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STRESS DISTRIBUTIONS IN AN ELASTIC SEMI-SPACE DUE TO POINT SOURCES

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Exact closed form expressions for the displacements and stresses are constructed for a linear isotropic elastic semi-space subjected to point sources applied at a finite distance h beneath its stress-free plane boundary. Point-sources considered are a single force, a double-force, a centre of rotation and a centre of dilatation. Equations of elasto-statics are solved using the potentialfunction approach of PAPKOVITCH and NEUBER [5], and explicit expressions for potentials are generated when the above forces are placed in an infinite-space. Expressions for the half-space potentials are developed from the full-space potentials by Aderogba's integro-differential formulae. Three-dimensional graphs depict variation of stresses in the interior of the elastic semi-space.

1. INTRODUCTION

Problems of variation of stresses due to point sources in the interior and on the surface of a semi-space and the interaction energy of an inclusion have been of great interest to geophysicists and metalurgists. A theorem due to MINDLIN and CHENG [2] states that: "If we know the displacements and stresses due to an inclusion undergoing a dilatational transformation in an isotropic homogeneous infinite solid, then the corresponding results when such an inclusion is embedded in a half-space with a stress-free plane boundary are directly deducible from the homogeneous infinite solid solutions by the application of suitably defined differential operators". Inspired by this theorem, ADEROGBA [3] generalized it for the case of an inclusion undergoing a general uniform transformation and expressed half-space Papkovitch potentials in terms of the full-space ones for an isotropic elastic solid under any loading conditions. SHARMA [1] dealt with the problem of point-forces in the interior of an elastic half-space employing multiple integral transforms and their inversions. A general solution of the equations of equilibrium was derived for any distribution of body forces and surface tractions.

The aim of this paper is to find general solution of elasto-static equations in a half-space. Explicit expressions for the displacements and the stresses developed due to point-sources embedded in a half-space with a stress-free boundary are obtained. The solutions are derived for the following cases of loading: i) a single horizontal force, ii) a single vertical force, iii) double force, iv) centre of rotation, and v) centre of dilatation. The method used in this work takes advantage of the generalized theorem given in [3]. In Sect. 3, for each case, we obtain the full-space solutions of the equations of equilibrium in terms of the Papkovitch potentials. In Sect. 4, the integro-differential operator formulae given in [3] are employed to generate half-space potentials from the corresponding full-space results. Further, from these potentials, the expressions for displacements and stresses inside the medium are derived.

The method presented in this work is advantageous we believe because it yields displacements inside the medium of a half-space regardless of any knowledge of the surface quantities as is the case in the integral transform method. Moreover, the difficult problem of integral transform inversion is avoided. Our expressions for the surface displacements are in agreement with the results found in [1]. In addition, we are able to obtain the displacements inside the medium in the cases of double force, centre of rotation and centre of dilatation, which are believed to be new.

Stress distributions around the point forces are depicted by various 3-*D* plots.

2. Basic equations

In a rectangular Cartesian coordinate (x, y, z) the displacement vector $\mathbf{u}$ of an isotropic elastic body in equilibrium satisfies

$$(\lambda+\mu)\nabla\nabla\mathbf{u}+\mu\nabla^2\mathbf{u}+\varrho\chi=\mathbf{0}, \tag{2.1}$$

where λ and μ are Lamé constants, and ϱ and χ are the density and body vector, respectively.

The Papkovitch and Neuber form for the displacement vector $\mathbf{u}$ is expressed as

$$2\mu\mathbf{u}=(\varkappa+1)\mathbf{\Psi}-\nabla(\Psi_0+\mathbf{r}\mathbf{\Psi}), \tag{2.2}$$

where $\mathbf{r}$ is the position vector of a point in space and $\varkappa=\lambda+3\mu/\lambda+\mu$. Equation (2.1), in terms of the unknown potentials, becomes

$$(\varkappa-1)\nabla^2\mathbf{\Psi}+2\nabla\nabla\mathbf{\Psi}-\nabla\nabla^2(\Psi_0+\mathbf{r}\mathbf{\Psi})+2\frac{\varkappa-1}{\varkappa+1}\varrho\chi=\mathbf{0}, \tag{2.3}$$

which is a coupled partial differential equation in $\mathbf{\Psi}$ and Ψ_0. Taking the curl of Eq. (2.3), we obtain

$$(\varkappa+1)\nabla^2\mathbf{\Psi}=-2\varrho\chi. \tag{2.4}$$

Substituting Eq. (2.4) into Eq. (2.3), we find

$$\nabla^2 \Psi_0 = -\mathbf{r}\nabla^2 \mathbf{\Psi}. \tag{2.5}$$

An elastostatic problem with a point source now reduces to determining the potentials $\mathbf{\Psi}$ and Ψ_0 which meet the necessary boundary, continuity and regularity conditions. For any given point source in an infinite space, Eq. (2.4), first, yields $\mathbf{\Psi}$ and then Ψ_0 is found from Eq. (2.5).

The stresses in terms of the Papkovitch Neuber potentials $\mathbf{\Psi}$ and Ψ_0 are expressed in index notation as

$$\sigma_{ij} = \frac{1}{2}(3-\varkappa)\,\delta_{ij}\,\Psi_{k,k} + \frac{1}{2}(\varkappa-1)\,(\Psi_{i,j}+\Psi_{j,i}) - \Psi_{0,ij} - x_k\,\Psi_{kij}, \tag{2.6}$$

where the usual Einstein summation convention for repeated indices is employed. Here, Latin indices assume the values 1, 2 and 3.

