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OPTIMAL DESIGN OF ELASTIC ARCHES WITH I CROSS-SECTION

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This paper concerns strength optimization of elastic arches with I cross-section. Arches are subjected to a dead weight and useful external load. The volume of the element or the deflection at the chosen point are the optimality criteria. Side conditions concern strength constraints (bending and shearing stress) and geometry is imposed on the dimensions of the cross-section. The effective Pontryagin method is used for solving the formulated tasks of optimization. The computer program has been designed and particular solutions for various forms of the arch centre line and for various kinds of support have been obtained.

Notation

$m=\dfrac{M}{\sigma_{\text{ad}}\,W_w}$	dimensionless bending moment,
$n=\dfrac{N}{\sigma_{\text{ad}}\,F_w}$	dimensionless longitudinal force,
$t=\dfrac{Q}{\sigma_{\text{ad}}\,F_w}$	dimensionless shearing force,
M	bending moment,
N	longitudinal force,
Q	shearing force,
$S(\xi)$	horizontal component of load,
$P(\xi)$	vertical component of load,
$s=\dfrac{S\cdot\bar{\varrho}_p}{\sigma_{\text{ad}}\,F_w}$	dimensionless horizontal component,
$q=\dfrac{P\cdot\bar{\varrho}_p}{\sigma_{\text{ad}}\cdot F_w}$	dimensionless vertical component,
$\bar{u}, \bar{w}$	tangent and normal displacements,
v, w	dimensionless tangent and normal displacements,
α	angle of rotation,
$\bar{\varrho}$	radius of curvature,
$\bar{\varrho}_p$	comparative radius of curvature,
B	width of cross-section,
b_0	comparative width,
h_s	height of cross-section,
$B_{\max}$	comparative height,
l	span of the arch,
E	Young's modulus,
F_w	cross-section area,
K_t, σ_{ad}	admissible stress,
W_w	modulus of section,
g_p, g_s	thickness of flanges and web.

1. Introduction

The paper concerns a strength optimization of elastic arches with I cross-section are subjected to a dead weight and useful load. The dead weight of deflection of the chosen point of an arch is established as the optimality criterion; moreover, the added constraints concern strength (bending, shearing, combined stresses) and overall dimensions. We restrict ourselves to an elastic range neglecting conditions connected with possible loss of stability. The width of the arch flange was taken as the control variable. We want to find such a width of the arch flange satisfying the arch equations and given constraint relations, that the criterion functional (cost function) obtains the minimal (as possible) value.

So far, as it is known from the literature of the subject (see for example [4]), the optimization problems of arches (especially in terms of stress restrictions) constitute a field relatively little explored, where the set of results is considerably more modest than the knowledge in optimization of other elements (beams for example).

The problem formulated in this paper is not trivial and not soved up to this time. By using the very effective Pontryagin's method of optimization, a general algorithm has been designed. The many solutions obtained are of theoretical and partly applicable value.

2. Formulation of the problem

A physical content of the tasks formulated in the introduction leads to the following optimization formalism. This is:

i) Equation of a physical systems i.e. the state equation describing statics and kinematics of an elastic arch (Fig. 1)

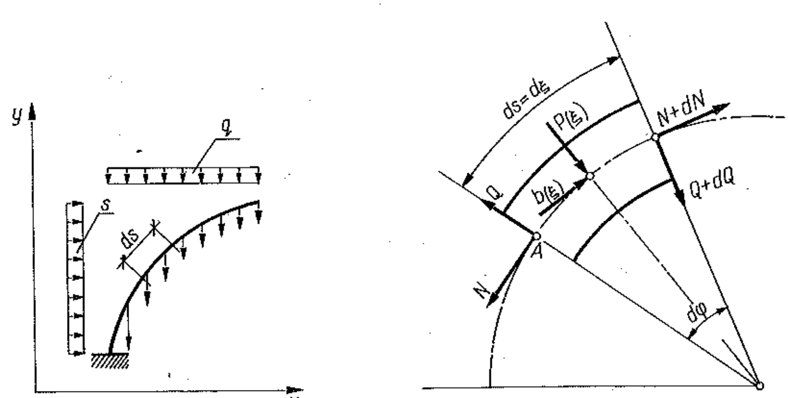

Fig. 1.

