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VIBRATIONS OF TIMOSHENKO BEAMS ON TWO-PARAMETER ELASTIC SOIL

N. M. Auciello

Department of Structural Engineering
University of Basilicata
Via dell'Ateneo Lucano, 10 – 85100, Potenza, Italy

In this paper the influence of the two-parameter elastic soil on the dynamic behaviour of a beam with variable cross-section is examined, in the presence of conservative axial loads. The beams are assumed to follow the well-known Timoshenko hypotheses, in order to take into account both the rotary inertia and shear deformation effect. The Rayleigh–Ritz approach is used and Boundary Characteristic Orthogonal Polynomials are chosen as trial functions; (BCOPs method [2]). The theory is concisely presented in a matrix form, so that the contribution of the rotary inertia and of the soil can be easily recognized. Various examples and comparisons are illustrated, in order to emphasize the influence of the soil properties and of the beam taper ratio. Finally, the results are also compared with the results given by other authors, using exact and approximate approaches.

Notations

H	depth of soil,
A, A_o	cross-sectional area of beam; cross-sectional area of beam in $x_1 = 0$,
$\mathbf{K}_U$, $\mathbf{K}_w$, $\mathbf{K}_G$	matrices in Eqs. (3.7), (3.8), (3.9),
E, G	Young's modulus; shear modulus of beam,
I, I_o	area moment of inertia; area moment of inertia in $x_1 = 0$,
k	shear factor,
k_w, k_G	Winkler, first coefficient of the elastic soil; second coefficient,
$\mathbf{K}$, $\mathbf{M}$	stiffness matrix; mass matrix,
L	length of the beam,
N, P	axial force; non-dimensional parameter,
P_c	critical buckling load parameter,
$\mathbf{q}_1$, $\mathbf{q}_2$	vector coefficients of trial function in Eqs. (3.1), (3.2),
r	radius of inertia of the beam, Eq. (4.4),
$\mathbf{v}$, $\mathbf{R}$	vectors Eq. (2.8),
U, U_P	strain energy; energy of axial force,
u_1, u_2, u_3	displacements of beam,
Ω_i	i-th non-dimensional eigenfrequency of beam; Eq. (4.4),
$\mathbf{\Phi}$, $\mathbf{\Psi}$	shape functions,
α	thickness ratio; Eq. (4.1),
φ	rotation of the cross-section,
γ_S	shear deformation,

γ	parameter of foundation; Eq. (2.13),
$\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$	stress and strain vectors,
υ_S	Poisson's coefficient,
ρ	mass density,
λ_w, λ_G	non-dimensional parameters of soil; Eq. (4.3),
ω_i	natural frequency.

1. Introduction

Various engineering problems can be traced back to the dynamic analysis of beams on elastic soil, and quite frequently the soil behaviour is approximated by the well-known Winkler model, according to which the soil is viewed as a distribution of mutually independent axial springs, thus neglecting the shear-contributed load causing constant displacements, and consequently – no bending of the beam. This drawback can be eliminated by adopting more refined two-parameter elastic models, which take into account the shear properties of the soil. Both the classical Filonenko and Pasternak models define an additional soil parameter in order to simulate an interaction between the springs, whereas Vlasov [8] aims to consider the influence of the elastic medium depth. According to this theory, Vallabhan and Das [9] proposed a variational procedure, which leads to a simplified form of the second elastic soil parameter.

Most contributions to the dynamic analysis of beams on a two-parameter elastic soil refer to slender beams, so that the classical Euler–Bernoulli hypotheses are usually accepted. Quite recently, a finite element procedure for the free vibration frequencies of slender beams on the Vlasov soil has been proposed by Franciosi and Masi [6].

If the beam cannot considered to be slender, it is convenient to adopt the Timoshenko theory, which takes into account both the shear deformations and the rotary inertia of the beam, and what nevertheless leads to a manageable differential problem. An exact solution for stiffness matrix for a Timoshenko beam on Winkler soil has been given by Chen and Panteldes [3], taking into account the effects of the axial forces, whereas De Rosa [5] has given the free vibration frequencies of Timoshenko beams with constant cross-section, resting on a two-parameter elastic soil, using two different models of the second soil parameter.

