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PLASTIC BEHAVIOUR OF FIBRE-REINFORCED COMPOSITES AND FRACTURE EFFECTS

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The work deals with the problem of longitudinal extension of a unidirectionally reinforced brittle fibres-ductile matrix composite. A model of the plastic deformation process is proposed with regard to a unit cell of the composite. The model accounts in an essential way for the matrix ductility and reveals the nature of the possible mechanisms of failure of the fibre-matrix interface. The model implies a relatively simple general scheme of analysis of the entire problem including, in particular, the determination of the current plastic zone size, the stress-strain curve for the considered composite as well as the important measure of a limiting elastic response of the matrix phase. The detailed solution of the problem associated with the proposed model is shown to require simple numerical calculations. However, certain important results are obtained in a closed form.

1. Introduction

The plastic behaviour of fibre-reinforced composites has been for long the subject of numerous investigations. A number of approaches to the problem have been developed in the works of R. Hill [1, 2], A. J. M. Spencer [3, 4], J. I. Mulhern *et al.* [5, 6], R. F. Thomason [7], G. J. Dvorak *et al.* [8, 9], I. M. Kopiov and A. S. Ovcinskij [10], M. R. Piggott [11]. An extensive review of progress in this field is given by G. A. Cooper and M. R. Piggott in [12]. Although the existing approaches account in different ways for the effects of fibre-reinforcement on the appearance and development of the plastic deformation and fracture processes in the composite materials, they focus generally attention on the strengthening effect of the fibres. But as known the fibres act at the same time as stress and strain concentrators as well and thus an attempt to a more precise account for the concentration effect of the fibres appears to be of obvious interest especially from the point of view of the fracture behaviour of the composites. Such an attempt forms the matter of the present work.

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2. Basic assumptions

The material under consideration is a unidirectionally reinforced fibrous composite with infinitely long continuous fibres and relatively small fibre volume fraction. The fibre material is linear elastic with Young's modulus E_f and Poisson's ratio ν_f. The material of the matrix is elastic (E_m, ν_m)-perfectly plastic and obeys the von Mises' yield condition with a tensile yield stress σ_y. As usual, the relations $E_m < E_f$, $\nu_f < \nu_m$ and $\sigma_y \ll E_m$ are supposed to apply.

A unit composite cell in the sense of the known model of coaxial fibre-matrix cylinders is studied in the following so that when referred to a cylindrical coordinate system (r, θ, z) the fibre and the matrix occupy the regions $(0 \leqq r \leqq r_f, 0 \leqq \theta \leqq 2\pi, -\infty \leqq z \leqq \infty)$ and $(r_f \leqq r \leqq r_m, 0 \leqq \theta \leqq 2\pi, -\infty \leqq z \leqq \infty)$, respectively.

Those loading conditions of the composite which result in axisymmetric stress and displacement fields within the unit cell only will be considered. Perfect contact is assumed to exist over the fibre-matrix interface $r = r_f$ and, as usually, the generalized plane strain condition is supposed to apply. The normal stresses $\sigma_i, i = r, \theta, z$ within the cell are thus principal ones and depend on the radial coordinate r only, i.e. $\sigma_i = \sigma_i(r), i = r, \theta, z$.

The loading is specified in the present analysis as longitudinal extension of the unit cell. The corresponding elastic solution of the problem with zero tractions over the outer matrix surface $r = r_m$ is known and can be found, for example, in [5, 10].

The condition of axial symmetry implies certain obvious features of the elastic-plastic state of the considered unit cell. These are that the plastic zone presents itself as an infinitely long cylinder $(r_f \leqq r \leqq r_c, 0 \leqq \theta \leqq 2\pi, -\infty \leqq z \leqq \infty : r_c \leqq r_m)$ and spreads with increasing loading or, equivalently, with increasing axial strin ε_z, into the matrix material. The equation of the current elastic-plastic boundary can thus be written in the form $r_c = r_c(\varepsilon_z)$.