(2.1)
$$\frac{dN}{d\bar{\xi}} = Q\frac{d\alpha}{d\bar{\xi}} - b(\bar{\xi}), \quad \frac{dQ}{d\bar{\xi}} = -N\frac{d\alpha}{d\bar{\xi}} - \bar{p}(\bar{\xi}), \quad \frac{dM}{d\bar{\xi}} = Q,$$
$$\varepsilon_0 = \frac{d\bar{u}}{d\bar{\xi}} + \frac{\bar{w}}{\bar{\varrho}}, \quad \varkappa = \frac{d\alpha}{d\bar{\xi}}, \quad \alpha = \frac{1}{\bar{\varrho}}\left(\bar{u} - \frac{d\bar{w}}{d\bar{\xi}}\right),$$
$$M = \varkappa EJ, \quad N = EF\varepsilon_0.$$

where N — longitudinal force, Q — shearing force, M — bending moment, $\bar{\xi}$ — independent variable measured along the arch center line, $\bar{p}(\bar{\xi})$ — normal component of the load, $b(\bar{\xi})$ — tangent component of the load, $\bar{u}, \bar{w}$ — tangent and normal displacements, α — angle of rotation, $\bar{\varrho}$ — radius of curvature.

Taking into account the method of solution, the set (2.1) will be disentangled with respect to the derivatives and written in dimensionless form due to the references of the geometrical quantities to the comparative radius $\bar{\varrho}_p$ and after introducing the auxiliary parameters.

Finally it takes the form

(2.2)
$$\frac{dn}{d\xi} = \frac{t}{\varrho} + a_1\left(U + \frac{1}{a_9}\right)\cos\varphi + (q-s)\sin\varphi\cos\varphi,$$
$$\frac{dt}{d\xi} = -\frac{n}{\varrho} - a_1\left(U + \frac{1}{a_9}\right)\sin\varphi - q\sin^2\varphi - s\cos^2\varphi,$$
$$\frac{dm}{d\xi} = a_2 t,$$
$$\frac{dv}{d\xi} = \frac{na_3}{Ua_4 + a_5} - \frac{w}{\varrho},$$
$$\frac{dw}{d\xi} = \frac{v}{\varrho} - a_2\alpha,$$
$$\frac{d\alpha}{d\xi} = \frac{ma_8}{a_7 + Ua_6},$$

where

$$a_1 = \frac{\gamma b_0 \bar{\varrho}_p B_{\max}}{F_w \sigma_{\mathrm{ad}}}, \quad a_2 = \frac{\bar{\varrho}_p F_w}{W_w}, \quad a_3 = \frac{F_w^2 \bar{\varrho}_p \sigma_{\mathrm{ad}}}{EW_w},$$
$$a_4 = 2g_p B_{\max}, \quad a_5 = h_s g_s, \quad a_6 = \left[\frac{g_p^3}{6} + \frac{g_p(h_s+g_s)^2}{4}\right]B_{\max},$$
$$a_7 = \frac{g_s h_s^2}{12}, \quad a_8 = \bar{\varrho}_p \varepsilon_{\mathrm{dop}} W_w, \quad a_9 = \frac{2g_p B_{\max}}{h_s g_s}, \quad U = \frac{B}{B_{\max}}$$

(for other quantities see the notations).

For concise notation of the state equations (2.2) we introduce the vector $\bar{y}(n, t, m, v, w, \alpha)$ and for the right-hand sides of this system a vector

$$\bar{\varphi} = \left[\frac{t}{\varrho} + a_1\left(U + \frac{1}{a_9}\right)\cos\varphi + (q-s)\sin\varphi\cos\varphi;\right.$$
$$-\frac{n}{\varrho} - \left(U + \frac{1}{a_9}\right)a_1\sin\varphi - s\cos^2\varphi - q\sin^2\varphi;\ a_2 t;$$
$$\left.\frac{na_3}{Ua_4+a_5} - \frac{w}{\varrho};\ \frac{v}{\varrho} - a_2\alpha;\ \frac{ma_8}{a_7+Ua_6}\right].$$

Thus the system (2.2) can be written short in the form

$$\frac{d\bar{y}}{d\xi} = \bar{\varphi}(\bar{y}, U, \xi). \tag{2.3}$$

ii) Constraints defining a set of admissible controls

For the control variable we introduce geometrical and strength constraints:

$$\begin{gathered} U_1 \leqslant U \leqslant U_2, \\ \sigma = \frac{|N|}{F} + \frac{|M|}{W} \leqslant \sigma_{\mathrm{ad}}, \\ t \leqslant K_t, \\ \sigma^2 + 3t^2 \leqslant \sigma_0^2, \end{gathered} \tag{2.4}$$

σ — normal stresses, τ — shearing stresses.