Semi-analytical and numerical approaches are obviously not limited to beams with constant cross-section. In the finite element context, a four-node element has been proposed by Yokoyama [10], for Rayleigh and Timoshenko beams, and the same author, in a later paper [11], considered the effects of axial forces and different boundary conditions. Finally, a refined cubic-quintic element has been implemented by Bruno *et al.* [4].

A different approach has been used by Filipich and Rosales [7], according to which the Rayleigh quotient is optimized and the fundamental frequency can be detected with great precision. On the other hand, the higher frequencies cannot be found whit sufficient accuracy.

In this paper we aim at a general method for estimation of the free vibration frequencies of Timoshenko beams with varying cross-section and non-classical boundary conditions, resting on varying two-parameter elastic soil. The analysis uses a variational Rayleigh-Ritz approach and sets of modified orthogonal polynomials, which can cope with different approximation degrees of displacements and rotations [2].

2. Formulation of the problem

Let us consider an isotropic beam with varying cross-section, resting on two-parameter elastic soil and subjected to a conservative axial load at the end. A Cartesian reference frame is x_1, x_2, x_3, such that x_1 becomes the beam axis, whereas x_2, x_3, are assumed to be the principal axes of the cross-section. If the Timoshenko model is assumed to be valid, then the displacements can be written as:

$$u_1 = -x_2\varphi(x_1,t), \qquad u_2 = u_2(x_1,t), \qquad u_3 = 0, \tag{2.1}$$

where $\varphi(x_1,t)$ is the rotation of the cross-section, which turns out to be different from the rotation θ of the neutral axis, so that the difference

$$\gamma_S = \frac{\partial u_2}{\partial x_1} - \varphi \tag{2.2}$$

gives the additional rotation due to the shear deformation.

According to (2.1), the strain components are given by:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} -x_2 \dfrac{d\varphi}{dx_1} \\ \dfrac{du_2}{dx_1} - \varphi \end{bmatrix}. \tag{2.3}$$

If the derivative with respect to x_1 is written as an apex, the Hooke's law for isotropic material gives the corresponding stress components:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \begin{bmatrix} EI & 0 \\ 0 & GA \end{bmatrix} \begin{bmatrix} -x_2\varphi' \\ u_2' - \varphi \end{bmatrix} = \begin{bmatrix} -EI\,x_2\varphi' \\ kGA\,(u_2' - \varphi) \end{bmatrix}, \tag{2.4}$$

where A is the cross-sectional area, I is the moment of inertia, E is the Young's modulus, G is the shear modulus, and k is the shear factor.

The strain energy can be written as:

$$U = \frac{1}{2}\int\limits_V \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}\, dx_1 = \frac{1}{2}\int\limits_V \boldsymbol{\varepsilon}^T \mathbf{D}\boldsymbol{\varepsilon}\, dx_1, \tag{2.5}$$

and using (2.3) and (2.4):

$$U = \frac{1}{2}\int_0^L \begin{bmatrix} -EI\varphi' \\ kGA(u_2' - \varphi) \end{bmatrix}^T \begin{bmatrix} -x_2\varphi' \\ u_2' - \varphi \end{bmatrix} dx_1 \tag{2.6}$$

$$= \frac{1}{2}\int_0^L \left[EI(\varphi')^2 + kGA(u_2' - \varphi)\right] dx_1,$$

after integration with respect to the cross-sectional area A.

The potential energy of the axial force N at the end is a quadratic function of the displacements, which can be written as:

$$U_P = \frac{N}{2}\int_0^L u_2' dx_1. \tag{2.7}$$

Finally, the kinetic energy of the system is equal to:

$$T = \frac{1}{2}\int_0^L \dot{\mathbf{v}}^T \mathbf{R}\, \dot{\mathbf{v}}\, dx_1, \tag{2.8}$$

where $\mathbf{v}$ and $\mathbf{R}$ are given by:

$$\mathbf{v} = \begin{bmatrix} u_2 \\ \varphi \end{bmatrix}, \tag{2.9}$$

$$\mathbf{R} = \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix}, \tag{2.10}$$

respectively, where ρ is the mass density of the beam.