3. Fibre-reinforcement effects

The elastic solution just mentioned implies two important consequences concerning the role of the fibre as stress and strain concentrator. Firstly, the presence of the fibre results in the known shrinkage effect, i.e. in the appearance of compressive radial stresses over the fibre-matrix interface. This effect is due on the whole to the above assumed relation $\nu_f < \nu_m$ and is influenced at the same time by the fibre volume fraction, that is by the ratio r_f/r_m. Secondly, because of the existence of the fibre plastic deformations appear first over the fibre-matrix interface at a value of the axial strain ε_z which is smaller than the ratio σ_y/E_m as in the case of absence of he fibre when, in addition, the whole matrix plastificates immediately. Since a composite with a relatively small fibre volume fraction is considered and since the effects of the stress concentrators are known to be generally local ones, it would be then reasonable to expect that intensive plastic deformations as well as fracture processes may develop within the part of the matrix immediately surrounding the fibre while at a certain distance from the fibre-matrix interface the matrix may still deform elastically.

As known, the strengthening effect of the fibre results in the fact that the behaviour of the composite "in the fibre direction" and of the unit cell as well is rather elastic-like than perfectly-plastic. A reasonable interpretation of this effect could be that the fibre, being linear elastic and possessing high stiffness, prevents, due to the assumed perfect fibre-matrix contact, the development of a relatively large plastic part ε_z^p of the total axial strain ε_z^t within the plastificated matrix region and contributes thus to the development of a relatively large elastic part ε_z^e. In accordance with the standard plasticity theory, the relation $\varepsilon_z = \varepsilon_z^e + \varepsilon_z^p$ applies at each instant of the deformation process within the plastic zone of the matrix where the total axial strain ε_z is itself a monotonously increasing function of the applied tensile load. The plastic deformation process developing within the matrix could be then viewed as a process of simultaneous increase in both the elastic ε_z^e and plastic ε_z^p parts of the total axial strain ε_z increasing itself. This reasoning in the whole interpretation of the process is obviously limited in the sense that the elastic response of the matrix material is limited itself. In other words, such an interpretation with the continuously increasing elastic part ε_z^e of the axial strain could actually apply only up to a certain stage of the process with a corresponding, say critical value $\overset{*}{\varepsilon}{}_z^e$ of the quantity ε_z^e. If, as it will be assumed, no unloading takes place in the course of the plastic deformation process, then, upon reaching this stage, only plastic strain increments $d\varepsilon_z^p$ will further develop while the elastic part of the axial strain keeps approximately a constant value $\overset{*}{\varepsilon}{}_z^e$. Thus the account for the limiting characteristic $\overset{*}{\varepsilon}{}_z^e$ of the elastic response of the matrix material implies in a natural way the necessity of distinguishing two stages of the plastic deformation process, the first one with increasing values of ε_z^e (up to the instant $\varepsilon_z^e = \overset{*}{\varepsilon}{}_z^e$) and the second one with the approximately constant elastic part ε_z^e of the axial strain ($\varepsilon_z^e = \overset{*}{\varepsilon}{}_z^e$).

4. The plastic deformation process

When taking into account the introduced limiting characteristic $\overset{*}{\varepsilon}{}_z^e$, the following description of the plastic deformation process appears as a reasonable one. In accordance with the known elastic solution of the considered problem and the assumed yield condition, plastic deformations occur at a given definite stage of the loading process within the matrix over the fibre-matrix interface. Consequently, a plastic zone $r_f \leqq r \leqq r_c$ appears and spreads monotonously into the matrix. The deformation process within this zone includes simultaneous increase in both the elastic and plastic parts of the total axial strain increasing itself up to the instant when $\varepsilon_z^e = \overset{*}{\varepsilon}{}_z^e$. One should expect due to the concentration effect of the fibre that the latter relation would be first satisfied over the fibre-matrix interface $r = r_f$. At this instant, another second plastic zone $r_f \leqq r \leqq R_c$, $R_c < r_c$ appears within which the relation $\varepsilon_z^e = \overset{*}{\varepsilon}{}_z^e$ holds true so that only plastic deformations in the axial direction develop further within this zone. The second plastic zone also spreads into the matrix material having the first zone, which occupies now the region $R_c \leqq r \leqq r_c$, at its front $r = R_c$ where the relation $\varepsilon_z^e = \overset{*}{\varepsilon}{}_z^e$ applies as well.