3. Infinite space solutions

In an infinite elastic space spanned by a three-dimensional Cartesian coordinate system (x, y, z), the concentrated body force $\chi = (X, Y, Z)$ is applied at $(0, 0, h)$. We now construct the potentials $\mathbf{\Psi}$ and Ψ_0 from Eqs. (2.4) and (2.5) for the follwing five different point-sources.

Case 1. Horizontal point-force at $0, 0, h$ *directed along x-axis*

For this case

$$\varrho X = F\delta(x)\,\delta(y)\,\delta(z-h), \qquad Y = 0, \qquad Z = 0, \tag{3.1}$$

where δ is the Dirac delta function

The solutions to Eqs. (2.4) and (2.5) are given by

$$\mathbf{\Psi}^0 = \frac{F}{2\pi(\varkappa+1)}\left(\frac{1}{R_1}, 0, 0\right), \tag{3.2}$$

$$\Psi_0^0 = 0, \tag{3.3}$$

where $R_1 = x^2+y^2+(z-h)^2$ and we have used the result

$$\frac{1}{\nabla^2}\left(-4\pi\delta(x)\,\delta(y)\,\delta(z-h)\right) = \frac{1}{R_1} \tag{3.4}$$

and the superscript 0 over the functions indicates that the solutions are for the infinite elastic space.

Case 2. Vertical point-force at $(0, 0, h)$ *directed along the z-axis*

In this case,

(3.5) $$X=0, \quad Y=0, \quad \varrho Z = F\delta(x)\,\delta(y)\,\delta(z-h).$$

Proceeding in the same manner as in Case 1, we obtain the potentials as

(3.6) $$(\Psi_1^0, \Psi_2^0, \Psi_3^0) = \frac{F}{2\pi(\varkappa+1)}\left(0, 0, \frac{1}{R_1}\right),$$

(3.7) $$\Psi_0^0 = \frac{F}{2\pi(\varkappa+1)}\left(-\frac{h}{R_1}\right).$$

Case 3. Double force with moment M at (0, 0, *h*) *about the y-axis*

In this case

(3.8) $$\varrho\chi = \left[\varepsilon F\,\delta(x)\,\delta(z-h)\,\frac{\delta(y+\varepsilon/2)-\delta(y-\varepsilon/2)}{\varepsilon}, 0, 0\right].$$

If εF remains finite and equals M while ε becomes infinitesimally small, then

(3.9) $$\varrho X = M\,\delta(x)\,\delta'(y)\,\delta(z-h), \quad Y=0, \quad Z=0.$$

The solutions for this case are given by

(3.10) $$\mathbf{\Psi}^0 = \frac{M}{2\pi(\varkappa+1)}\left(-\frac{y}{R_1^3}, 0, 0\right),$$

(3.11) $$\Psi_0^0 = 0,$$

where we have used the result

(3.12) $$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

for any continuous function $f(x)$.

Case 4. Centre of rotation at (0, 0, *h*) *about the z-axis*

In this case

(3.13) $$\begin{aligned} \varrho X &= M\,\delta(x)\,\delta'(y)\,\delta(z-h),\\ \varrho Y &= -M\,\delta'(x)\,\delta(y)\,\delta(z-h),\\ Z &= 0. \end{aligned}$$

The solutions for the potentials are found to be

(3.14) $$\mathbf{\Psi}^0 = \frac{M}{2\pi(\varkappa+1)}\left(\frac{-y}{R_1^3}, \frac{x}{R_1^3}, 0\right)$$

and

$$\Psi_0^0 = 0. \tag{3.15}$$

Case 5. Centre of dilatation at (0, 0, *h*)

In this case

$$\begin{aligned} \varrho X &= -M\, \delta'(x)\, \delta(y)\, \delta(z-h), \\ \varrho Y &= -M\, \delta(x)\, \delta'(y)\, \delta(z-h), \\ \varrho Z &= -M\, \delta(x)\, \delta(y)\, \delta'(z-h). \end{aligned} \tag{3.16}$$

The solutions constructed for this case are given by

$$\mathbf{\Psi}^0 = \frac{M}{2\pi(\varkappa+1)}\left(\frac{x}{R_1^3}, \frac{y}{R_1^3}, \frac{z-h}{R_1^3}\right) \tag{3.17}$$

and

$$\Psi_0^0 = \frac{-M}{2\pi(\varkappa+1)}\left[\frac{3}{R_1} + \frac{h(z-h)}{R_1^3}\right], \tag{3.18}$$

where we have used the result

$$x\delta'(x) = -\delta(x) \tag{3.19}$$

and the relation (3.12).

The potentials derived above correspond to various point-forces and the point-moments applied at (0, 0, *h*). Here, some of the nonvanishing Papkovitch potentials generated in the infinite space agree with the results in [4].

4. Displacements and stresses in the interior of the half-space

We now divide the infinite space into two half-spaces by introducing the plane $z = 0$ with the z-axis directed downward, and assuming that the half-space $z > 0$ contains all the point-sources at (0, 0, *h*). The problem then reduces to determining the field quantities in the half-space $z > 0$ with the stress-free plane boundary $z = 0$ and with no singularities in the half-space $z < 0$.

