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RESISTANCE OF RC ANNULAR CROSS-SECTIONS WITH OPENINGS SUBJECTED TO AXIAL FORCE AND BENDING

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This paper presents ultimate limit state analysis of the resistance of reinforced concrete (RC) annular cross-sections with openings, subjected to the axial force and the bending moment. Based on nonlinear material laws for concrete and reinforcing steel and using the method of mathematical induction, analytical formulae are derived in the case when the cross-section is weakened by an arbitrary number of openings located symmetrically with respect to the bending direction. In this approach, the additional reinforcement at openings is also taken into account. The results of numerical calculations are presented in the form of interaction diagrams with the design values of the normalized cross-sectional forces n_u and m_u for the sections weakened by openings as well as for the closed ones. This approach has been applied to investigate the influence of different parameters such as the size and the number of openings, the reinforcement ratio, the additional reinforcement at the opening, the form of stress-strain relationships for concrete and the thickness/radius ratio, on the section resistance.

Notations

E_s	modulus of elasticity of steel,
F_{ad1}	area of the additional reinforcement at the opening specified by α_1,
F_{ad2}	area of the additional reinforcement at the opening specified by α_2,
M_{Sd}	design bending moment,
M_u	ultimate bending moment,
N_{Sd}	design axial force,
N_u	ultimate axial force,
R	outer radius of ring,
c_{cs}	coefficient of concrete softening,
c_{sh}	coefficient of steel hardening,
f_{ck}	characteristic strength of concrete in compression,
f_{yk}	yield stress of steel,
$m_{Rd} = m_u$	design normalized ultimate bending moment,
$n_{Rd} = n_u$	design normalized ultimate axial force,
r	inner radius of the ring,
r_m	mean radius of the ring,
r_s	radius of the circumference on which reinforcing steel is located,
$t = R - r$	thickness of the cross-section,

α angle describing the location of the neutral axis ($\alpha_1 \leq \alpha \leq \alpha_2$), rad,
α_1 angle describing the location of the first opening, rad,
α_2 angle describing the location of the second opening, rad,
m number of openings,
(α_1, α_2), (α_3, α_4), ..., (α_{m-1}, α_m) couples of angular coordinates determining the locations of openings, rad,
α_b angle determining the depth of the zone of plastified concrete, rad,
α_{a1} angle determining the depth of the zone of plastified steel in compression, rad,
α_{a2} angle determining the depth of the zone of plastified steel in tension, rad,
ε strain expressed in ‰,
ε_c strain in concrete, [‰],
ε_{cu} ultimate strain in concrete, [‰],
ε_s strain in steel, [‰],
ε_{su} ultimate strain in steel, [‰],
ε_{sy} strain related to the yield stress of steel, [‰],
ε_0 the given numerical parameter,
γ_c partial safety factor for concrete,
γ_s partial safety factor for steel,
μ the ratio of cross-sectional areas, steel to concrete,
$\mu_{\alpha 1}$, $\mu_{\alpha 2}$ the ratios of cross-sectional areas, additional reinforcement located at the openings specified by α_1, α_2 to concrete,
$\mu_{\alpha i}$ the ratio of cross-sectional areas, additional reinforcement located at the opening side specified by α_i to concrete,
σ_c compressive stress in concrete,
σ_s stress in steel.

1. Introduction

Structures and members with the annular cross-section weakened by openings subjected to the axial force and bending moment are frequently encountered in engineering practice (towers, chimneys, lamp posts, columns etc.).

Determination of the resistance of the cross-sections of RC chimneys and tower structures has been reported in the literature by several authors. The ultimate load analysis of a shell with a circular cross-section weakened by one and two openings is presented in the monograph by Pinfold [1]. A similar approach is also used by Nieser and Engel [2], in DIN 1056 and CICIND codes [3, 4], assuming the central layout of steel reinforcement in the wall of tower or chimney structures and ignoring the effect of additional reinforcement at the sides of openings. The generalized linear section model for analysis of RC chimneys weakened by openings was proposed by Lechman and Lewiński [5].

