-\frac{d}{dt}\int_{B-\omega_r}\varrho_0(k+\varepsilon)\,d\Omega-\int_{\partial\omega_r}\varrho_0(k+\varepsilon)\,\dot{\mathbf{c}}\cdot\mathbf{N}\,d\Sigma+ \tag{2.10}$$

(2.10) [cont.]

$$+\int_{B-\omega_r}(\varrho_0\,\theta\dot{\eta}-\mathbf{Q}\cdot\mathrm{Grad}\,\theta/\theta)\,d\Omega+\int_{B-\omega_r}\varrho_0\,\mathbf{b}\cdot\mathbf{v}\,d\Omega+$$

$$+\int_{\Sigma_r}\mathbf{S}^T\mathbf{v}\cdot\mathbf{N}\,d\Sigma-\int_{\partial\omega_r}\mathbf{S}^T\mathbf{v}\cdot\mathbf{N}\,d\Sigma.$$

Let us observe that Eq. (2.9) is a general energy balance law for a body with a propagating crack, while Eq. (2.10) gives an expression for the dissipation in the regular domain $B-\omega_r$ during the crack propagation.

3. Energy release rates during crack propagation

To take into account the influence of the crack on the form of the balance laws (2.9) and (2.10) we shall introduce the energy release rate associated with the region ω_r at the time t by

$$E=\int_{\partial\omega_r}[\varrho_0\,(k+\varepsilon)\,\dot{\mathbf{c}}\cdot\mathbf{N}+\mathbf{SN}\cdot\mathbf{v}-\mathbf{Q}\cdot\mathbf{N}]\,d\Sigma. \tag{3.1}$$

With Eq. (3.1) the laws (2.9) and (2.10) can be written, respectively,

$$\frac{d}{dt}\int_{B-\omega_r}\varrho_0\,(k+\varepsilon)\,d\Omega+E=\int_{B-\omega_r}\varrho_0\,(\mathbf{b}\cdot\mathbf{v}+p)\,d\Omega+\int_{\Sigma_r}(\mathbf{SN}\cdot\mathbf{v}-\mathbf{Q}\cdot\mathbf{N})\,d\Sigma, \tag{3.2}$$

$$\int_{B-\omega_r}\varrho_0\,\theta\delta\,d\Omega=-E-\frac{d}{dt}\int_{B-\omega_r}\varrho_0\,(k+\varepsilon)\,d\Omega+\int_{B-\omega_r}(\varrho_0\,\theta\dot{\eta}-$$

$$-\mathbf{Q}\cdot\mathrm{Grad}\,\theta/\theta)\,d\Omega+\int_{B-\omega_r}\varrho_0\,\mathbf{b}\cdot\mathbf{v}\,d\Omega+\int_{\Sigma_r}\mathbf{SN}\cdot\mathbf{v}\,d\Sigma-\int_{\partial\omega_r}\mathbf{Q}\cdot\mathbf{N}\,d\Sigma. \tag{3.3}$$

Equation (3.2) justifies why E is defined as energy release generated by the crack propagation in the region ω_r.

Besides the energy release rate E, it is necessary to consider the rate at which the energy is absorbed at the crack tip as

$$E_c=\lim_{r\to 0}\int_{\partial w_r}[\varrho_0\,(k+\varepsilon)\,\dot{\mathbf{c}}\cdot\mathbf{N}+\mathbf{SN}\cdot\mathbf{v}-\mathbf{Q}\cdot\mathbf{N}]\,d\Sigma. \tag{3.4}$$

Taking into account the properties, given in the Appendix, of the fracture fields appearing in Eq. (3.2) we obtain

$$E_c=-\frac{d}{dt}\int_{B}\varrho_0\,(k+\varepsilon)\,d\Omega+\int_{B}\varrho_0\,(\mathbf{b}\cdot\mathbf{v}+p)\,d\Omega+\int_{\Sigma}(\mathbf{SN}\cdot\mathbf{v}-\mathbf{Q}\cdot\mathbf{N})\,d\Sigma. \tag{3.5}$$

REMARK 1. From Eq. (3.4) it is clear that in the right hand side of Eq. (3.5) instead of B and Σ we may take any fixed region D and $(\Sigma \cap D) \cup \partial D$ respectively, where D surrounds the tip at every time $t \in [t_0, t_1]$.