5. Analysis

It should be recognized that an exact analysis of all the stages of the plastic deformation process within the frames of the description given above of the latter seems to be impossible for many reasons. Nevertheless the accepted interpretation of the process leads to certain important qualitative conclusions and quantitative estimations which are of definite interest from the point of view of the fracture behaviour of the composite. As already mentioned, the relation

$$\varepsilon_z = \varepsilon_z^e + \varepsilon_z^p \tag{5.1}$$

applies at each instant of the plastic deformation process. Since the condition of generalized plane strain applies, then the total axial strain ε_z does not depend on the radial coordinate r within the entire matrix region $r_f \leqq r \leqq r_m$. Upon expressing, as usual, the elastic part ε_z^e of the axial strain by means of Hooke's law, one comes up with the relation

$$\sigma_z = E_m\, \varepsilon_z^e + \nu_m (\sigma_r + \sigma_\theta), \tag{5.2}$$

where σ_i, $i = r, \theta, z$ are the normal and at the same time the principal stresses within the plastic zone where Eq. (5.2) actually applies together with the von Mises' yield condition

$$(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 = 2\sigma_y^2. \tag{5.3}$$

Substituting here for σ_z from Eq. (5.2) gives

$$\left(\frac{\sigma_\theta - \sigma_r}{2}\right)^2 + \left(\frac{\sigma_r + \sigma_\theta}{2} - \frac{E_m\, \varepsilon_z^e}{1 - 2\nu_m}\right)^2 \frac{(1 - 2\nu_m)^2}{3} = \frac{\sigma_y^2}{3}. \tag{5.4}$$

Equation (5.4) is identically satisfied by presenting the stresses in the form

$$\left.\begin{matrix}\sigma_r \\ \sigma_\theta\end{matrix}\right\} = \frac{E_m\, \varepsilon_z^e}{1 - 2\nu_m} + \frac{\sigma_y}{\sqrt{3}\sin\varphi} \cos(\omega \pm \varphi), \tag{5.5}$$

where the following notations are used:

(5.6) $$\sin \omega = \frac{\sigma_\theta - \sigma_r}{2} \bigg/ \frac{\sigma_y}{\sqrt{3}},$$

(5.7) $$\operatorname{ctg} \varphi = \frac{\sqrt{3}}{1 - 2\nu_m}.$$

Equations (5.5) reflect the assumption that within the plastic zone the relation $\sigma_\theta \geqq \sigma_r$ is valid and the angle ω as defined by Eq. (5.6) is thus $0 \leqq \omega \leqq \pi$. Substituting for σ_r and σ_θ from Eq. (5.5) into the equilibrium equation

(5.8) $$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

results into the equation

(5.9) $$\frac{E_m}{1-2\nu_m} \frac{d\varepsilon_z^e}{dr} - \frac{\sigma_y}{\sqrt{3}\sin\varphi} \sin(\omega+\varphi) \frac{d\omega}{dr} - \frac{2\sigma_y}{\sqrt{3}} \frac{\sin\omega}{r} = 0,$$

where ε_z^e is an obviously unknown function of the radial coordinate r and therefore the integration of Eq. (5.9) cannot be performed. Nevertheless an approximate relation between the angle ω and the quantity ε_z^e is obtainable at least within the immediate surrounding of the fibre upon the assumption that the elastic parts of the ε_r and ε_θ strain components within this region are negligible with respect to the corresponding plastic parts. This assumption together with the standard condition of plastic incompressibility of the matrix material implies that just around the fibre-matrix interface the relation

(5.10) $$\varepsilon_z^e = \varepsilon_{\text{vol}}$$

holds true where ε_{vol} is the relative volume change, i.e. $\varepsilon_{\text{vol}} = \varepsilon_r + \varepsilon_\theta + \varepsilon_z$. Expressing ε_{vol} with the aid of Eqs. (5.2) and (5.5) and Hooke's law gives