When RC cross-sections under consideration are subjected to the given design axial force N_{Sd} and bending moment M_{Sd} and a nonlinear behavior of concrete and steel reinforcement is assumed, the problem is described mathematically by a set of equations which are nonlinear and difficult to solve. Therefore, a numerical strategy must be used. For this purpose the modified BFGS method has been successfully applied by Lechman and Stachurski [6].

Despite the generality of the papers mentioned, there are no appropriate analytical formulae for determining the resistance of the annular sections weakened by an arbitrary number of openings and taking into consideration the physical nonlinearity of concrete and reinforcing steel. Such a task has been undertaken in the present paper.

2. Derivation of formulae for the section with one or two diametrically opposite openings

As the first step of the proposed approach, the annular cross-section, described by the outer radius – R and the inner radius – r, is assumed to be weakened by one or two openings. The locations of the openings are determined by couples of the angular coordinates $(0, \alpha_1)$, (α_2, π), $0 \leq \alpha_1 \leq \alpha_2 \leq \pi$. The reinforcing steel spaced in a general case continuously at l layers can be replaced by a continuous ring of equivalent area located on the reference circumference of radius r_s (Fig. 1a). The section under consideration is subjected to the axial force N_u and the bending moment M_u at ultimate limit state. If $\alpha_1 \neq 0$ and $\alpha_2 = \pi$, it forms the cross-section weakened by a single opening, while $\alpha_1 = 0$ and $\alpha_2 = \pi$ describe the closed annular one.

In the present derivation, the following assumptions have been introduced:

(i) plane cross-sections remain plane,

(ii) the tensile strength of concrete is ignored,

(iii) the reinforcement in both the tension and compression zone is taken into account,

(iv) the thickness of the section is thin compared with its diameter,

(v) elasto-plastic stress/strain relationships for concrete and steel are used,

(vi) the ultimate strain for concrete is defined as $-3.5‰$ or $-2‰$, while for reinforcement as $5‰$ (tension) and $-5‰$ (compression).

For determining the resistance of cross-sections, the stress-strain relationships for concrete in compression with softening in the plastic range is given by (Fig. 1b):

$$(2.1)\qquad \begin{aligned} \sigma_c &= \frac{f_{ck}}{\gamma_c}\frac{2}{\varepsilon_0}\left(1+\frac{\varepsilon}{2\varepsilon_0}\right)\varepsilon, && \text{for } -\varepsilon_0 \leq \varepsilon \leq 0, \\ \sigma_c &= -\frac{f_{ck}}{\gamma_c}\left(1-c_{cs}\frac{\varepsilon+\varepsilon_0}{\varepsilon_{cu}+\varepsilon_0}\right), && \text{for } -3.5 \leq \varepsilon \leq -\varepsilon_0, \end{aligned}$$

where ε_0 – the given numerical parameter, f_{ck} – characteristic strength of concrete in compression, γ_c – partial safety factor for concrete, $c_{cs} = (f_{cd} - f_{cu})/f_{cd}$ – coefficient of concrete softening in the plastic range, f_{cd} – design value of the compressive strength of concrete, $f_{cu} = \sigma_c(\varepsilon_{cu})$.