From (2.8) it is possible to separate the variables, and the kinetic energy becomes

$$T = \frac{\omega^2}{2}\int_0^L \rho(u_2^2\, A + \varphi^2\, I)\, dx_1. \tag{2.11}$$

According to Winkler, the pressure at the generic point is linearly proportional to the corresponding displacement, but quite often this Winkler hypothesis cannot be considered to be valid and more refined pressure-displacement relationships must be accepted, as for example:

$$p(x_1) = k_w\, u_2 - k_G \frac{d^2 u_2}{dx_1^2}, \tag{2.12}$$

where the physical interpretation of the second parameter k_G varies according to the different model proposed. For example, the Filonenko–Borodich soil parameter k_G is the tensile force of an ideal membrane connecting the Winkler spring, whereas Pasternak assumes that the second parameter is equal to the shear force between the foundation and the soil.

A more refined model is considered by Vlasov, assuming that the foundation rests on an elastic half-plane, and some simplifying hypothesis allow us to express the second soil parameter as:

$$k_G = \gamma \frac{E_S}{(1 + \upsilon_S)}, \tag{2.13}$$

where E_S is the elastic modulus of the soil, υ_S is the Poisson coefficient, and γ is a coefficient which depends on the foundation geometry. If E_S and υ_S are assumed to vary linearly with the depth H, a variational procedure, as suggested by VALLABHAN and DAS [9], gives a simple expression for the elastic soil parameters.

In any case, the strain energy of the soil can be calculated by using (2.12), and regardless of the particular model, it can be written as:

$$U_S = \int_0^L \left[k_w\, u_2^2 + k_G \left(\frac{d^2 u_2}{dx_1^2} \right)^2 \right] dx_1. \tag{2.14}$$

3. Approximate analytical solution

An approximate solution for the problem at hand can be obtained by assuming that the displacements u_2 and the rotations ϕ can be expressed as

$$\hat{u}_2(x_1) = \sum_{i=1}^{n} a_i \varPhi_i = \mathbf{\Phi}^T \mathbf{q}_1, \tag{3.1}$$

$$\hat{\varphi}(x_1) = \sum_{i=1}^{n} b_i \varPsi_i = \mathbf{\Psi}^T \mathbf{q}_2, \tag{3.2}$$

where $\mathbf{q}_1$ and $\mathbf{q}_2$ play the role of generalized coordinates, whereas $\varPhi_i$ and $\varPsi_i$ are the shape functions which must obey the only geometric boundary conditions. If this expression are inserted into the strain energy formulae, then a discrete structural system is obtained, with a finite number of degrees of freedom. The strain energy (2.6) becomes:

$$(3.3)\quad U = \frac{1}{2}\int_0^L EI(\mathbf{q}_2^T \boldsymbol{\Psi}'\boldsymbol{\Psi}'^T \mathbf{q}_2)dx_1$$
$$= \frac{1}{2}\int_0^L kGA\left[\boldsymbol{\Phi}'^T\mathbf{q}_1 - \boldsymbol{\Psi}^T\mathbf{q}_2\right]^T\left[\boldsymbol{\Phi}'^T\mathbf{q}_1 - \boldsymbol{\Psi}^T\mathbf{q}_2\right]dx_1,$$

where the strain energy (2.14) due to the elastic soil is given by

$$(3.4)\qquad U_S = \frac{1}{2}\int_0^L k_w\left(\boldsymbol{\Phi}^T\mathbf{q}_1\right)^T\left(\boldsymbol{\Phi}^T\mathbf{q}_1\right)dx_1 = \frac{1}{2}\int_0^L k_G\left(\boldsymbol{\Phi}'^T\mathbf{q}_1\right)^T\left(\boldsymbol{\Phi}'^T\mathbf{q}_1\right)dx_1$$

and the potential energy of the axial load (2.7) transform as follows:

$$(3.5)\qquad U_P = \frac{N}{2}\int_0^L \mathbf{q}_1\boldsymbol{\Phi}'\boldsymbol{\Phi}'^T\mathbf{q}_1 dx_1.$$