(5.11) $$\frac{\sigma_y (1+\nu_m)}{E_m} \cos\omega = -\varepsilon_z^e.$$

Note once more that Eq. (5.11) applies approximately within a thin layer around the fibre and over the fibre-matrix interface $r = r_f$ in particular where the condition $\varepsilon_z^e = \overset{*}{\varepsilon}{}_z^e$ is first achieved. The corresponding value $\omega^* = \omega(\overset{*}{\varepsilon}{}_z^e)$ of the angle ω follows from Eq. (5.11) to be

(5.12) $$\omega^* = \arccos\left[-\frac{E_m \overset{*}{\varepsilon}{}_z^e}{\sigma_y (1+\nu_m)}\right].$$

According to the model of the plastic deformation process proposed above further increase in the applied external load results in the appearance of a second plastic zone $r_f \leqq r \leqq R_c$ over the outer boundary $r = R_c$ of which Eq. (5.12) is valid. Finally, assuming that for a given matrix material under the considered scheme of loading the quantity $\overset{*}{\varepsilon}{}^e_z$, respectively ω^* (cf. Eq. (5.12)), is approximately constant and introducing the angle ω_{R_c} as

$$\omega_{R_c} = \omega(R_c), \tag{5.13}$$

one comes up with the relation

$$\omega_{R_c} = \omega^*. \tag{5.14}$$

Equations (5.2), (5.5), (5.12), (5.13) and (5.14) define now the stress state over the outer surface $r = R_c$ of the second plastic zone entirely through the still unknown quantity R_c. Moreover, these equations together with Eq. (5.9) define the stress state within the whole second plastic zone again by means of its unknown outer radius R_c.

Really, as it was assumed, in the course of the enlargement of the second plastic zone the elastic part ε^e_z of the axial strain keeps constant value $\overset{*}{\varepsilon}{}^e_z$ so that within this zone Eq. (5.9) is valid again with $\varepsilon^e_z = \overset{*}{\varepsilon}{}^e_z$ now. Upon integration Eq. (5.9) gives

$$r^2 \sin\omega = C \exp\left(-\frac{\sqrt{3}}{1-2\nu_m}\,\omega\right),$$

where the integration constant C is determined from the condition $\omega|_{r=R_c} = \omega_{R_c}$ with ω_{R_c} defined by Eqs. (5.14) and (5.12). This condition implies finally the relation

$$\frac{R_c^2}{r^2} = \frac{\sin\omega}{\sin\omega_{R_c}} \exp\left[\frac{\sqrt{3}}{1-2\nu_m}(\omega - \omega_{R_c})\right]. \tag{5.15}$$

Now it is easily observed that the latter equation together with Eqs. (5.2), (5.5), (5.12), (5.13) and (5.14) defines the stresses within the whole region $r_f \leqq r \leqq R_c$, where R_c is still to be determined.

6. The shrinkage effect

Consider now the radial stress σ_r acting over the fibre-matrix interface. From Eqs. (5.5) with $\varepsilon^e_z = \overset{*}{\varepsilon}{}^e_z$ one has

$$\sigma_r|_{r=r_f} = \frac{E_m \overset{*}{\varepsilon}{}^e_z}{1-2\nu_m} + \frac{\sigma_y}{\sqrt{3}\sin\varphi}\cos(\omega_{r_f} + \varphi), \tag{6.1}$$

where $\omega_{r_f} = \omega|_{r=r_f}$.

It would be reasonable to expect that the developing plastic deformation prosess will further contribute to the shrinkage effect. Really, as already mentioned above, the latter is due to the difference in the lateral contractions of the fibre and the matrix materials and this difference may actually only increase with the development of the plastic deformation process because of the plastic incompressibility of the matrix material. One should consequently expect that the stress $\sigma_r|_{r=r_f}$ as given by Eq. (6.1) would decrease with increasing loading, i.e. the angle ω_{r_f} would increase remaining obviously larger than the angle ω^* as given by Eq. (5.12). Thus Eq. (6.1) implies that a maximum shrinkage effect is achieved at a value of $\omega_{r_f} = \pi - \varphi$. The radius R_c^* at this instant, that is