Expressing the constraints (2.3) by the dimensionless variables, we define the set of the admissible controls as follows:

$$U_{\mathrm{ad}} = \left\{ U:\ \begin{aligned} & U_1 \leqslant U \leqslant U_2, \\ & \frac{|n|\,a_{10}}{a_9 U + 1} + \frac{|m|\,a_{11}}{a_6 U + a_7} \leqslant 1, \\ & \frac{t\,[Ua_{12} + a_{13}]}{a_{14} + Ua_6 g_s} \leqslant \frac{K_t}{\sigma_{\mathrm{ad}}}, \\ & \frac{m^2 a_{15}^2}{(a_7 + Ua_6)^2} + \frac{3t^2 a_{16}^2 U^2}{(Ua_6 g_s + a_{14})^2} \leqslant 1, \end{aligned} \right.$$

where

$$a_{10} = \frac{F_w}{h_s g_s}, \quad a_{11} = W_w\left(g_p + \frac{h}{2}\right), \quad a_{12} = \frac{1}{2} g_p (h_s + g_s) B_{\max} F_w,$$

$$a_{13} = \frac{g_s h_s^2 F_w}{8}, \quad a_{14} = \frac{g_s^2 h_s^3}{12}, \quad a_{15} = W_w \frac{h}{2},$$

$$a_{16} = F_w\, g_p \frac{h_s + g_s}{2} B_{\max}.$$

Existance of the constraints (2.4) secures the solutions sought for in the suitable physical meaning.

iii) Cost functions defined by the functionals

(2.5) $I(u) = w_k$ displacement of a chosen point K,

(2.6) $I(u) = \int\limits_C \left[U(\xi) + \frac{a_5}{a_4} \right] d\xi.$ volume of an element (arch).

We want to determine such $U(\xi)$ which renders the minimum of the functional (2.5) (problem 1) or (2.6) (problem 2), satisfying the state equations (2.2) with proper boundary conditions and the constraints (2.4).

3. General solution of the problem 1

In order to solve the formulated problems, we use the Pontryagin maximum principle. Therefore for the problem 1 we introduce the Hamiltonian

$$\begin{aligned}(3.1)\qquad H(\bar{y}, \bar{\lambda}, U) = \bar{\lambda} \cdot \bar{\varphi} = \lambda_1 \left[\frac{t}{\varrho} + a_1 \left(U + \frac{1}{a_{14}} \right) \cos\varphi + \right. \\ \left. + (q - s) \sin\varphi \cos\varphi \right] + \lambda_2 \left[-\frac{n}{\varrho} - a_1 \left(U + \frac{1}{a_9} \right) \sin\varphi - \right. \\ \left. - q \sin^2\varphi - s \cos^2\varphi \right] + \lambda_3\, a_2 t + \lambda_4 \left[\frac{n a_3}{U a_4 + a_5} - \frac{w}{\varrho} \right] + \\ + \lambda_5 \left(\frac{v}{\varrho} - a_2\, \alpha \right) + \lambda_6 \frac{m a_8}{a_7 + U a_6}.\end{aligned}$$

According to the Pontryagin maximum principle [5] for the optimal control $U^*(\xi)$, there is $H(y^*, \lambda^*, U^*) = \max H(\bar{y}, \bar{\lambda}, U)$ and hence invoking the condition $\partial H/\partial U = 0$ we have the equation

$$\begin{aligned}(3.2)\qquad U^4 [a_4^2\, a_6^2\, a_1 (\lambda_1 \cos\varphi - \lambda_2 \sin\varphi)] + U^3 [2a_4^2\, a_6\, a_7 + \\ + 2a_4\, a_5\, a_6^2\, a_1 (\lambda_1 \cos\varphi - \lambda_2 \sin\varphi)] + U^2 [(a_4^2\, a_7^2 + 4a_4\, a_5\, a_6\, a_7 + \\ + a_5^2\, a_6^2)(\lambda_1\, a_1 \cos\varphi - \lambda_2\, a_1 \sin\varphi - \lambda_4\, n a_3\, a_4\, a_6^2 - \lambda_6\, m a_8\, a_6\, a_4^2)] + \\ + U [(2a_4\, a_5\, a_7^2 + 2a_5^2\, a_6\, a_7)(\lambda_1\, a_1 \cos\varphi - \lambda_2\, a_1 \sin\varphi) + \\ + (-2a_6\, a_7\, \lambda_4\, n a_3\, a_4 - \lambda_6\, m a_8\, a_6\, 2a_4\, a_5)] + \\ + a_5^2\, a_7^2 (\lambda_1\, a_1 \cos\varphi - \lambda_2\, a_1 \sin\varphi) - \lambda_4\, n a_3\, a_4\, a_7^2 - \lambda_6\, m a_8\, a_6\, a_5^2 = 0\end{aligned}$$

from which we shall obtain the optimal control $U(\xi)$.

In the formulated optimization problems the constraints (2.4) depend on the state of the system, then not the whole space is an admissible region. We shall utilize the modified version of the maximum principle concerning the case of constraints of phase variables [5].