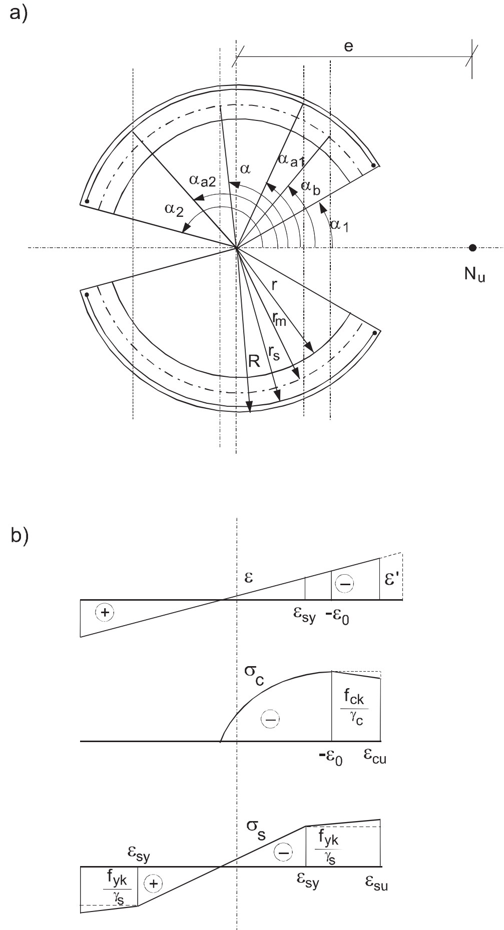

FIG. 1. a) The cross-section weakened by two openings, b) distribution of strains ε, stresses in concrete σ_c and in steel σ_s across the section.

To determine the resistance of the cross-sections, the stress-strain relations for reinforcing steel with hardening in the plastic range are assumed as (Fig. 1b):

$$
\begin{aligned}
\sigma_s &= \frac{f_{yk}}{\varepsilon_{ss}}\varepsilon, && \text{for} \quad -\varepsilon_{sy} \le \varepsilon \le \varepsilon_{sy},\\
\sigma_s &= \frac{f_{yk}}{\gamma_s}\left(1 + c_{sh}\frac{\varepsilon - \varepsilon_{sy}}{\varepsilon_{su} - \varepsilon_{sy}}\right), && \text{for} \quad \varepsilon_{sy} \le \varepsilon \le 10,\\
\sigma_s &= -\frac{f_{yk}}{\gamma_s}\left(1 - c_{sh}\frac{\varepsilon + \varepsilon_{sy}}{\varepsilon_{su} - \varepsilon_{sy}}\right), && \text{for} \quad -10 \le \varepsilon \le -\varepsilon_{sy},\\
\varepsilon_{ss} &= \frac{f_{yk}}{E_s}, \qquad \varepsilon_{sy} = \frac{\varepsilon_{ss}}{\gamma_s}, &&
\end{aligned}
\tag{2.2}
$$

where f_{yk} – yield stress of steel, E_s – modulus of elasticity of steel, γ_s – partial safety factor for steel, c_{sh} – coefficient of steel hardening in the plastic range expressed as:

$$c_{sh} = \frac{f_{su} - f_{yd}}{f_{yd}},$$

f_{yd} – design value of the yield stress of steel, $f_{su} = \sigma_s(\varepsilon_{su})$.

Let us consider the cross-section under combined compression and bending. Due to the Bernoulli assumption we obtain:

$$\text{(2.3)} \qquad \begin{aligned} \varepsilon_c &= \frac{\cos\varphi - \cos\alpha}{\rho_R - \cos\alpha}\varepsilon' = (\cos\varphi - \cos\alpha)\varepsilon'_\alpha, \\ \varepsilon_s &= \frac{\rho\cos\varphi - \cos\alpha}{\rho_R - \cos\alpha}\varepsilon' = (\rho\cos\varphi - \cos\alpha)\varepsilon'_\alpha, \end{aligned}$$

where ε' – the maximum compressive strain in concrete at the point $(0, R)$, [‰], α – angle describing the location of the neutral axis, rad, φ – angular coordinate, rad, ρ – coefficient, $\rho = r_s/r_m$, ρ_R – coefficient, $\rho_R = R/r_m$, $\varepsilon'_\alpha = \dfrac{\varepsilon'}{\rho_R - \cos\alpha}$.

The conditions of the strain conformity for the concrete and the steel in compression and in tension are expressed, respectively, by

$$\text{(2.4)} \qquad (\cos\alpha_b - \cos\alpha)\varepsilon'_\alpha = -\varepsilon_0,$$

$$\text{(2.5)} \qquad (\rho\cos\alpha_{a1} - \cos\alpha)\varepsilon'_\alpha = -\varepsilon_{sy},$$

$$\text{(2.6)} \qquad (\rho\cos\alpha_{a2} - \cos\alpha)\varepsilon'_\alpha = \varepsilon_{sy},$$

where α_b – angle determining the depth of the zone of the plastified concrete, α_{a1} – angle determining the depth of the zone of the plastified steel in compression, α_{a2} – angle determining the depth of the zone of the plastified steel in tension.