(6.2) $$R_c^* = R_c|_{\omega_{r_f} = \pi - \varphi},$$

is defined from Eq. (5.15) as

(6.3) $$R_c^{*2} = r_f^2 \frac{\sin\varphi}{\sin\omega_{R_c}} \exp\left[\frac{\sqrt{3}}{1-2\nu_m}(\pi - \varphi - \omega_{R_c})\right].$$

Thus the whole scheme of analysis leads to the conclusion that a further decrease in $\sigma_r|_{r=r_f}$ as well as an increase in R_c is impossible. The occurrence of such a critical state with $\omega_{r_f} = \pi - \varphi$ within the considered unit cell of the composite could be expected to affect much the corresponding velocity field within the second plastic zone.

7. Velocity field

In order to investigate the velocity field within the second plastic zone, the associated flow rule will be applied with the yield function in Eq. (5.4) serving as a plastic potential. This implies for the plastic strain rates ξ_r and ξ_θ the relation

(7.1) $$\xi_r - \xi_\theta \frac{\Sigma_r}{\Sigma_\theta} = 0,$$

where

$$\left.\begin{matrix}\Sigma_r\\ \Sigma_\theta\end{matrix}\right\} = \frac{\sigma_r}{2}\left[\pm 1 + \frac{(1-2\nu_m)^2}{3}\right] + \frac{\sigma_\theta}{2}\left[\mp 1 + \frac{(1-2\nu_m)^2}{3}\right] - \frac{E_m \overset{*}{\varepsilon}{}_z^e (1-2\nu_m)}{3}$$

or, equivalently (cf. Eqs. (5.5)),

(7.2) $$\left.\begin{matrix}\Sigma_r\\ \Sigma_\theta\end{matrix}\right\} = \frac{\sigma_y}{\sqrt{3}} \sin(\varphi \mp \omega).$$

Equation (7.1) together with the incompressibility condition $\xi_r + \xi_\theta + \xi_z = 0$ implies for the axial plastic strain rate ξ_z the expression

(7.3) $$\xi_z = -\xi_\theta \frac{\Sigma_\theta + \Sigma_r}{\Sigma_\theta}.$$

Now it is easily seen from Eqs. (7.2) and (7.3) that $\Sigma_\theta|_{\omega=\omega_{r_f}} \to 0$ when $\omega_{r_f} \to (\pi - \varphi)$ so that in accordance ith Eq. (7.3) the critical tate considered above is characterized by the relation $\xi_z|_{r=r_f} \to +\infty$. The latter means physically that at this state free plastic flow of the matrix material tends to take place within a thin layer immediately surrounding the fibre.

The tendency for the occurrence of such a singular velocity field is a result of the stress redistribution which develops within the second plastic zone simultaneously with the expansion of the latter.

8. Fracture effects

It is quite obvious that at the state just considered the behaviour of the composite unit cell will depend upon the interaction between the occurrence of a singular velocity field and the strengthening effect of the fibre. Because of the assumed perfect fibre-matrix contact the latter effect tends to prevent the development of such a singularity of the velocity field. The very nature of these two effects implies the natural assumption that their interaction results in the appearance of shearing stresses over the fibre-matrix interface where the interaction process actually develops. Moreover, these shearing stresses should, for obvious reasons, be equal to the shear yield stress $\tau_y = \sigma_y/\sqrt{3}$ of the matrix material.

Let now τ_s be the shear strength of the fibre-matrix interface. If $\tau_s \leqq \tau_y$, then the very reaching of the considered critical state of the composite will obviously result in immediate failure of the fibre-matrix interface in the form of the so-called fibre-matrix debonding. If, on the contrary, $\tau_s > \tau_y$, then the well-known mechanism of fibre pull-out (see, for example [12]) will develop, most probably simultaneously with a process of fibre breaking.

Since the strength properties of the fibre-matrix interface in the real composites are as a rule high enough, then the pull-out effect is the one to be expected with these composites. The effect has been observed experimentally by B. Harris *et al.* [13] in ductile epoxy resins reinforced with continuous elastic carbon fibres, i.e. in a real composite which in much corresponds to the model employed in the present work.