According to a theorem in [3], the half-space Papkovitch potentials can be generated from the full-space ones with the application of the following intergo-differential operator formulae:

$$\Psi_i = \Psi_i^0 + \bar{\Psi}_i^0 + \delta_{i3}(\varkappa-1)\frac{\partial}{\partial x_j}\int \bar{\Psi}_j^0\, dz + 2\delta_{i3}\left\{\frac{\partial}{\partial z}\bar{\Psi}_0^0 + \sum_{j=1}^{2}\left(x_j\frac{\partial}{\partial z} - z\frac{\partial}{\partial x_j}\right)\bar{\Psi}_j^0\right\} \tag{4.1}$$

and

$$(4.2)\qquad \Psi_0 = \Psi_0^0 + \bar{\Psi}_0^0 + \frac{1}{2}(\varkappa^2-1)\frac{\partial}{\partial x_j}\iint \bar{\Psi}_j^0\, dz\, dz + (\varkappa-1)\left\{\bar{\Psi}_0^0 + \sum_{j=1}^{2}\left(x_j\frac{\partial}{\partial z} - z\frac{\partial}{\partial x_j}\right)\int \bar{\Psi}_j^0\, dz\right\},$$

where the arbitrary constants of integration must be set equal to zero in order that the displacements and stresses vanish at infinity, while a bar placed over a function denotes an image quantity with respect to the plane $z=0$, i.e.,

$$\bar{\Psi}(x, y, z) = \Psi(x, y, -z).$$

Equations (4.1) and (4.2) along with Eq. (2.6) satisfy the stress free boundary conditions, viz.,

$$(4.3)\qquad \sigma_{31} = \sigma_{32} = \sigma_{33} = 0$$

when $z=0$.

Equation (2.2) can be expressed in terms of the index notation as

$$(4.4)\qquad 2\mu u_i = (\varkappa+1)\,\Psi_i - (\Psi_0 + x_j\,\Psi_j)_{,i}.$$

For displacements inside the medium, Eqs. (4.1) and (4.2) are introduced into Eq. (4.4) for each case in Sect. 3. Similarly, stresses inside the medium can be obtained from Eq. (2.6). We give here the explicit expressions for the potentials and the displacement vector components. The expressions for stresses are listed in Appendix A.

Case 1. Buried horizontal point-force at $(0, 0, h)$ *along the z-axis*

The nonvanishing infinite space Papkovitch potential is given by Eq. (3.2). Substituting Eqs. (3.2) and (3.3) into Eqs. (4.1) and (4.2), the resulting half-space potentials and displacement vector components are given by

$$(4.5)\qquad \Psi_1 = \frac{F}{2\pi(\varkappa+1)}\left[\frac{1}{R_1} + \frac{1}{R_2}\right],$$

$$(4.6)\qquad \Psi_2 = 0,$$

$$(4.7)\qquad \Psi_3 = \frac{Fx}{2\pi(\varkappa+1)}\left[\frac{\varkappa-1}{R_2(R_2+z+h)} - \frac{2h}{R_2^3}\right],$$

$$(4.8)\qquad \Psi_0 = \frac{-Fx}{2\pi(\varkappa+1)}\left[\frac{\frac{1}{2}(\varkappa-1)^2}{R_2+z+h} - \frac{(\varkappa-1)\,h}{R_2(R_2+z+h)}\right],$$

$$(4.9)\qquad 2\mu u=\frac{F}{2\pi(\varkappa+1)}\left\{\frac{\varkappa}{R_1}+\frac{1}{R_2}+\frac{\frac{1}{2}(\varkappa^2-1)}{R_2+z+h}+\frac{2zh}{R_2^3}+\right.$$
$$\left.+x^2\left[\frac{1}{R_1^3}+\frac{\varkappa}{R_2^3}-\frac{\frac{1}{2}(\varkappa^2-1)}{R_2(R_2+z+h)^2}-\frac{6zh}{R_2^5}\right]\right\},$$

$$(4.10)\qquad 2\mu v=\frac{Fxy}{2\pi(\varkappa+1)}\left[\frac{1}{R_1^3}+\frac{\varkappa}{R_2^3}-\frac{\frac{1}{2}(\varkappa^2-1)}{R_2(R_2+z+h)^2}-\frac{6zh}{R_2^5}\right],$$

$$(4.11)\qquad 2\mu w=\frac{Fx}{2\pi(\varkappa+1)}\left[\frac{\frac{1}{2}(\varkappa^2-1)}{R_2(R_2+z+h)}+\frac{z-h}{R_1^3}+\frac{\varkappa(z-h)}{R_2^3}-\frac{6zh(z+h)}{R_2^5}\right],$$

where

$$(4.12)\qquad R_2^2=x^2+y^2+(z+h)^2.$$

Case 2. Buried vertical point-force at $(0, 0, h)$ *along the z-axis*

For a vertical point-force, the potentials and the displacement vector components are found by substituting Eqs. (3.6) and (3.7) into Eqs. (4.1) and (4.2) and the resulting potentials into Eqs. (4.4). We then find