The resistance of the cross-section is reached when either the ultimate strain in concrete ε_{cu} or in steel ε_{su} is reached anywhere in that section, i.e. the following conditions must be satisfied:

$$\text{(2.7)} \qquad (\cos\alpha_1 - \cos\alpha)\varepsilon'_\alpha = \varepsilon_{cu},$$

$$\text{(2.8)} \qquad (\rho\cos\alpha_2 - \cos\alpha)\varepsilon'_\alpha = \varepsilon_{su}.$$

On the basis of a combinatorial approach, eight possible forms of the stress distribution in the section are to be considered:

1) elastic phase of the concrete and steel,
2) plastic phase of the concrete, elastic phase of the steel,

3) plastic phase of the concrete and the steel in compression, elastic phase of the steel in tension,
4) plastic phase of the concrete and the steel in tension, elastic phase of the steel in compression,
5) elastic phase of the concrete and the steel in compression, plastic phase of the steel in tension,
6) elastic phase of the concrete and the steel in tension, plastic phase of the steel in compression steel,
7) elastic phase of the concrete, plastic phase of the steel in compression and the steel in tension,
8) plastic phase of the concrete and steel.

Let us consider the case 8). The equilibrium equation of the normal forces in the cross-section weakened by one or two openings at ultimate limit state takes the following form:

$$2\left(\int_{\alpha_1}^{\alpha_b} \sigma_c^{pl} dA_c + \int_{\alpha_b}^{\alpha} \sigma_c^{e} dA_c\right) + 2\left(\int_{\alpha_1}^{\alpha_{a1}} \sigma_s^{pl} dA_s + \int_{\alpha_{a1}}^{\alpha_{a2}} \sigma_s^{e} dA_s + \int_{\alpha_{a2}}^{\alpha_2} \sigma_s^{pl} dA_s\right) + 2F_{a\alpha1}\sigma_s^{pl}(\alpha_1) + 2F_{a\alpha2}\sigma_s^{pl}(\alpha_2) + N_u = 0, \tag{2.9}$$

where σ_c^{pl} – the stress function of concrete in the plastic range given by $(2.1)_2$, σ_c^e – the stress function of concrete in the elastic range given by $(2.1)_1$, σ_s^{pl} – the stress function of steel in the plastic range given by $(2.2)_2$, $(2.2)_3$, σ_s^e – the stress function of steel in the elastic range given by $(2.2)_1$, dA_c – element of the concrete area, dA_s – element of the steel area, $F_{a\alpha1}$ – area of the additional reinforcement at the opening specified by α_1, $F_{a\alpha2}$ – area of the additional reinforcement at the opening specified by α_2.

Using the relation $dA_c + dA_s = dA = r_m t d\varphi$, the equilibrium equation of the sectional bending moments at ultimate limit state with respect to the line perpendicular to the symmetry axis and crossing it at the centre of the section, can be written in the form

$$r_m t(1-\mu)\left(\int_{\alpha_1}^{\alpha_b} \sigma_c^{pl} r_m \cos\varphi d\varphi + \int_{\alpha_b}^{\alpha} \sigma_c^{e} r_m \cos\varphi d\varphi\right) + r_m t\mu\left(\int_{\alpha_1}^{\alpha_{a1}} \sigma_s^{pl} r_s \cos\varphi d\varphi + \int_{\alpha_{a1}}^{\alpha_{a2}} \sigma_s^{e} r_s \cos\varphi d\varphi + \int_{\alpha_{a2}}^{\alpha_2} \sigma_s^{pl} r_s \cos\varphi d\varphi\right) + F_{a\alpha1}\sigma_s^{pl}(\alpha_1) r_s \cos\alpha_1 + F_{a\alpha2}\sigma_s^{pl}(\alpha_2) r_s \cos\alpha_2 + M_u = 0. \tag{2.10}$$