9. Plastic zone size and associated problems

Certain questions concerning the problem considered above have still to be solved. Of particular interest to this regard is the determination of the quantity $\overset{*}{\varepsilon}{}_z^e$ as well as the current R_c- and r_c-values. A possible qualitative approach to the problem is described in this section.

The quantity $\overset{*}{\varepsilon}{}_z^e$ as it has been introduced obviously depends on the matrix material as well as on the current stress state or, equivalently, on the radial coordinate r. The latter dependence has been assumed to be negligible in the present analysis and therefore the question arises about the determination of the specific value of $\overset{*}{\varepsilon}{}_z^e$ which one should actually use within the framework of the analysis described above.

A possible approach to this question could be based upon the simple formal assumption that the $\overset{*}{\varepsilon}{}_z^e$-value is just a part of the limiting elastic strain σ_y/E_m of the matrix material in simple tension, i.e. $\overset{*}{\varepsilon}{}^e_z = \alpha\sigma_y/E_m$, $0 < \alpha \leqq 1$. Let $R^*_c(\alpha)$ be the corresponding value of the maximum plastic zone radius, in accordance with Eqs. (5.12), (5.14) and (6.3). Then a simple comparison between this theoretical $R^*_c(\alpha)$-value and the experimentally observed value of R^*_c implies the actual α and thus the desired actual $\overset{*}{\varepsilon}{}^e_z$-value for the considered problem. It follows from the equations just mentioned that in the case $\alpha = 1$, for example, the theoretical R^*_c-value for a composite unit cell with $v_m = 0.35$ is $R^*_c = 2.15\ r_f$ provided at the critical state both plastic zones as well as an elastically deformed annulus $r_c \leqq r \leqq r_m$ are still present.

A more sophisticated approach to the problem is based on the assumption that the first plastic zone $R_c \leqq r \leqq r_c$ presents itself as a thin layer and thus the relation $R_c = r_c$ approximately applies. Such an assumption seems to be a reasonable one for the following reasons. Firstly, since a low fibre concentration composite is considered, one should expect both R_c- and r_c-radii to be much smaller than r_m because of the local nature of the fibre concentration effect. Secondly, the matrix material is a typical ductile one with low resistance to the occurrence of developed plastic deformations such as just like the deformations within the second plastic zone. Therefore the transition region between this zone and the elastically-deformed zone might be really viewed as a thin layer.

The first plastic zone could be now considered to act as an elastic-plastic boundary where the latter has the form of a thin layer. The layer itself has the shape of a thin-walled circular cylinder with the mean radius R_c.

As usual, the standard transition conditions of continuity for the stresses and displacements should be satisfied over the elastic-plastic boundary which is classically presented by a certain mathematical surface. Because of the specific thin layer shape of the elastic-plastic boundary, a softened version of satisfying the continuity requirements could be applied in the considered case. This version involves the exact satisfaction of only two of these conditions. The first one is the requirement of continuity of the radial stress. The necessity of this requirement is obvious since the layer is assumed to be thin and thus one should not expect substantial change of the stress acting normal to the layer, i.e. of the radial stress within the layer itself. Let $\sigma^e_i, i = r, \theta, z$ be the normal stresses within the elastic region of the matrix and let $\sigma_i, i = r, \theta, z$ denote as above the corresponding stresses within the plastic zone. This requirement could then be written in the form

$$\sigma_r|_{R_c} = \sigma^e_r|_{R_c}. \tag{9.1}$$

The second requirement which has to be exactly satisfied is the fulfillment of the yield condition (5.3) over the elastic-plastic boundary. With the aid of the notations introduced this requirement could be written in the form

$$[(\sigma_r^e-\sigma_\theta^e)^2+(\sigma_\theta^e-\sigma_z^e)^2+(\sigma_z^e-\sigma_r^e)^2]|_{R_c}=2\sigma_y^2. \tag{9.2}$$