The constraints are imposed in the form

$$B = \{\bar{y}: g_i(\bar{y}, U) = 0\} \quad i = 1, 2, 3, \tag{3.3}$$

where, from Eq. (2.4),

$$g_1 = \frac{|m|\, a_{11}}{a_6\, U + a_7} + \frac{|n|\, a_{10}}{U a_g + 1} - 1,$$

$$g_2 = \frac{|t|\,(U a_{12} + a_{13})}{a_{14} + U a_6\, b_s} - \frac{K_t}{\sigma_{ad}},$$

$$g_3 = \frac{m^2\, a_{15}^2}{(a_7 + U a_6)^2} + \frac{3t^2\, a_{16}^2\, U^2}{(U a_6\, b_s + a_{14})^2} - 1.$$

We are especially interested in these parts of the optimal solution which belong to the boundary ∂B. The set ∂B is surface determined by the equations

$$g_i(\bar{y}, U) = 0, \quad i = 1,\ 2,\ 3. \tag{3.4}$$

The optimal control at the entering point into the boundary of the admissible region ∂B is obtained from the relations (3.4). At the entering point the variable $\xi = \xi^*$ satisfies, moreover, the equations

$$\begin{aligned} \lambda_i(\xi^* - 0) - \lambda_i(\xi^* + 0) &= 0, \\ (H)_{\xi^* + 0} - (H)_{\xi^* - 0} &= 0. \end{aligned} \tag{3.5}$$

Hence the optimal solution for the I-section has the final form

$$U^* = \{U_1 \vee U_2 \vee U_3 \vee U_4 \vee U_5 \vee \hat{U}\}, \tag{3.6}$$

where $\hat{U}$ is root of Eq. (3.2) and

$$U_3 = \frac{-\{a_9\, a_7 + a_6 + |n|\,(-a_{10}\, a_6 - a_{11}\, a_9)\}}{2a_9\, a_6} + \frac{\sqrt{\{a_9\, a_7 + a_6 + |n|\,(-a_{10}\, a_6 - a_{11}\, a_9)\}^2 - 4a_9\, a_6\,(a_7 - |n|\, a_{10}\, a_7 - |m|\, a_{11}}}{2a_9\, a_6},$$

$$U_4 = \frac{K_t\, a_{14} - |t|\, a_{13}\, \sigma_{ad}}{|t|\, a_{12}\, \sigma_{ad} - K_t\, a_6\, b_s}, \tag{3.7}$$

U_5 is the root of Eq. $(3.7)_1$ which follows from Eqs. $(3.3)_1$, $(3.3)_2$ and $(3.3)_3$

(3.7′)
$$U^4(3t^2 a_{16}^2 a_6^2 - a_6^4 b_s^2) + U^3(6t^2 a_6 a_7 a_{16}^2 - 2a_6^3 a_7 b_s^2 - 2a_6^3 a_{14} b_s) + U^2(m^2 a_6^2 a_{15}^2 b_s^2 + 3t^2 a_7^2 a_{16}^2 - a_6^2 a_7^2 b_s^2 - 4a_6^2 a_7 a_{14} b_s - a_{14}^2 a_6^2) + U(2m^2 a_4 a_6 a_{15}^2 b_s - 2a_6 a_7^2 a_{14} b_s - 2a_6 a_7 a_{14}^2) + m^2 a_{15}^2 a_{14}^2 - a_7^2 a_{14}^2 = 0.$$

Though the optimal solution (3.6) is exact still it has a purely formal character because we know neither the functions $(\bar{y}, \bar{\lambda})$ nor the ranges in which the individual relations (3.6) hold.

To decide which formula for $U(\xi)$ holds, we test the constraints $(2.4)_{2,3,4}$. If these inequalities are strong, we chose amongst the relations $(3.6)_{1,2,6}$ as the optimal control this function which results from the supremum of the Hamiltonian (3.1). When the constraints $(2.4)_{2,3,4}$ become equalities, the optimal design is determined by Eqs. $(3.6)_{3,4,5}$.

To find the optimal control effectively, we construct a system of the adjoint equations having the form [5]

$$\frac{d\bar{\lambda}}{d\xi} = -\frac{\partial H}{d\bar{y}}$$

it is

(3.8)
$$\begin{aligned}
\frac{d\lambda_1}{d\xi} &= \frac{\lambda_2}{\varrho} - \frac{\lambda_4 a_3}{Ua_4 + a_5},\\
\frac{d\lambda_2}{d\xi} &= -\left(\frac{\lambda_1}{\varrho} + \lambda_3 a_2\right),\\
\frac{d\lambda_3}{d\xi} &= -\frac{\lambda_6 a_8}{a_7 + Ua_6},\\
\frac{d\lambda_4}{d\xi} &= -\frac{\lambda_5}{\varrho},\\
\frac{d\lambda_5}{d\xi} &= \frac{\lambda_4}{\varrho},\\
\frac{d\lambda_6}{d\xi} &= \lambda_5 a_2.
\end{aligned}$$

We show an analysis of the boundary conditions for the case of a clamped arch.