Taking into account the relationships (2.1)–(2.3), after integration and rearrangement of (2.9), (2.10) we obtain

$$\alpha = \text{arc }\cos\left(\frac{\rho_R(\varepsilon_{su}\cos\alpha_1 - \varepsilon_{cu}\cos\alpha_2)}{\varepsilon_{su}-\varepsilon_{cu}}\right), \tag{2.11}$$

$$\varepsilon' = \varepsilon_{cu}\frac{\rho_R - \cos\alpha}{\rho_R\cos\alpha_1 - \cos\alpha}, \tag{2.12}$$

$$\alpha_b = \begin{cases} \alpha_1 & \text{elastic phase,} \\ \text{arc }\cos\left(\cos\alpha - \varepsilon_0\dfrac{1}{\varepsilon'_\alpha}\right) & \text{plastic phase,} \end{cases} \tag{2.13}$$

$$\alpha_{a1} = \begin{cases} \alpha_1 & \text{elastic phase,} \\ \text{arc }\cos\left[\dfrac{1}{\rho}\left(\cos\alpha - \varepsilon_{sy}\dfrac{1}{\varepsilon'_\alpha}\right)\right] & \text{plastic phase,} \end{cases} \tag{2.14}$$

$$\alpha_{a2} = \begin{cases} \alpha_2 & \text{elastic phase,} \\ \text{arc }\cos\left[\dfrac{1}{\rho}\left(\cos\alpha + \varepsilon_{sy}\dfrac{1}{\varepsilon'_\alpha}\right)\right] & \text{plastic phase,} \end{cases} \tag{2.15}$$

$$\begin{aligned} n_u = -\frac{1}{\pi}\Bigg\{ & -\frac{1-\mu}{\gamma_c}[X_7(\alpha_b) + \frac{c_{cs}}{\varepsilon_{cu}+\varepsilon_0}[\varepsilon'_\alpha X_4(\alpha,\alpha_b) + \varepsilon_0 X_7(\alpha_b)] \\ & + \frac{1-\mu}{\gamma_c}\varepsilon'_\alpha\frac{2}{\varepsilon_0}\left[X_1(\alpha,\alpha_b) + \frac{1}{2\varepsilon_0}\varepsilon'_\alpha X_2(\alpha,\alpha_b)\right] + \mu\frac{f_{yk}}{f_{ck}}\Bigg\{-\frac{1}{\gamma_s}\,[X_8(\alpha_{a1}) \\ & + \frac{c_{sh}}{\varepsilon_{su}-\varepsilon_{sy}}[\varepsilon'_\alpha X_5(\alpha,\alpha_{a1}) - \varepsilon_{sy}X_8(\alpha_{a1})] + \frac{1}{\varepsilon_{ss}}\varepsilon'_\alpha X_3(\alpha,\alpha_{a1},\alpha_{a2}) \\ & + \left\{\frac{1}{\gamma_s}[X_9(\alpha_{a2}) + \frac{c_{sh}}{\varepsilon_{su}-\varepsilon_{sy}}[\varepsilon'_\alpha X_6(\alpha,\alpha_{a2}) - \varepsilon_{sy}X_9(\alpha_{a2})]\right\} \\ & + \frac{f_{yk}}{f_{ck}}\mu_{\alpha 1}\Bigg\{-\delta_{k1}\frac{1}{\gamma_s}\left[1 - \frac{c_{sh}}{\varepsilon_{su}-\varepsilon_{sy}}[\varepsilon'_\alpha(\rho\cos\alpha_1 - \cos\alpha) + \varepsilon_{sy}]\right] \\ & + \delta_{k1+1}\,\frac{\varepsilon'_\alpha}{\varepsilon_{ss}}(\rho\cos\alpha_1 - \cos\alpha)\Bigg\} \\ & + \frac{f_{yk}}{f_{ck}}\mu_{\alpha 2}\Bigg\{\delta_{k2}\frac{1}{\gamma_s}\left[1 + \frac{c_{sh}}{\varepsilon_{su}-\varepsilon_{sy}}[\varepsilon'_\alpha(\rho\cos\alpha_2 - \cos\alpha) - \varepsilon_{sy}]\right] \\ & + \delta_{k2+1}\,\frac{\varepsilon'_\alpha}{\varepsilon_{ss}}(\rho\cos\alpha_2 - \cos\alpha)\Bigg\}\Bigg\}. \end{aligned} \tag{2.16}$$