$$(4.13)\qquad \Psi_1=\Psi_2=0,$$

$$(4.14)\qquad \Psi_3=\frac{F}{2\pi(\varkappa+1)}\left[\frac{1}{R_1}+\frac{\varkappa}{R_2}+\frac{2h(z+h)}{R_2^3}\right],$$

$$(4.15)\qquad \Psi_0=\frac{-F}{2\pi(\varkappa+1)}\left[\frac{h}{R_1}+\frac{\varkappa h}{R_2}-\frac{1}{2}(\varkappa^2-1)\ln(R_2+z+h)\right],$$

$$(4.16)\qquad 2\mu u=\frac{Fx}{2\pi(\varkappa+1)}\left[\frac{z-h}{R_1^3}+\frac{\varkappa(z+h)}{R_2^3}-\frac{\frac{1}{2}(\varkappa^2-1)h}{R_2(R_2+z+h)}+\frac{6zh(z+h)}{R_2^5}\right],$$

$$(4.17)\qquad 2\mu v=\frac{Fy}{2\pi(\varkappa+1)}\left[\frac{z-h}{R_1^3}+\frac{\varkappa(z+h)}{R_2^3}-\frac{\frac{1}{2}(\varkappa^2-1)}{R_2(R_2+z+h)}+\frac{6zh(z+h)}{R_2^5}\right],$$

$$(4.18)\qquad 2\mu w=\frac{F}{2\pi(\varkappa+1)}\left[\frac{\varkappa}{R_1}+\frac{\frac{1}{2}(\varkappa^2-1)}{R_2}+\frac{(z-h)^2}{R_1}+\frac{\varkappa(z+h)^2-2hz}{R_2^3}+\right.$$
$$\left.+\frac{6hz(z+h)^2}{R_2^5}\right].$$

Case 3. Buried double force at $(0, 0, h)$ *with moment* M *along the x-axis*

The potentials and displacement vector components are constructed in a similar way to Cases 1 and 2 and are found as

$$\Psi_1=\frac{-My}{2\pi(\varkappa+1)}\left[\frac{1}{R_1^3}+\frac{1}{R_2^3}\right],$$

$$\Psi_2 = 0,$$

$$\Psi_3 = \frac{Mxy}{2\pi(\varkappa+1)}\left[\frac{2(\varkappa-1)(z+h)}{R_2(x^2+y^2)^2}+\frac{(\varkappa-1)(z+h)}{R_2^3(x^2+y^2)}+\frac{6h}{R_2^5}\right],$$

$$\Psi = \frac{-(\varkappa-1)Mxy}{2\pi(\varkappa+1)}\left[\frac{\frac{1}{2}(\varkappa+1)}{R_2(x^2+y^2)}-\frac{(\varkappa+1)R_2}{(x^2+y^2)^2}+\frac{1}{R_2^3}+\right.$$
$$\left.+\frac{2z(z+h)}{R_2(x^2+y^2)^2}+\frac{z(z+h)}{R_2^3(x^2+y^2)}\right],$$

(4.19)
$$2\mu\bar{u} = \frac{My}{2\pi(\varkappa+1)}\left\{\varkappa\left(\frac{1}{R_1^3}+\frac{1}{R_2^3}\right)+z(z+h)\left(\frac{1}{r^4R_2}+\frac{\varkappa-1}{r^2R_2^3}\right)-\right.$$
$$-\frac{6zh}{R_2^5}+x^2\left[\frac{3(\varkappa-1)}{R_2^5}+\frac{\varkappa^2-1}{2}\left(\frac{-8R_2}{r^6}+\frac{4}{r^4R_2}+\right.\right.$$
$$\left.\left.\left.+\frac{1}{r^2R_2^3}\right)-z(z+h)\left(\frac{3(2\varkappa-3)}{r^2R_2^5}+\frac{(9\varkappa+11)}{r^4R_2^3}\right)\right]\right\},$$

$$2\mu\bar{v} = -\frac{Mx}{2\pi(\varkappa+1)}\left\{\varkappa\left(\frac{1}{R_1^3}+\frac{1}{R_2^3}\right)+z(z+h)\left(\frac{1}{r^4R_2}+\frac{\varkappa-1}{r^2R_2^3}\right)-\right.$$
$$-\frac{6zh}{R_2^5}+y^2\left[\frac{3(\varkappa-1)}{R_2^5}+\frac{\varkappa^2-1}{2}\left(-\frac{8R_2}{r^6}+\frac{4}{r^4R_2}+\right.\right.$$
$$\left.\left.\left.+\frac{1}{r^2}\frac{1}{R_2^3}\right)-z(z+h)\left(\frac{3(2\varkappa-3)}{r^2R_2^5}+\frac{(9\varkappa+11)}{r^4R_2^3}\right)\right]\right\},$$

$$2\mu w = \frac{Mxy}{2\pi(\varkappa+1)}\left\{\frac{1}{2}(\varkappa^2-1)(z+h)\left[\frac{2}{R_2(x^2+y^2)^2}+\frac{1}{R_2^3(x^2+y^2)}\right]-\right.$$
$$\left.-3\left[\frac{z-h}{R_1^5}-\frac{\varkappa h}{R_2^5}\right]+\frac{30zh(z+h)}{R_2^7}\right\}.$$