Then

(3.9)
$$\begin{array}{ll}
\text{for } \xi = 0 & \text{for } \xi = \xi_L \\
v(0) = 0;\ w(0) = 0;\ \alpha(0) = 0, & v(\xi_L) = 0;\ t(\xi_L) = 0;\ \alpha(\xi_L) = 0.
\end{array}$$

The vector $\bar{\lambda}\{\lambda_i\}$ is normal to the manifold θ_0 and θ_k defined by the relations

(3.10)
$$\theta_0 = \left\{ \bar{y}_0 : \begin{array}{c} n(0) = n_0 \\ t(0) = t_0 \\ m(0) = m_0 \\ v(0) = 0 \\ w(0) = 0 \\ \alpha(0) = 0 \end{array} \right\}, \quad \theta_K = \left\{ \bar{y}_K : \begin{array}{c} n(\xi_L) = n_L \\ t(\xi_L) = 0 \\ m(\xi_L) = m_L \\ v(\xi_L) = 0 \\ w(\xi_L) = w_L \\ \alpha(\xi_L) = 0 \end{array} \right\}.$$

The values n_0, t_0, m_0 and n_L, m_L w_L are unknown. The vector $\bar{\lambda}$ is normal to the initial manifold θ_0

$$\bar{\lambda}\,\bar{y}_0 = 0$$

what leads to the following boundary values:

(3.11) $$\lambda_1(0) = 0, \quad \lambda_2(0) = 0, \quad \lambda_3(0) = 0.$$

The cost function in the form (2.5) causes that the terminal transversality condition undergoes the modification [5]. It has the form

(3.12) $$\frac{\partial \psi}{\partial y_s(\xi_L)} + \lambda_s(\xi_L) = 0, \quad s = 1,\ 3,\ 5, \quad \psi = w.$$

Because

$$\frac{\partial \psi}{\partial n} = 0, \quad \frac{\partial \psi}{\partial m} = 0, \quad \frac{\partial \psi}{\partial w} = 1,$$

we obtain

(3.13) $$\lambda_1(\xi_L) = 0, \quad \lambda_3(\xi_L) = 0, \quad \lambda_5(\xi_L) = -1.$$

Finally the relations (3.11) and (3.13) determine the boundary values for the vector $\bar{\lambda}$ in the case of the bilaterally clamped arch.

To find effectively the optimal control $U^*(\xi)$, it is necessary to solve the system of the state equations (2.2) and the adjoint equations (3.8) with the conditions (3.9), (3.11) and (3.13), taking into account the solution (3.6). The system (2.2) and (3.8) consists of twelve nonlinear ordinary differential equations of the first order. The solution of this system is possible only in a numerical way.

The algorithm of the numerical solution of the problem will be as follows:

1. We start by accepting a constant value of the control U_0

$$U_1 \leqslant U_0 \leqslant U_2.$$

2. Using the standard procedure of numerical integration (the Runge-Kutta method) we solve the system (2.2) and (3.8) with the boundary conditions are of such a type that the two-point boundary value problem considered here is of the type «6+6», i.e. only 6 initial and 6 terminal

conditions are known. The two-point boundary value problem of the type «6+6» is solved using the method of the adjoint equation [2],

3. As the results of integration of the system we obtain the vectors $\bar{y}_0$, $\bar{\lambda}_0$ corresponding to the value of the control chosen at the point 1.

4. The optimal control is determined from the relation (3.6). We decide, moreover, which formula for $U(\xi)$ holds and what range of validity it is. Therefore at each step of integration we test the constraints, so:

i) if

$$\frac{|n|\, a_{10}}{a_g\, U+1}+\frac{|m|\, a_{11}}{a_6\, U+a_7}<1,$$

$$\frac{|t|\,[Ua_{12}+a_{13}]}{a_{14}+Ua_6\, g_s}<\frac{K_t}{\sigma_{ad}},$$

$$\frac{m^2\, a_{15}^2}{(a_7+Ua_6)^2}+\frac{3t^2\, a_{16}^2\, U^2}{(Ua_6\, g_s+a_{14})^2}<1,$$

then this formula of the relations $(3.6)_{1,2}$ and (3.2) is valid, which gives the supremum of the Hamiltonian.

ii) when

$$\frac{|n|\, a_{10}}{a_9\, U+1}+\frac{|m|\, a_{11}}{a_6\, U+a_7}>1,$$

$$\frac{|t|\,[Ua_{12}+a_{13}]}{a_{14}+Ua_6\, g_s}>\frac{K_t}{\sigma_{ad}},$$

$$\frac{m^2\, a_{15}^2}{(a_7+Ua_6)^2}+\frac{3t^2\, a_{16}^2\, U^2}{(Ua_6\, g_s+a_{14})^2}<1,$$

then $U(\xi)$ is obtained from the relation $(3.6)_3$.