$$(2.17)\quad m_u = -\frac{1}{\pi}\Bigg\{-0.5\frac{1-\mu}{\gamma_c}\left[Y_7(\alpha_b)+\frac{c_{cs}}{\varepsilon_{cu}+\varepsilon_0}(\varepsilon'_\alpha Y_4(\alpha,\alpha_b)+\varepsilon_0 Y_7(\alpha_b))\right]$$

$$+0.5\frac{1-\mu}{\gamma_c}\varepsilon'_\alpha\frac{2}{\varepsilon_0}\left[Y_1(\alpha,\alpha_b)+\frac{1}{2\varepsilon_0}\varepsilon'_\alpha Y_2(\alpha,\alpha_b)\right]+0.5\mu\frac{f_{yk}}{f_{ck}}\Bigg[-\frac{1}{\gamma_s}\rho\Bigg[Y_8(\alpha_{a1})$$

$$+\frac{c_{sh}}{\varepsilon_{su}-\varepsilon_{sy}}(\varepsilon'_\alpha Y_5(\alpha,\alpha_{a1})-\varepsilon_{sy}Y_8(\alpha_b))\Bigg]+\frac{1}{\varepsilon_{ss}}\varepsilon'_\alpha Y_3(\alpha_{a1},\alpha_{a2})$$

$$+\frac{1}{\gamma_s}\rho\left[Y_9(\alpha_{a2})-\sin\alpha_{a2}+\frac{c_{sh}}{\varepsilon_{su}-\varepsilon_{sy}}(\varepsilon'_\alpha Y_6(\alpha,\alpha_{a2})-\varepsilon_{sy}Y_9(\alpha_{a2}))\right]\Bigg]$$

$$+\frac{f_{yk}}{f_{ck}}\rho\mu_{\alpha 1}\Bigg\{-\delta_{k1}\frac{1}{\gamma_s}\left[1-\frac{c_{sh}}{\varepsilon_{su}-\varepsilon_{sy}}[\varepsilon'_\alpha(\rho\cos\alpha_1-\cos\alpha)+\varepsilon_{sy}]\right]\cos\alpha_1$$

$$+\delta_{k1+1}\frac{\varepsilon'_\alpha}{\varepsilon_{ss}}(\rho\cos\alpha_1-\cos\alpha)\cos\alpha_1\Bigg\}$$

$$+\frac{f_{yk}}{f_{ck}}\rho\mu_{\alpha 2}\Bigg\{\delta_{k2}\frac{1}{\gamma_s}\left[1+\frac{c_{sh}}{\varepsilon_{su}-\varepsilon_{sy}}[\varepsilon'_\alpha(\rho\cos\alpha_2-\cos\alpha)-\varepsilon_{sy}]\right]\cos\alpha_2$$

$$+\delta_{k2+1}\frac{\varepsilon'_\alpha}{\varepsilon_{ss}}(\rho\cos\alpha_2-\cos\alpha)\cos\alpha_2\Bigg\}\Bigg\},$$

where:

$$(2.18)\qquad n_u = \frac{N_u}{2\pi r_m t f_{ck}}$$

denotes the normalized ultimate normal force,

$$(2.19)\qquad m_u = \frac{M_u}{4\pi r_m^2 t f_{ck}}$$

denotes the normalized ultimate bending moment, $\mu = dA_s/dA$ – the ratio of areas, steel to concrete, $\mu_{\alpha 1} = \dfrac{F_{a\alpha 1}}{r_m t}$, $\mu_{\alpha 2} = \dfrac{F_{a\alpha 2}}{r_m t}$ – the ratios of areas, additional reinforcement located at the openings specified by α_1, α_2 to concrete, t - thickness of the cross-section $t = R - r$, $d\varphi$ – element of the angle measured from the axis in the compressive zone.