If the elastic soil parameters k_w and k_G are assumed to be constant along the beam axis, then the total potential energy of the system can be written as

$$(3.6)\qquad U_t = \frac{1}{2}\mathbf{q}^T\left[\mathbf{K}_U + k_w\mathbf{K}_w + (k_G - N)\mathbf{K}_G\right]\mathbf{q} = \frac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q},$$

where the coordinates have been substituted into the column vector $\mathbf{q} = [\mathbf{q}_1\ \mathbf{q}_2]^T$ and:

$$(3.7)\quad \mathbf{K}_U = kG\int_0^L A\begin{bmatrix}\boldsymbol{\Phi}'\boldsymbol{\Phi}'^T & -\boldsymbol{\Psi}\boldsymbol{\Phi}'^T\\ -\boldsymbol{\Psi}\boldsymbol{\Psi}'^T & \boldsymbol{\Psi}\boldsymbol{\Psi}^T\end{bmatrix}dx_1 + E\int_0^L I\begin{bmatrix}0 & 0\\ 0 & \boldsymbol{\Psi}'\boldsymbol{\Psi}'^T\end{bmatrix}dx_1,$$

$$(3.8)\qquad \mathbf{K}_w = \int_0^L\begin{bmatrix}\boldsymbol{\Phi}\boldsymbol{\Phi}^T & 0\\ 0 & 0\end{bmatrix}dx_1,$$

$$(3.9)\qquad \mathbf{K}_G = \int_0^L\begin{bmatrix}\boldsymbol{\Phi}'\boldsymbol{\Phi}'^T & 0\\ 0 & 0\end{bmatrix}dx_1.$$

The $\mathbf{K}_U$ matrix is the sum of the bending and shear stiffness matrices, whereas $\mathbf{K}_w$ is the Winkler soil stiffness matrix. The stiffening effect of the second soil parameter is clearly indicated in (3.6), because k_G and the axial force N both multiply the same geometric stiffness matrix $\mathbf{K}_G$.

Finally, if the mass density is assumed to be constant along the beam, then the assumptions (3.1) and (3.2) lead to the following matrix form of the kinetic energy (2.11):

$$T = \frac{\omega^2}{2}\int_0^L \rho\left[A\left(\mathbf{q}_1\boldsymbol{\Phi}\boldsymbol{\Phi}^T\mathbf{q}_1\right) + I\left(\mathbf{q}_2\boldsymbol{\Psi}\boldsymbol{\Psi}^T\mathbf{q}_2\right)\right]dx_1 = \frac{\omega^2}{2}\mathbf{q}^T\mathbf{M}\mathbf{q}, \tag{3.10}$$

where

$$\mathbf{M} = \int_0^L \rho\begin{bmatrix} A\boldsymbol{\Phi}\boldsymbol{\Phi}^T & 0 \\ 0 & I\boldsymbol{\Psi}\boldsymbol{\Psi}^T \end{bmatrix} dx_1. \tag{3.11}$$

The mass matrix $\mathbf{M}$ can be divided into the mass matrix due to the transverse displacements and the mass matrix due to the rotary inertia. A trivial application of the well-known Hamilton principle leads to the following eigenvalue problem:

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{q} = 0, \tag{3.12}$$

which in turn leads to the frequency equation

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = 0. \tag{3.13}$$

It has been already mentioned that the shape functions must obey only the geometric boundary conditions, so that it will be possible to write:

$$\varPhi_1(x_1) = \sum_{j=0}^{n_u} a_j x_1^j, \tag{3.14}$$

$$\varPsi_1(x_1) = \sum_{j=0}^{n_\varphi} b_j x_1^j, \tag{3.15}$$

where n_u and n_φ are the geometric conditions which must be imposed on the vertical displacements and rotations, respectively. The coefficients a_i and b_i can be determined imposing the boundary conditions, whereas the higher-order functions can be sought by means of the *Gram–Schmidt* [12] iterative method.