Case 4. Buried centre of rotation at (0, 0, h) *about the z-axis*

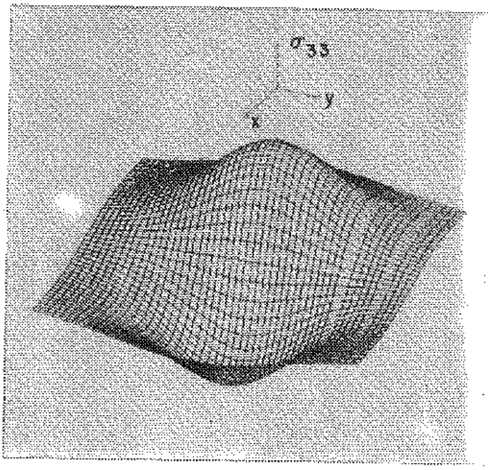
The potentials and the displacement vector components are found as

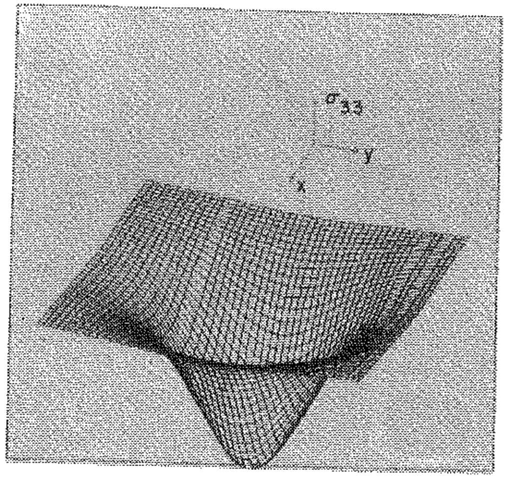
$$\Psi_1 = \frac{-My}{2\pi(\varkappa+1)}\left[\frac{1}{R_1^3}+\frac{1}{R_2^3}\right],$$

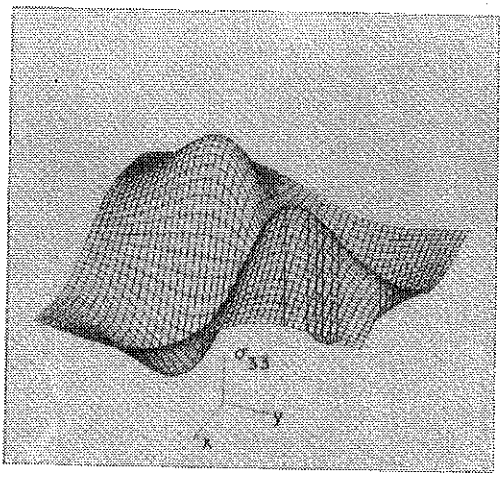
$$\Psi_2 = \frac{Mx}{2\pi(\varkappa+1)}\left[\frac{1}{R_1^3}+\frac{1}{R_2^3}\right],$$