Using the equations of Sect. 5 and the known general form of the elastic solution [5, 10], one may rewrite the latter conditions in the form

$$\frac{E_m\,\overset{*}{\varepsilon}{}_z^e}{1-2\nu_m}+\frac{\sigma_y}{\sqrt{3}\sin\varphi}\cos(\omega_{R_c}+\varphi)=\frac{E_m\,C}{(1+\nu_m)\,r_m^2}\left(1-\frac{r_m^2}{R_c^2}\right), \tag{9.3}$$

$$\frac{3C^2}{R_c^4}=\frac{\sigma_y^2\,(1+\nu_m)^2}{E_m^2}-\left[\frac{C\,(1-2\nu_m)}{r_m^2}-(1+\nu_m)\,\varepsilon_z\right]^2, \tag{9.4}$$

where the constant C_0 has to be determined together with R_c.

It is important to mention at this point that with the conditions (9.1) and (9.2) satisfied or, equivalently, with the set (9.3), (9.4) solved for the quantities R_c and C, the stresses in both the elastic and plastic regions of the matrix are entirely determined by means of the current total axial strain ε_z and the quantity $\overset{*}{\varepsilon}{}_z^e$. One could now consider the remaining elastic-plastic transition conditions of continuity to be satisfied as well in the sense that the corresponding quantities (stresses and displacements) change continuously within the layer between their values over its elastic and plastic surfaces.

Moreover, Eqs. (9.1) and (9.2) or, equivalently, Eqs. (9.3) and (9.4) imply the desired dependence of the current plastic zone radius R_c on the current total axial strain ε_z. This dependence is, as easily seen, of the form

$$R_c=R_c\,(\varepsilon_z;\,\overset{*}{\varepsilon}{}_z^e,\,E_m,\,\nu_m,\,\sigma_y,\,r_m). \tag{9.5}$$

Provided the actual value of $\overset{*}{\varepsilon}{}_z^e$ is already known (for example from the experiment described above in this section) the problem of determination of the current $R_c\,(\varepsilon_z)$-value is solved. Moreover, assuming that the relation (9.5) is reversible and applying its inverse version with regard to the critical state of the composite unit cell, one obtains the critical value ε_z^* of the total axial strain at which the fibre-matrix interface fails. With account for Eq. (6.3) this value is given by an equation of the form

$$\varepsilon_z^*=\varepsilon_z^*\,(R_c^*,\,\overset{*}{\varepsilon}{}_z^e,\,E_m,\,\nu_m,\,\sigma_y,\,r_m). \tag{9.6}$$

To obtain the dependence of R_c on the current composite axial stress $\bar{\sigma}_z=P/\pi r_m^2$, P being the applied axial force, is now a matter of simple computation. Really, it could be easily verified that using again the equations of Sect. 5 and the general form of the elastic solution [5, 10] as well as the continuity condition for the radial stress at the fibre-matrix interface $r=r_f$ one may construct the expressions for the axial stress σ_z acting in each of the regions $0\leqq r\leqq r_f$, $r_f\leqq r\leqq R_c$, $R_c\leqq r\leqq r_m$. These expressions are, respectively,

$$\sigma_z^f=E_f\,\varepsilon_z+2\nu_f\left[\frac{E_m\,\overset{*}{\varepsilon}{}_z^e}{1-2\nu_m}+\frac{\sigma_y}{\sqrt{3}\sin\varphi}\cos(\omega_{r_f}+\varphi)\right], \tag{9.7}$$

(9.8) $$\sigma_z = \frac{1}{1-2\nu_m}(E_m \overset{*}{\varepsilon}{}^e_z + 2\sigma_y \cos \omega),$$

(9.9) $$\sigma^e_z = E_m\left(\varepsilon_z + \frac{2\nu_m}{1+\nu_m}\frac{C}{r^2_m}\right),$$

where $C = C(\varepsilon_z; \overset{*}{\varepsilon}{}^e_z, E_m, \nu_m, \sigma_y, r_m)$ is now known from the solution of the set (9.3), (9.4) and σ^f_z means the axial stress acting within the fibre.