The geometric boundary conditions at the ends of the beam can be written as follows:

Pinned–Pinned:

$$x_1 = 0, \qquad x_1 = L \Rightarrow u_1 = 0\,; \tag{3.16}$$

Pinned–Clamped:

$$x_1 = 0 \Rightarrow u_1 = 0, \qquad x_1 = L \Rightarrow \left[\begin{array}{c} u_1 = 0 \\ u_{1,1} = 0 \end{array} \right. ; \tag{3.17}$$

Clamped–Free:

$$x_1 = 0 \Rightarrow \left[\begin{array}{c} u_1 = 0 \\ u_{1,1} = 0 \end{array} \right. , \qquad x_1 = L \Rightarrow \left[\begin{array}{c} u_1 \neq 0 \\ u_{1,1} \neq 0 \end{array} \right. . \tag{3.18}$$

4. Numerical examples

In order to test the method suggested above, some numerical examples have been performed, for a beam with arbitrarily varying cross-section, with the area and moment of inertia given by the general relationships:

$$A(x_1) = A_0 \left[1 + \alpha \frac{x_1}{L} \right], \tag{4.1}$$

$$I(x_1) = I_0 \left[1 + \alpha \frac{x_1}{L} \right]^3, \tag{4.2}$$

where A_0 and I_0 are the cross-sectional area and moment of at the abscissa $x_1 = 0$. It is also usual to introduce the following non-dimensional parameters:

$$P = \frac{NL^2}{\pi^2 EI_0}, \qquad \lambda_w = \frac{k_w L^2}{EI_0}, \qquad \lambda_G = \frac{k_G L^4}{\pi^2 EI_0}, \tag{4.3}$$

whereas the free vibration frequencies are usually written as:

$$\Omega_i^2 = \omega_i^2 L^4 \frac{\rho A_0}{EI_0}, \qquad r^2 = \frac{I_0}{A_0}. \tag{4.4}$$

As the first comparison, let us consider the beams with constant cross-section, subjected to axial forces as studies by YOKOYAMA [11] by means of a finite element approach. The Poisson coefficient is equal to 0.25, $E/G = 2.5$, the cross-section is assumed to be rectangular, and consequently, the shear factor is given by $k = 2/3$. In the following we have used 5 polynomial trial function in order to approximate both the displacements and rotations, so that the resulting problem has 10 degrees of freedom. The first three frequency coefficients Ω_i have been calculated for pinned-pinned (P-P) beam and for an pinned-clamped (P-C) beam. The results are given in Table 1 together with the free frequencies as given in [11] for a finite element mesh with 16 elements. The full agreement with the exact frequencies is quite evident, small discrepancies can be noticed only for the higher frequencies, but the error turns out to be smaller than 0.2%.

Table 1. First three non-dimensional frequencies for beams (constant cross-section).

P	λ_W	λ_G	P-P			P-C		
			Exact	Present	[11]	Exact	Present	[11]
0	0	0	8.210	8.214	8.220	10.630	10.626	10.630
			24.230	24.228	24.310	25.620	25.616	25.710
			41.540	41.545	41.960	42.030	42.035	42.460
0.6			3.470	3.466	3.470	7.320	7.323	7.330
			19.220	19.280	19.310	20.930	20.931	21.030
			35.080	35.352	35.480	35.700	35.750	36.160
	$0.6\,\pi^4$		8.210	8.214	8.220	10.460	10.481	10.490
			20.590	20.645	20.670	22.200	22.207	22.300
			35.860	36.126	36.250	36.500	36.508	36.900
		1		12.638	12.640		14.419	14.420
				28.075	28.100		29.248	29.340
				46.191	46.340		46.281	46.710