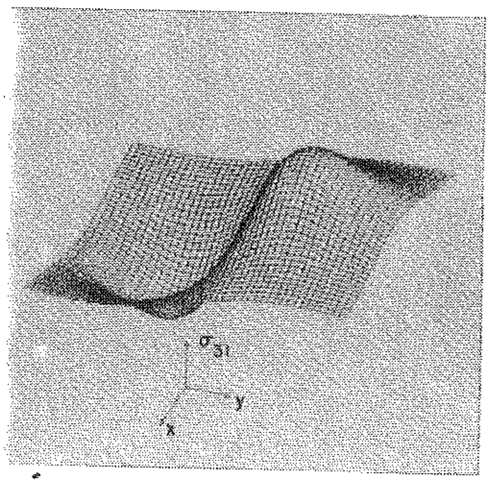
(4.20)
$$\Psi_3 = 0,$$

$$2\mu u = \frac{-My}{2\pi}\left[\frac{1}{R_1^3}+\frac{1}{R_2^3}\right],$$









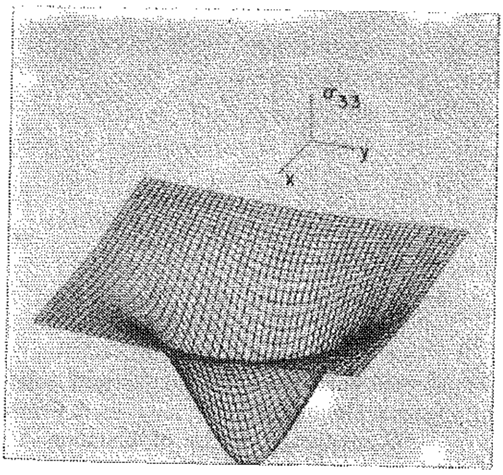
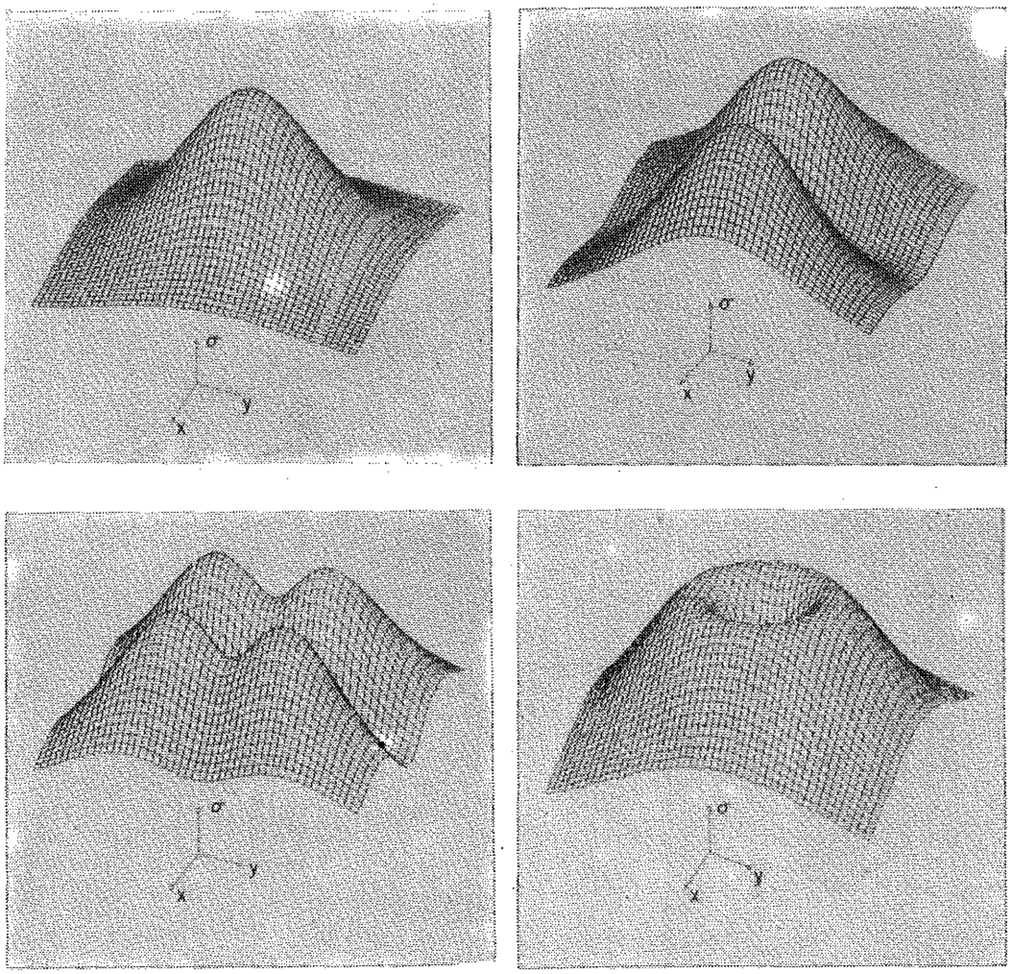


Fig. 1. Variation of stresses along the indicated direction on the $z = 22$ m plane while the point-source is applied at $h = 20$ m. a. Horizontal point-force. b. Vertical point-force. c. Double force with moment. d. Centre of rotation. e. Centre of dilatation.



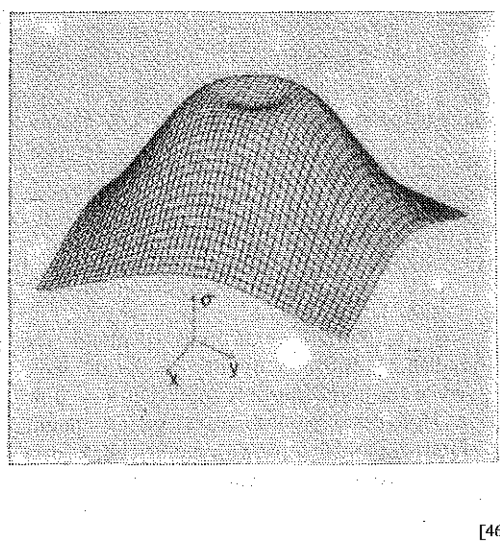


Fig. 2. Variation of total stress $\sigma = (\sigma_{31}^2 + \sigma_{32}^2 + \sigma_{33}^2)^{1/2}$ on the $z = 22$ m plane while the point-source applied at $h = 20$ m.
a. Horizontal point-force. b. Vertical point-force. c. Double force with moment. d. Centre of rotation. e. Centre of dilatation.

$$2\mu v = \frac{Mx}{2\pi}\left[\frac{1}{R_1^3}+\frac{1}{R_2^3}\right],$$

$$2\mu w = 0.$$

Case 5. Buried centre of dilatation

The potentials and the displacement vector components are given by

$$\Psi_1 = \frac{Mx}{2\pi(\varkappa+1)}\left[\frac{1}{R_1^3}+\frac{1}{R_2^3}\right],$$

$$\Psi_2 = \frac{My}{2\pi(\varkappa+1)}\left[\frac{1}{R_1^3}+\frac{1}{R_2^3}\right],$$

$$\Psi_3 = \frac{M}{2\pi(\varkappa+1)}\left[\frac{z-h}{R_1^3}-(2\varkappa-3)\frac{z+h}{R_2^3}\right],$$