The condition of equilibrium of forces acting in the axial direction is

(9.10) $$r^2_m \bar{\sigma}_z = r^2_f \sigma^f_z + 2\int_{r_f}^{R_c} \sigma_z \, r dr + (r^2_m - R^2_c)\,\sigma^e_z.$$

Introducing here the stresses from expressions (9.7), (9.8) and (9.9) one obtains, upon integration and using the relation (9.5), the desired dependence

(9.11) $$R_c = R_c(\bar{\sigma}_z; \overset{*}{\varepsilon}{}^e_z, E_m, \nu_m, E_f, \nu_f, \sigma_y, r_f, r_m).$$

When applied to the critical state of the unit composite cell, the inverse version of the latter equation defines in accordance with Eq. (6.3) the failure composite stress $\bar{\sigma}^*_z$ in the form

(9.12) $$\bar{\sigma}^*_z = \bar{\sigma}^*_z(\overset{*}{\varepsilon}{}^e_z, E_m, \nu_m, E_f, \nu_f, \sigma_y, r_f, r_m).$$

Equation (9.12) implies immediately a simple criterion of failure of the fibre-matrix interface of the form

$$\bar{\sigma}_z = \bar{\sigma}^*_z.$$

A similar criterion concerning the total axial strain follows from Eq. (9.6).

Finally, note that upon eliminating R_c from Eqs. (9.5) and (9.11) one obtains the equation of the $\bar{\sigma}_z$ versus ε_z curve for the considered composite in the form

(9.13) $$\bar{\sigma}_z = \bar{\sigma}_z(\varepsilon_z; \overset{*}{\varepsilon}{}^e_z, E_m, \nu_m, E_f, \nu_f, \sigma_y, r_f, r_m).$$

As it should be expected this curve depends upon the mechanical and geometrical properties of the composite as well as on the specific quantity $\overset{*}{\varepsilon}{}^e_z$. The comparison between this theoretically predicted curve and the experimentally-obtained one forms now the basis for another possible approach to the problem of determining of the $\overset{*}{\varepsilon}{}^e_z$-value for the considered composite material.

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Резюме

ПЛАСТИЧНОСТЬ КОМПОЗИТОВ АРМИРОВАННЫХ ВОЛОКНАМИ И ЯВЛЕНИЯ РАЗРУШЕНИЯ

Работа посвящена проблеме продольного растяжения однонаправленно армированного композита с хрупкими волокнами и растяхимой атрицей. Предложена модель процесса пластических деформаций по отношению к единичной ячейке композита. Модель воспроизводит в принципе растяжимость матрицы и выявляет сущность возможных механизмов разрушения на границ между матрицей и волокном. Модель приводит тоже к сравнительно простой схеме анализа целой проблемы, учитывая в частности определение границ пластической области, кривой напряжение-деформация для рассматриваемого композита, как и установления существенной меры предельной упругой реакции матрицы. Показано, что детальные решения, связанные с предлагаемой моделью, требуют простых численных расчетов. Некоторые существенные результаты допускают однако аналитического представления.

Streszczenie

PLASTYCZNOŚĆ KOMPOZYTÓW ZBROJONYCH WŁÓKNAMI I ZJAWISKA PĘKANIA

Praca poświęcona jest problemowi rozciągania podłużnego jednokierunkowo zbrojonego kompozytu o kruchych włóknach i ciągliwej matrycy. Zaproponowano model procesu odkształceń plastycznych w odniesieniu do pojedynczej komórki kompozytu. Model oddaje zasadniczo ciągliwość matrycy i wyjawia istotę możliwych mechanizmów zniszczenia na granicy między matrycą a włóknem. Model prowadzi również do stosunkowo prostego schematu analizy całego problemu uwzględniając w szczególności określenie granic obszaru plastycznego, krzywej naprężenie–odkształcenie dla rozważanego kompozytu jak i ustalenie istotnej miary granicznej reakcji sprężystej matrycy. Pokazano, że szczegółowe rozwiązania związane z zaproponowanym modelem wymagają prostych obliczeń numerycznych. Pewne istotne wyniki udało się nawet uzyskać w postaci zamkniętej.

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