In order to study the influence of the soil parameters, let us consider, as a first example, the pinned-pinned beam and the clamped-free (C-F) beam with constant cross-section in the absence of axial forces. The first two frequency parameters are given in Figs. 1, 2 and in Figs. 3, 4 for the (P-P) beam and for the cantilever beam, respectively, where the solid lines refer to Ω_1 and the dashed

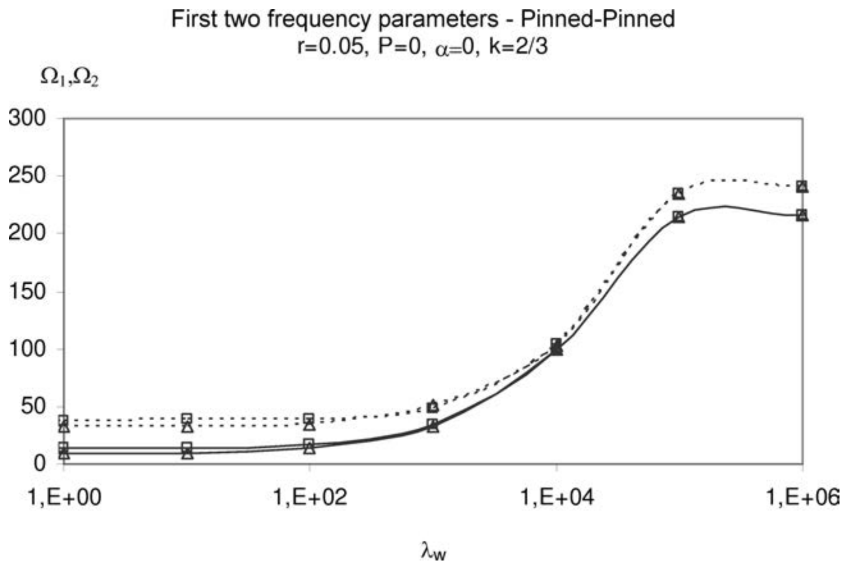

FIG. 1.

lines refer to Ω_2. Two different λ_G values have been considered, i.e. $\lambda_G = 0$ (△) and $\lambda_G = 1$ (□). It is worth noting that, regardless of the r value, the influence of λ_G is reduced as λ_w increases, and the stiffening effect for large λ_w values causes the coalescence of the first two vibration frequencies.

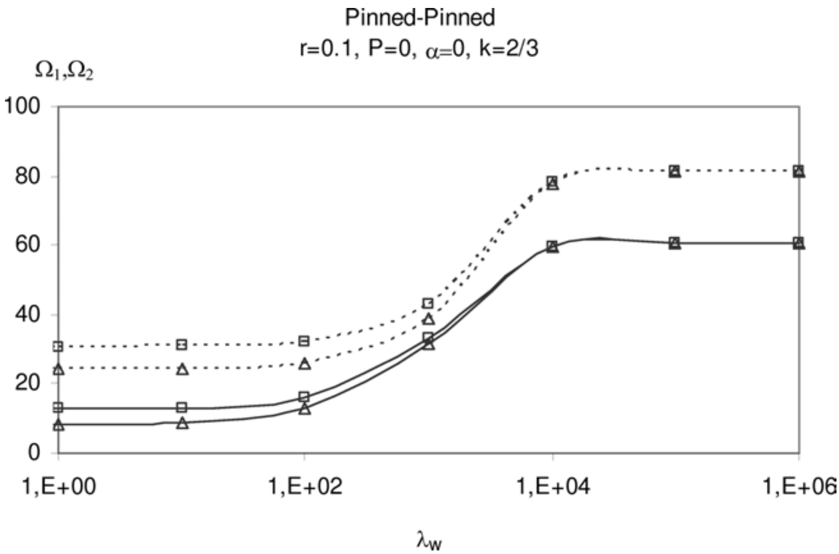

Fig. 2.