$$\delta_k = \frac{1}{2}((-1)^k + 1), \qquad k = 1, 2, 3;$$

$$k1 = 1, 2; \qquad k2 = 2, 3.$$

The functions X_1–X_9 and Y_1–Y_9 are defined by the following formulae:

$$
\begin{aligned}
X_1(\alpha,\alpha_b) &= \sin\alpha - \sin\alpha_b - \cos\alpha(\alpha-\alpha_b),\\
X_2(\alpha,\alpha_b) &= (0.5+\cos^2\alpha)(\alpha-\alpha_b) + 0.25(\sin 2\alpha - \sin 2\alpha_b)\\
&\qquad - 2\cos\alpha(\sin\alpha - \sin\alpha_b),\\
X_3(\alpha_{a1},\alpha_{a2}) &= \rho(\sin\alpha_{a2} - \sin\alpha_{a1}) - \cos\alpha(\alpha_{a2}-\alpha_{a1}),\\
X_4(\alpha,\alpha_b) &= \sin\alpha_b - \sin\alpha_1 - \cos\alpha(\alpha_b-\alpha_1),\\
X_5(\alpha,\alpha_{a1}) &= \rho(\sin\alpha_{a1} - \sin\alpha_1) - \cos\alpha(\alpha_{a1}-\alpha_1),\\
X_6(\alpha,\alpha_{a2}) &= \rho(\sin\alpha_2 - \sin\alpha_{a2}) - \cos\alpha(\alpha_2-\alpha_{a2}),\\
X_7(\alpha_b) &= \alpha_b - \alpha_1,\\
X_8(\alpha_{a1}) &= \alpha_{a1} - \alpha_1,\\
X_9(\alpha_{a2}) &= \alpha_2 - \alpha_{a2}.
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
Y_1(\alpha,\alpha_b) &= 0.5(\alpha-\alpha_b) + 0.25(\sin 2\alpha - \sin 2\alpha_b)\\
&\qquad - \cos\alpha(\sin\alpha - \sin\alpha_b),\\
Y_2(\alpha,\alpha_b) &= (1+\cos^2\alpha)(\sin\alpha - \sin\alpha_b) - \frac{1}{3}(\sin^3\alpha - \sin^3\alpha_b)\\
&\qquad - \cos\alpha[\alpha-\alpha_b + 0.5(\sin 2\alpha - \sin 2\alpha_b)],\\
Y_3(\alpha_{a1},\alpha_{a2}) &= \rho\{\rho[0.5(\alpha_{a2}-\alpha_{a1}) + 0.25(\sin 2\alpha_{a2} - \sin 2\alpha_{a1})]\\
&\qquad - \cos\alpha(\sin\alpha_{a2} - \sin\alpha_{a1})\},\\
Y_4(\alpha,\alpha_b) &= 0.5(\alpha_b-\alpha_1) + 0.25(\sin 2\alpha_b - \sin 2\alpha_1)\\
&\qquad - \cos\alpha(\sin\alpha_b - \sin\alpha_1),\\
Y_5(\alpha,\alpha_{a1}) &= \rho[0.5(\alpha_{a1}-\alpha_1) + 0.25(\sin 2\alpha_{a1} - \sin 2\alpha_1)]\\
&\qquad - \cos\alpha(\sin\alpha_{a1} - \sin\alpha_1),\\
Y_6(\alpha,\alpha_{a2}) &= \rho[0.5(\alpha_2-\alpha_{a2}) + 0.25(\sin 2\alpha_2 - \sin 2\alpha_{a2})]\\
&\qquad - \cos\alpha(\sin\alpha_2 - \sin\alpha_{a2}),\\
Y_7(\alpha_b) &= \sin\alpha_b - \sin\alpha_1,\\
Y_8(\alpha_{a1}) &= \sin\alpha_{a1} - \sin\alpha_1,\\
Y_9(\alpha_{a2}) &= \sin\alpha_2 - \sin\alpha_{a2}.
\end{aligned}
\tag{2.21}
$$

In a similar way one can analyze the section wholly being in compression.