(4.21) $$\Psi_0 = \frac{-M}{2\pi(\varkappa+1)}\left[\frac{3}{R_1}+\frac{h(z-h)}{R_1^3}-\frac{h(z+h)}{R_2^3}-\frac{(\varkappa-1)^2+3}{R_2}\right],$$

$$2\mu\bar{u} = \frac{Mx}{2\pi(\varkappa+1)}\left[\frac{\varkappa-1}{R_1^3}+\frac{\varkappa^2-\varkappa+6}{R_2^3}-\frac{6z(z+h)(\varkappa-1)}{R_2^5}\right],$$

$$2\mu v = \frac{My}{2\pi(\varkappa+1)}\left[\frac{\varkappa-1}{R_1^3}+\frac{\varkappa^2-\varkappa+6}{R_2^3}-\frac{6z(z+h)(\varkappa-1)}{R_2^5}\right],$$

$$2\mu w = \frac{(\varkappa-1)M}{2\pi(\varkappa+1)}\left[\frac{z-h}{R_1^3}-\frac{\varkappa(z+h)-2z}{R_2^3}-\frac{6z(z+h)^2}{R_2^5}\right].$$

The displacement vector components given by Eqs. (4.9)—(4.11) and (4.16)—(4.18) for the horizontal and the vertical point-forces are in complete agreement with the results found in [1] when they are expressed in cylindrical co-ordinates. The expressions for displacements and stresses in Cases 3, 4 and 5 are believed to be new. Surface displacements in all 5 cases can be obtained by simply letting $z=0$ in the expressions where R_1 and R_2 become equal and

(4.22) $$R_1^2 = R_2^2 = R^2 = r^2+h^2$$

and r is the radial distance in cylindrical coordinates. By a few simple algebraic manipulations on the surface displacements, it can be shown that they are in agreement with the results in [1].

When the elastic semi-space is glued to a rigid base, the displacement vector components satisfy

(4.23) $$u = v = w = 0$$

on $z=0$. For this case the integro-differential formulae (4.1) and (4.2) take simpler forms as given in [3].

The stresses and the 3-D plots of σ_{33} and the total stress $\sigma=(\sigma_{31}^2+\sigma_{32}^2+\sigma_{33}^2)^{1/2}$ for each case are given in Appendix A.

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Appendix A

The stresses for each of the five point souces found from Eq. (2.6) and by substituting the corresponding potentials are given as follows:

Case 1. Buried horizontal force at $(0,0,h)$ *along the x-axis.*

$$\text{(A.1)}\quad \sigma_{31}=\frac{-F}{2\pi(\varkappa+1)}\left\{\frac{1}{2}(\varkappa-1)(z-h)\left(\frac{1}{R_1^3}-\frac{1}{R_2^3}\right)+\frac{6zh(z+h)}{R_2^5}+ +3x^2\left[\frac{z-h}{R_1^5}+\frac{\varkappa z+h}{R_2^5}-\frac{10zh(z+h)}{R_2^7}\right]\right\},$$

$$\text{(A.2)}\quad \sigma_{32}=\frac{-3Fxy}{2\pi(\varkappa+1)}\left[\frac{z-h}{R_1^5}+\frac{\varkappa z+h}{R_2^5}-\frac{10zh(z+h)}{R_2^7}\right],$$

$$\text{(A.3)}\quad \sigma_{33}=\frac{-Fx}{2\pi(\varkappa+1)}\left\{\frac{1}{2}(\varkappa-1)\left(\frac{1}{R_2^3}-\frac{1}{R_1^3}\right)+3\left[\frac{(z-h)^2}{R_1^5}- -\frac{(z+h)^2+2z^2}{R_2^5}+(\varkappa+3)\frac{z(z+h)}{R_2^5}\right]-\frac{30zh(z+h)^2}{R_2^7}\right\}.$$

Case 2. Buried vertical force at $(0,0,h)$

$$\text{(A.4)}\quad \sigma_{31}=\frac{-Fx}{2\pi(\varkappa+1)}\left\{\frac{1}{2}(\varkappa-1)\left[\frac{1}{R_1^3}-\frac{1}{R_2^3}\right]+\frac{3(z^2-h^2)}{R_1^5}-\frac{3h(z-h)}{R_2^5}+ +\frac{3\varkappa z(z+h)}{R_2^5}-\frac{30zh(z+h)^2}{R_2^7}\right\},$$

(A.5) $$\sigma_{32} = \frac{-Fy}{2\pi(\varkappa+1)}\left\{\frac{1}{2}(\varkappa-1)\left[\frac{1}{R_1^3}-\frac{1}{R_2^3}\right]+\frac{3(z^2-h^2)}{R_1^5}-\frac{3h(z-h)}{R_2^5}+ +\frac{3\varkappa z(z+h)}{R_2^5}-\frac{30zh(z+h)^2}{R_2^7}\right\},$$

(A.6) $$\sigma_{33} = \frac{-F}{4\pi}\left[\frac{z+h}{R_2^3}+\frac{z-h}{R_1^3}\right]+\frac{F}{2\pi(\varkappa+1)}\left[\frac{z+h}{R_1^3}+\frac{\varkappa z-h}{R_2^3}- -\frac{3(z+h)(z-h)^2}{R_1^5}-\frac{3(z+h)[(z+h)(\varkappa z-h)-6hz]}{R_2^5}-\frac{30hz(z+h)^3}{R_2^7}\right].$$