iii) in the case

$$\frac{|n|\, a_{10}}{a_9\, U+1}+\frac{|m|\, a_{11}}{a_6\, U+a_7}<1,$$

$$\frac{|t|\,[Ua_{12}+a_{13}]}{a_{14}+Ua_6\, g_s}<\frac{K_t}{\sigma_{ad}},$$

$$\frac{m^2\, a_{15}^2}{(a_7+Ua_6)^2}+\frac{3t^2\, a_{16}^2\, U^2}{(Ua_6\, g_s+a_{14})^2}>1,$$

$U(\xi)$ is obtained from Eq. (3.7)

5. Thus we determine the first approximation for $U(\xi)$.

6. This algorithm is repeated many tims until

$$|U_i^{r+1}(\xi)-U_i^r(\xi)|<\varepsilon$$

for each i, where i is the number of the range in the **Runge-Kutta** discretization.

4. Problem 2 design for minimization of volume

The Hamiltonian constructed for this case

$$(4.1)\qquad H=\sum_{i=0}^{N}\lambda_i\,\varphi_i=\lambda_0\left(U+\frac{a_5}{a_4}\right)+\lambda_1\left[\frac{t}{\varrho}+a_1\left(U+\frac{1}{a_9}\right)\cos\varphi+\right.$$
$$\left.+(q-s)\sin\varphi\cos\varphi\right]+\lambda_2\left[-\frac{n}{\varrho}-a_2\left(U+\frac{1}{a_9}\right)\sin\varphi-q\sin^2\varphi+\right.$$
$$\left.+s\cos^2\varphi\right]+\lambda_3\,a_2\,t+\lambda_4\left[\frac{na_3}{Ua_4+a_5}-\frac{w}{\varrho}\right]+$$
$$+\lambda_5\left(\frac{v}{\varrho}-a_2\,\alpha\right)+\lambda_6\,\frac{ma_6}{a_7+Ua_6}.$$

leads to the equations which determine the optimal control

$$(4.2)\qquad U(\xi)=\{U_1\vee U_2\vee \text{from } (3.6)_{3,4,5}\vee \text{from } (4.3)\},$$

where Eq. (4.3) follows from the condition $\partial H/\partial U=0$ and has the form

$$(4.3)\qquad U^4\left[a_4^2\,a_6^2\,(\lambda_0+\lambda_1\,a_1\cos\varphi-\lambda_2\,a_1\sin\varphi)\right]+U^3\left[(2a_4^2\,a_6\,a_7+2a_4\,a_5\cdot\right.$$
$$\left.\cdot a_6^2)(\lambda_0+\lambda_1\,a_1\cos\varphi-\lambda_2\,a_1\sin\varphi)\right]+U^2\left[(a_4^2\,a_7^2+4a_4\,a_5\,a_6\,a_7+\right.$$
$$\left.+a_5^2\,a_6^2)(\lambda_0+\lambda_1\,a_1\cos\varphi-\lambda_2\,a_1\sin\varphi)-\lambda_4\,na_3\,a_4\,a_6^2-\lambda_6\,ma_8\,a_6\,a_4^2\right]+$$
$$+U\left[(2a_4\,a_5\,a_7^2+2a_5^2\,a_6\,a_7)(\lambda_0+\lambda_1\,a_1\cos\varphi-\lambda_2\,a_1\sin\varphi)-\right.$$
$$\left.-2a_6\,a_7\,\lambda_4\,na_3\,a_4-\lambda_6\,ma_8\,a_6\,2a_4\,a_5\right]+a_5^2\,a_7^2\,(\lambda_0+\lambda_1\,a_1\cos\varphi-$$
$$-\lambda_2\,a_1\sin\varphi)-\lambda_4\,na_3\,a_4\,a_7^2-\lambda_6\,ma_8\,a_6\,a_5^2=0.$$

The constraints appearing in both problems are identical; therefore the complete analysis of the problem 2 requires to solve the system (2.2) and (3.8) with the direction (4.2) for the optimal design.

5. Numerical examples

The numerical solution of the formulated problems requires universal testing of the designed computer program because a general proof of convergence, uniqueness and stability is usually impossible.

The examples concern arches having the following static schemes (Fig. 2):

Fig. 2.