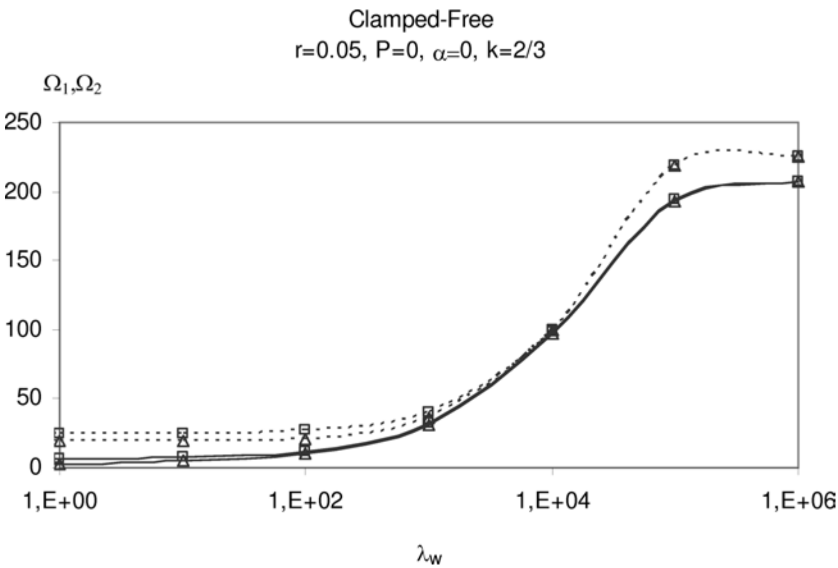

Fig. 3.

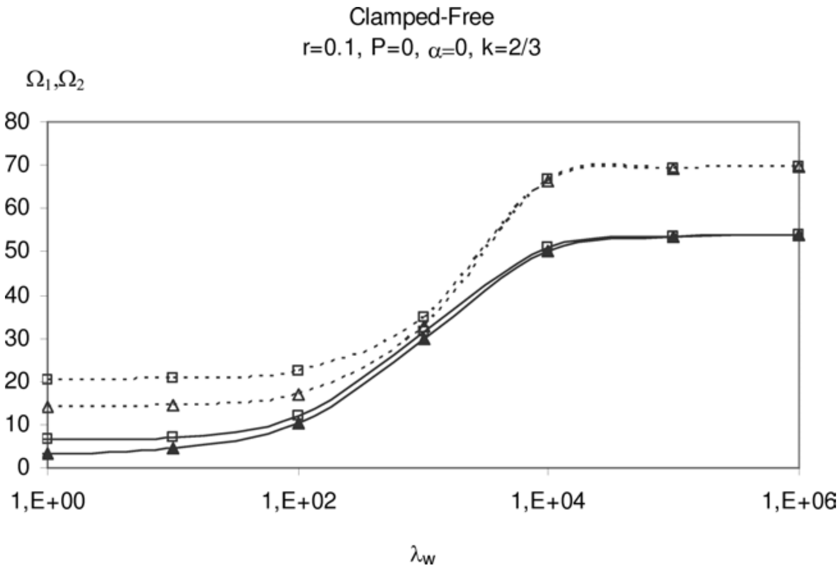

Fig. 4.

The influence of axial load and of the taper ratio (α) on the fundamental frequencies is illustrated by the graphs in Figs. 5–7. For all the boundary conditions the frequency parameter Ω goes to zero as $P/P_c \to -1$. Finally, the influence of the taper ratio α seems to be relevant for the cantilever beam, whereas it is less important for simply-supported beam and pinned-clamped beam. In Table 2 the non-dimensional critical loads are given, which have been used to obtain the previous pictures.

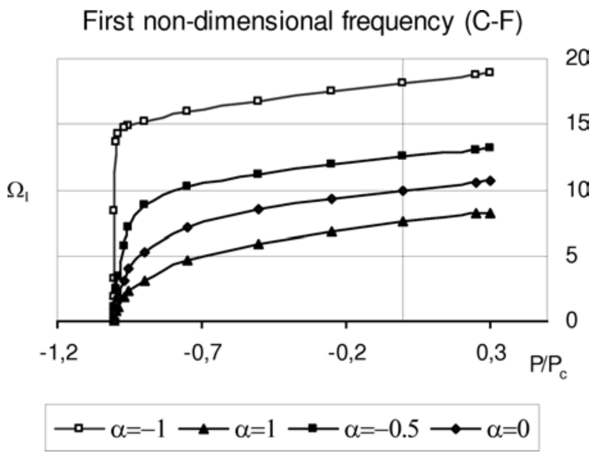

Fig. 5.