REMARK 2. Equation (3.5) generalizes the dynamic energy release rate defined by GURTIN and YATOMI [3] in the case of elasto-dynamic crack propagation, where $\mathbf{b}$, p and $\mathbf{Q}$ are neglected.

REMARK 3. If a process zone is considered within the region ω_r and the velocity of $\partial\omega_r$ is equal to $\dot{\mathbf{c}}$, then from Eq. (3.1) it results that E is just the energy release rate given by AOKI, KISHIMOTO and SAKATA [4] in the case of infinitesimal strains.

4. J-TYPE INTEGRALS

For a nonintersecting path γ which begins and ends on the crack and surrounds the tip, we define the $\tilde{\mathbf{J}}$-integral and $\tilde{\mathbf{J}}_c$-integral as the following vectors respectively:

$$\tilde{\mathbf{J}} = \int_{\gamma} [\varrho_0 (k+\varepsilon)\,\mathbf{N} - (\operatorname{Grad} \mathbf{u})^T \mathbf{S}\mathbf{N}]\, d\Sigma, \tag{4.1}$$

$$\tilde{\mathbf{J}}_c = \lim_{r\to 0} \int_{\partial\omega_r} [\varrho_0 (k+\varepsilon)\,\mathbf{N} - (\operatorname{Grad} \mathbf{u})^T \mathbf{S}\mathbf{N}]\, d\Sigma. \tag{4.2}$$

To obtain the relation between $\tilde{\mathbf{J}}_c$ and E_c, we need the following result:

$$\lim_{r\to 0} \int_{\partial\omega_r} \mathbf{S}\mathbf{N}\cdot \mathbf{v}\, d\Sigma = -\lim_{r\to 0} \dot{\mathbf{c}} \int_{\partial\omega_r} (\operatorname{Grad} \mathbf{u})^T \mathbf{S}\mathbf{N}\, d\Sigma, \tag{4.3}$$

where it is supposed that the displacement $\mathbf{u}(\mathbf{X}, t)$, taken as a function $\mathbf{g}(\mathbf{X}-\mathbf{c}(t), t)$, has the properties

$$\mathbf{V} = \left.\frac{\partial g}{\partial t}\right|_{\mathbf{X}-\mathbf{c}(t)} \quad \text{is continuous at the tip,} \tag{4.4}$$

and

$$\int_{\partial\omega_r} \mathbf{S}\mathbf{N}\, d\Sigma \to 0, \quad \int_{\partial\omega_r} |\mathbf{S}\mathbf{N}|\, d\Sigma \quad \text{is bounded as} \quad r\to 0. \tag{4.5}$$

Now Eq. (3.4) can be written as

$$E_c = \dot{\mathbf{c}}\tilde{\mathbf{J}}_c - \lim_{r\to 0} \int_{\partial\omega_r} \mathbf{Q}\cdot\mathbf{N}\, d\Sigma. \tag{4.6}$$

It is clear from Eq. (4.6) that $E_c = \dot{\mathbf{c}}\tilde{\mathbf{J}}_c$ if and only if $\lim_{r\to 0} \int_{\partial\omega_r} \mathbf{Q}\mathbf{N}\, d\Sigma = 0$.

Taking into account Eq. (4.4), the following expression can be obtained for E in terms of the $\tilde{\mathbf{J}}$-integral

$$E = \dot{\mathbf{c}}\cdot \tilde{\mathbf{J}}(\partial\omega_r) + \int_{\partial\omega_r} (\mathbf{S}\mathbf{N}\cdot\mathbf{V} - \mathbf{Q}\cdot\mathbf{N})\, d\Sigma, \tag{4.7}$$

where $\tilde{\mathbf{J}}(\partial\omega_r)$ is calculated as an integral over $\partial\omega_r$.