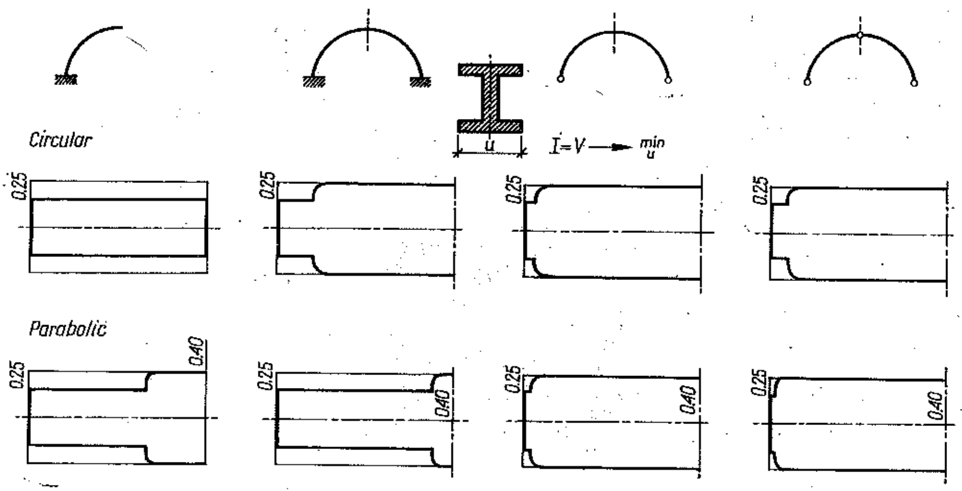

Fig. 3.

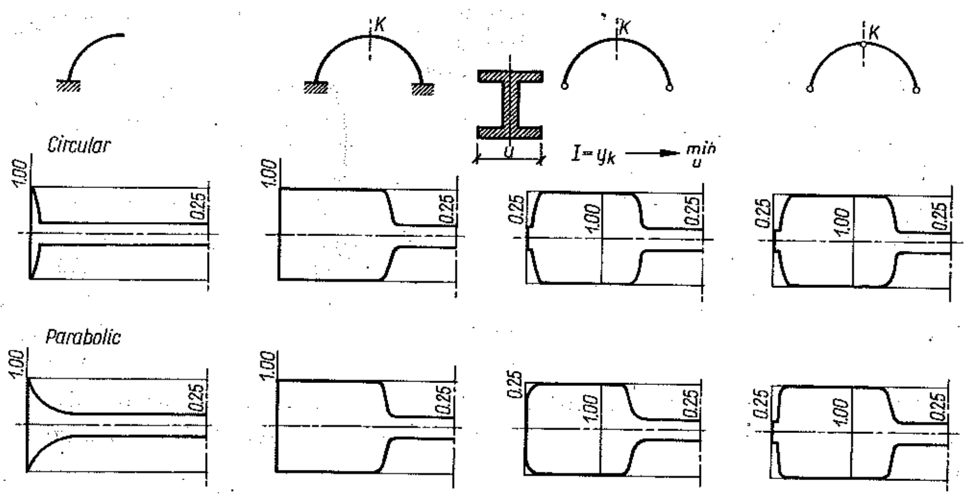

Fig. 4.

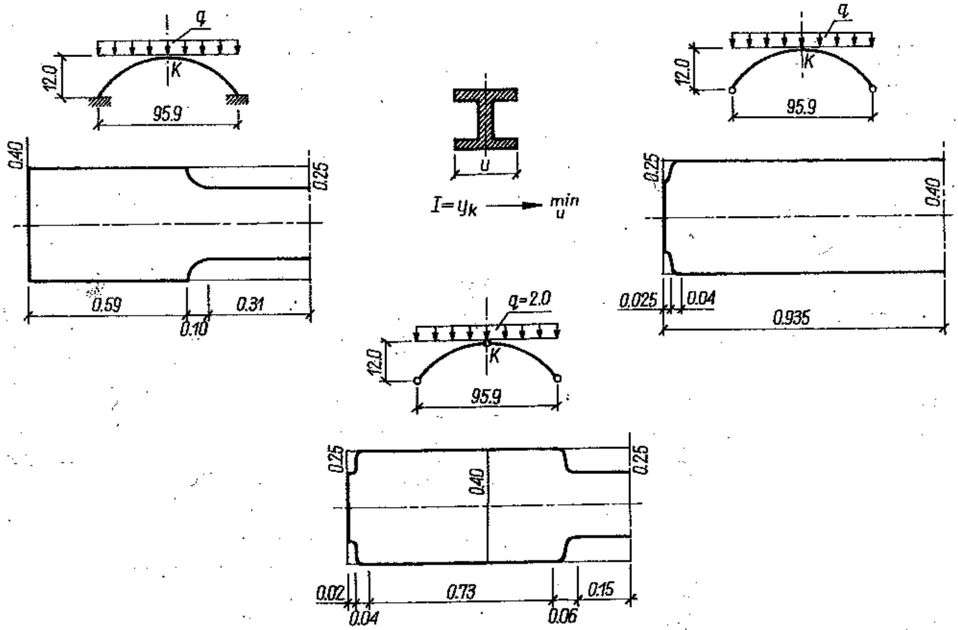

Fig. 5.