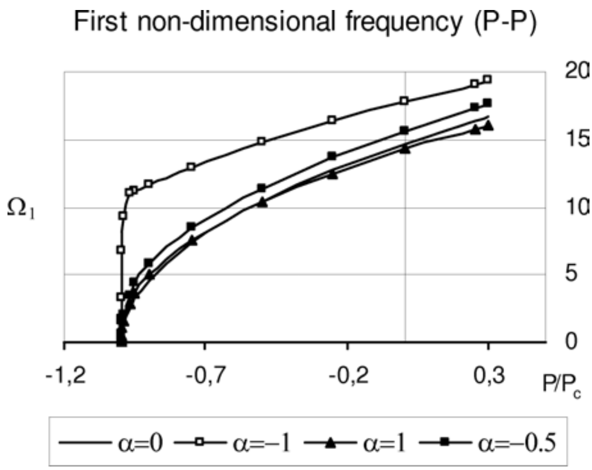

FIG. 6.

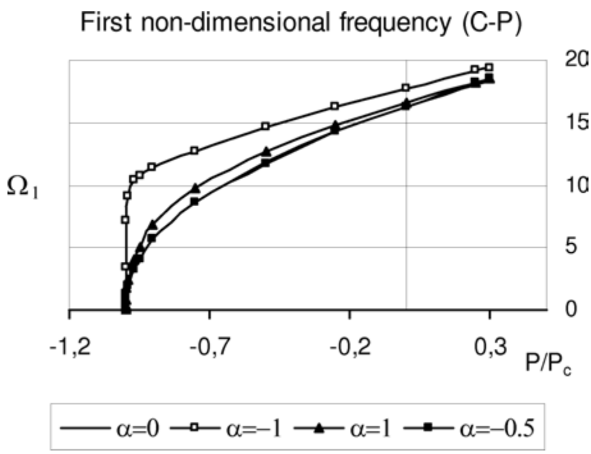

FIG. 7.

Table 2. Critical parameter (P_c) for different boundary conditions.

$r = 0.1$ $\lambda_G = 1$ $\lambda_W = 0.6\ \pi^4$ $k = 2/3$										
α	-1	-0.9	-0.75	-0.5	0	0.25	0.5	0.75	1	1.25
C-F	1.0255	1.0643	1.178	1.373	1.7415	1.904	2.056	2.19	2.323	2.434
P-F	1.0254	1.0665	1.1779	1.362	1.7269	1.894	2.045	2.177	2.289	2.382
P-P	1.0612	1.1463	1.3946	1.785	2.3298	2.546	2.749	2.935	3.104	3.253
P-C	1.1467	1.3443	1.5951	2.036	2.5908	2.827	3.031	3.204	3.359	3.464

The use of BCOPs method to calculate free vibration frequencies and critical load is always influenced by trial functions, and a careful choice leads to well approximated results. In turn, the trial functions depend on the boundary conditions, so that it seems to be convenient to use polynomials.

5. Conclusion

In this paper, a powerful version of the Rayleigh–Ritz variational method has been applied to the vibration analysis of Timoshenko beams on a two-parameter elastic soil. The influence of various structural parameters on the behaviour of the free vibration frequencies has been illustrated in various numerical examples.

The proposed approach belongs to the so-called semi-analytical methods (SAN methods), and as such it can be considered as a useful tool in purely numerical approaches (finite element methods, differential quadrature methods etc.), in which all the parameters must be defined from the very beginning.

The numerical examples show the reliability of the method, and the particular efficiency of the chosen trial functions.

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