If we denote by D_g the global dissipation given by

$$(4.8)\qquad D_g = -\frac{d}{dt}\int_B \varrho_0 (k+\varepsilon)\, d\Omega + \int_B \left(\varrho_0\, \theta\dot{\eta} - \frac{1}{\theta}\,\mathbf{Q}\cdot \mathrm{Grad}\,\theta\right) d\Omega + \int_B \varrho_0\, \mathbf{b}\cdot\mathbf{v}\, d\Omega + \int_\Sigma \mathbf{SN}\cdot\mathbf{v}\, d\Sigma \geqslant 0,$$

then we obtain from Eqs. (3.3) and (4.6)

$$(4.9)\qquad D_g = \int_B \varrho_0\, \theta\delta\, d\Omega + \dot{\mathbf{c}}\cdot\tilde{\mathbf{J}}_c.$$

A similar form to Eq. (4.9) was obtained by Q. S. NGUYEN [6] by a different approach and for infinitesimal strains.

APPENDIX

In the following we shall specify the properties of the fields used throughout the paper (see [7]).

The field $\Gamma = \Gamma(\mathbf{X}, t)$ defined in $\{B - \mathbf{c}(I_t)\} \times [t_0, t_1]$, where $I_t = [t_0, t]$, $\forall t \in (t_0, t_1]$, is called a C^n fracture field if the derivatives of Γ of order less or equal than n exist and are continuous away from the crack and, except at the tip, are continuous up to the crack from either side.

A fracture density is a scalar-valued C^0 fracture field Γ such that

$$\int_B \Gamma\, d\Omega = \lim_{r\to 0} \int_{B-\omega_r} \Gamma\, d\Omega.$$

Γ is called a regular fracture density if

i) Γ is a C^1 fracture field;

ii) $\int_B \Gamma\, d\Omega$ is differentiable with respect to time;

iii) $\displaystyle \frac{d}{dt}\int_B \Gamma\, d\Omega = \lim_{r\to 0} \frac{d}{dt}\int_{B-\omega_r} \Gamma\, d\Omega.$

A fracture flux is a vector-valued C^0 fracture field γ which satisfies

i) $\displaystyle \int_{c(I_t)} [\gamma]\cdot\mathbf{N}\, d\Sigma = \lim_{\substack{t'\to t\\ t'<t}} \int_{c(I_{t'})} [\gamma]\cdot\mathbf{N}\, d\Sigma;$

ii) $\displaystyle \lim_{r\to 0} \int_{\partial\omega_r} \gamma\cdot\mathbf{N}\, d\Sigma$ exists;

where $[\gamma] = \gamma^+ - \gamma^-$ is the jump of the field γ across the faces of the crack.

Throughout the paper we have assumed that the thermomechanical fields $\mathbf{u}$, ε, η, $\mathbf{S}$, θ, ϱ_0, δ, p, $\mathbf{b}$ and $\mathbf{Q}$ satisfy the following conditions:

a) $\mathbf{u}$ is a C^2 fracture field;

b) θ, $\mathbf{S}$ and $\mathbf{Q}$ are C^1 fracture fields;

c) $\varrho_0 \varepsilon$, $\varrho_0 \eta$ and $\varrho_0 k$ are regular fracture densities;

d) $\mathbf{S}^T\mathbf{v}$ and $\mathbf{Q}$ are fracture fluxes;

e) $\varrho_0 \theta\delta$, $\varrho_0 \theta\dot{\eta}$, $\varrho_0 p$, $\varrho_0 \mathbf{b}\cdot\mathbf{v}$ and $-\frac{1}{\theta}\mathbf{Q}\cdot\mathrm{Grad}\,\theta$ are fracture densities.

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Streszczenie

WSPÓŁCZYNNIK WYZWALANIA ENERGII W MATERIAŁACH DYSYPATYWNYCH

Podano wyrażenia na współczynniki wyzwalania energii dla szczeliny poruszającej się w materiale z dysypacją. Pokazano związek między tymi współczynnikami i całkami typu J i wykorzystano go do obliczania dysypacji globalnej w procesie pękania.

Резюме

КОЭФФИЦИЕНТ ОСВОБОЖДЕНИЯ ЭНЕРГИИ В ДИССИПАТИВНЫХ МАТЕРИАЛАХ

Работа содержит формулы для определения коэффициентов освобождения энергии в случае трещины движущейся в материале с диссипацией. Указана связь этих коэффициентов с интегралами типа J. Это позволило расчитать общую диссипацию в процессе разрушения.

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Received February 4, 1986.