Case 3. Bouried double force with moment M along the x-axis

(A.7) $$\sigma_{31} = \frac{3My}{2\pi(\varkappa+1)}\left\{\frac{1}{2}(\varkappa-1)(z-h)\left(\frac{1}{R_1^5}-\frac{1}{R_2^5}\right)+\frac{10zh(z+h)}{R_2^7}+ +5x^2\left[\frac{z-h}{R_1^7}+\frac{\varkappa z+h}{R_2^7}-\frac{14zh(z+h)}{R_2^9}\right]\right\},$$

(A.8) $$\sigma_{32} = \frac{-3Mx}{2\pi(\varkappa+1)}\left\{\frac{z-h}{R_1^5}+\frac{\varkappa z+h}{R_2^5}-\frac{10zh(z+h)}{R_2^7}-5y^2\left[\frac{z-h}{R_1^7}+ +\frac{\varkappa z+h}{R_2^7}-\frac{14zh(z+h)}{R_2^9}\right]\right\},$$

(A.9) $$\sigma_{33} = \frac{3Mxy}{2\pi(\varkappa+1)}\left\{\frac{1}{2}(\varkappa-1)\left(\frac{1}{R_2^5}-\frac{1}{R_1^5}\right)+5\left[(\varkappa+3)\frac{z(z+h)}{R_2^7}+ +\frac{(z-h)^2}{R_1^7}-\frac{(z+h)^2+2z^2}{R_2^7}-\frac{14zh(z+h)^2}{R_2^9}\right]\right\}.$$

Case 4. Buried centre of rotation

(A.10) $$\sigma_{31} = \frac{3My}{4\pi}\left[\frac{z-h}{R_1^5}+\frac{z+h}{R_2^5}\right],$$

(A.11) $$\sigma_{32} = \frac{-3Mx}{4\pi}\left[\frac{z-h}{R_1^5}+\frac{z+h}{R_2^5}\right],$$

(A.12) $$\sigma_{33} = 0.$$

Case 5. Buried centre of dilatation

(A.13) $$\sigma_{31} = \frac{3Mx}{2\pi(\varkappa+1)}\left[-(\varkappa-1)\frac{(z-h)}{R_1^5}-2(\varkappa-1)\frac{z}{R_2^5}+10\varkappa\frac{z(z+h)^2}{R_2^7}+ +\frac{10z(z^2-h^2)}{R_2^7}-(\varkappa-1)\frac{z+h}{R_2^5}\right],$$

$$\text{(A.14)} \quad \sigma_{32} = \frac{3My}{2\pi(\varkappa+1)}\left[-(\varkappa-1)\frac{(z-h)}{R_1^5}-(\varkappa-1)\frac{z+h}{R_2^5}+10\varkappa\frac{z(z+h)^2}{R_2^7}+\right.$$
$$\left.+\frac{10z(z^2-h^2)}{R_2^7}-2(\varkappa-1)\frac{z}{R_2^5}\right],$$

$$\text{(A.15)} \quad \sigma_{33} = \frac{(\varkappa-1)M}{4\pi(\varkappa+1)}\left[\frac{1}{R_1^3}-\frac{1}{R_2^3}-\frac{3(z-h)^2}{R_1^5}+\frac{3(z+h)^2}{R_2^5}\right]+$$
$$+\frac{3(1-2\varkappa)M}{2\pi(\varkappa+1)}\left[\frac{3z(z+h)}{R_2^5}-\frac{5z(z+h)^3}{R_2^7}\right]+\frac{315Mz^2(z+h)^4}{2\pi(\varkappa+1)R_2^9},$$

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Streszczenie

ROZKŁADY NAPRĘŻEŃ W PÓŁPRZESTRZENI SPRĘŻYSTEJ POD DZIAŁANIEM ŹRÓDEŁ PUNKTOWYCH

Wyprowadzono ścisłe rozwiązanie w postaci zamkniętej dla przemieszczeń i naprężeń wywołanych w liniowej izotropowej półprzestrzeni sprężystej poddanej działaniu źródeł punktowych umieszczonych na skończonej głębokości h pod nieobciążoną powierzchnią. Rozważono źródła punktowe w postaci sił pojedynczych, podwójnych, centrum obrotu i dylatacji. Równania elasto-statyki rozwiązano za pomocą potencjałów Papkowicza i Neubera [5], które przedstawić można w postaci jawnej w przypadku, gdy wymienione siły działają w pełnej przestrzeni sprężystej. Rozwiązania dla półprzestrzeni uzyskuje się z nich za pomocą całkowo-różniczkowych wzorów Aderogby. Rozwiązania zilustrowano trójwymiarowymi wykresami.

Резюме

РАСПРЕДЕЛЕНИЯ НАПРЯЖЕНИЙ В УПРУГОМ ПОЛУПРОСТРАНСТВЕ ПОД ДЕЙСТВИЕМ ТОЧЕЧНЫХ ИСТОЧНИКОВ

Выведены точные решения в замкнутом виде для перемещений и напряжений, вызванных в линейном изотропном упругом полупространстве, подвергнутом действию точечных источников, помещенных на конечной глубине h под ненагруженной поверхностью. Рассмотрены точечные источники в виде единичных сил, двойных сил, центра вращения и дилатации. Уравнения эластостатики решены при помощи потенциалов Папковича и Неубера [5], которые можно представить в явном виде в случае, когда упомянутые силы действуют в полном пространстве. Решения для полупространства получаются из них при помощи интегро-дифференциальных формул Адергоби. Решения иллюстрированы трехмерными диаграммами.

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