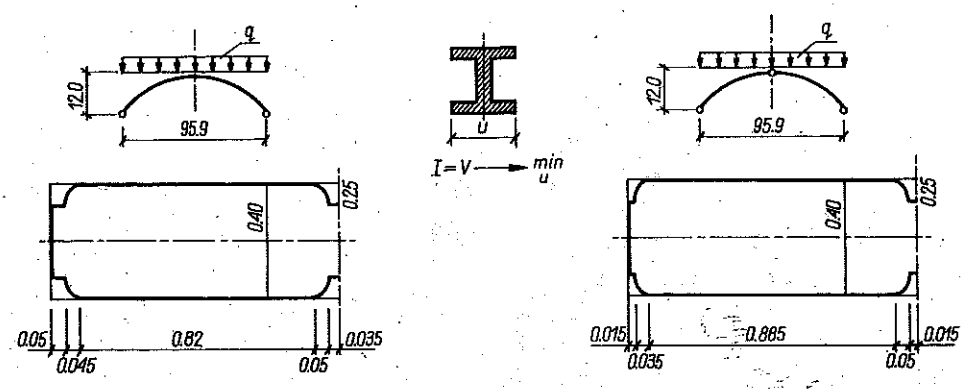

Fig. 6.

The arch line can be an arbitrary curve. Computations were carried out for the circular (Variant 1), elliptic (Variant 2) and parabolic (Variant 3) line.

Calculations were made for the following data:

$$\gamma = 78\,000 \text{ N/m}^3, \quad L = 95.9 \text{ m}, \quad q = 20\,000 \text{ N/m}, \quad B_{\max} = 1.0 \text{ m}.$$

On the basis of a main flowchart the program written in FORTRAN

EXTENDED and accomodated on the computer CYBER 72 was designed. This program was used in a series of computations (ca 80 examples). Some results of these calculations are presented below (Figs. 3–6).

References

1. A. K. Aziz, *Numerical solution of boundary value problems for differential equations*, Academic Press, New York 1975.
2. R. Bulirsch, *Die Mehrzielmethode zur numerischen Lösung von nichtlinearen Randwertproblemen und Aufgaben der optimalen, Steuerung*, Raport der Carl-Grnz-Gesellschaft, 1971.
3. R. Bulirsch, J. Stoer, P. Deuflhard, *Numerical solution of nonlinear two-point boundary value problems*, Handbook, series Approximation Num. Math., 1972.
4. В. Немировский, В. Мазалов, *Оптимальное проектирование конструкций*, библиограф. указатель., Новосибирск 1975.
5. L. S. Pontriagin, V. G. Boltianskij, R. V. Gramkrelidze, E. F. Miscenko, *Mathematische Theorie optimaler Prozesse*, Verlag R. Oldenbourg, München 1964.
6. S. M. Roberts, J. S. Shipman, *Two-point boundary value problems. Shooting methods*, Elsevier, New York, London, Amsterdam 1972.

Streszczenie

OPTYMALNE PROJEKTOWANIE ŁUKÓW SPRĘŻYSTYCH O PRZEKROJU DWUTEOWYM

Praca dotyczy optymalizacji wytrzymałościowej łuków sprężystych o przekroju dwuteowym poddanych obciążeniom własnym i użytkowym. Jako kryteria optymalizacji przyjęto objętość elementu lub jego ugięcie w danym punkcie. Warunki poboczne dotyczą ograniczeń wytrzymałościowych (naprężenia styczne lub pochodzące od zginania) oraz geometrycznych (wymiary przekroju poprzecznego). Do rozwiązania postawionego zagadnienia optamalizacji zastosowano efektywną metodę Pontriagina. Opracowano program komputerowy i przedstawiono rozwiązania przypadków szczególnych dla różnych kształtów linii środkowej łuku i różnych warunków podparcia.

Резюме

ОПТИМАЛЬНОЕ ПРОЕКТИРОВАНИЕ УПРУГИХ АРОК С ДВУТАВРОВЫМ СЕЧЕНИЕМ

Работа касается прочностной оптимизации упругих с двутавровым сечением, подвергнутых собственным и эксплуатационным нагрузкам. В качестве критерия оптимизации принят объем элемента или его прогиб в данной точке. Дополнительные условия касаются ограничений прочности (статическое напряжение или от изгиба), а также геометрических (размеры поперечного сечения). Для решения поставленной задачи оптимизации был применен эффективный метод Понтрягина. Была разработана программа для компьютера и представлено рещение особых случаев для разного вида осевой линии арки и разных условий опоры.

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