3. Generalization of the obtained formulae for the section with m openings

The presented model can be generalized for the cross-section weakened by more than two openings. Let us consider the annular cross-section weakened by m openings situated symmetrically with respect to the bending direction. By this assumption, the locations of the openings are determined by couples of the angular coordinates (α_1, α_2), (α_3, α_4), ..., (α_{m-1}, α_m), $0 \le \alpha_1 \le \alpha_2 \le ... \le \alpha_{m-1}, \le \alpha_m \le \pi$ (Fig. 2). Using the principle of mathematical induction one can obtain a solution that takes a similar form as (2.11)–(2.17). It covers all locations of the neutral axis and takes account of possible plastic ranges of concrete and reinforcing steels. The functions X_1–X_9 and Y_1–Y_9 are given in this case by:

$$
\begin{aligned}
X_1(\alpha,\alpha_b) &= \sum_{l+1}^{k} (-1)^{i-1} \sin\alpha_i - \delta_l \sin\alpha_b + \delta_k \sin\alpha \\
&\qquad - \cos\alpha \left(\sum_{l+1}^{k} (-1)^{i-1} \alpha_i - \delta_l \alpha_b + \delta_k \alpha \right), \\
X_2(\alpha,\alpha_b) &= (0.5 + \cos^2\alpha) \left(\sum_{l+1}^{k} (-1)^{i-1} \alpha_i - \delta_l \alpha_b + \delta_k \alpha \right) \\
&\qquad + 0.25 \left(\sum_{l+1}^{k} (-1)^{i-1} \sin 2\alpha_i + \delta_l \sin 2\alpha_b + \delta_k \sin 2\alpha \right) \\
&\qquad - 2\cos\alpha \left(\sum_{l+1}^{k} (-1)^{i-1} \sin\alpha_i - \delta_l \sin\alpha_b + \delta_k \sin\alpha \right), \\
X_3(\alpha,\alpha_{a1},\alpha_{a2}) &= \rho \left(\sum_{k1+1}^{k2} (-1)^{i-1} \sin\alpha_i - \delta_{k1} \sin\alpha_{a1} + \delta_{k2} \sin\alpha_{a2} \right) \\
&\qquad - \cos\alpha \left(\sum_{k1+1}^{k2} (-1)^{i-1} \alpha_i - \delta_{k1} \alpha_{a1} + \delta_{k2} \alpha_{a2} \right), \\
X_4(\alpha,\alpha_b) &= \sum_{l=1}^{l} (-1)^{i-1} \sin\alpha_i + \delta_l \sin\alpha_b \\
&\qquad - \cos\alpha \left(\sum_{i=1}^{l} (-1)^{i-1} \alpha_i + \delta_l \alpha_b \right),
\end{aligned}
\tag{3.1}
$$

$$X_5(\alpha,\alpha_{a1}) = \rho\left(\sum_{i=1}^{k1}(-1)^{i-1}\sin\alpha_i + \delta_{k1}\sin\alpha_{a1}\right) - \cos\alpha\left(\sum_{i=1}^{k1}(-1)^{i-1}\alpha_i + \delta_{k1}\alpha_{a1}\right),$$

$$X_6(\alpha,\alpha_{a2}) = \rho\left(\sum_{k2+1}^{m}(-1)^{i-1}\sin\alpha_i + \delta_{k2}\sin\alpha_{a2}\right) - \cos\alpha\left(\sum_{k2+1}^{m}(-1)^{i-1}\alpha_i + \delta_{k2}\alpha_{a2}\right),$$

(3.1) [cont.]

$$X_7(\alpha_b) = \sum_{i=1}^{l}(-1)^{i-1}\alpha_i + \delta_l\alpha_b,$$

$$X_8(\alpha_{a1}) = \sum_{i=1}^{k1}(-1)^{i-1}\alpha_i + \delta_{k1}\alpha_{a1},$$

$$X_9(\alpha_{a2}) = \sum_{k2+1}^{m}(-1)^{i-1}\alpha_i - \delta_{k2}\alpha_{a2} + \pi.$$

$$Y_1(\alpha,\alpha_b) = 0.5\left(\sum_{l+1}^{k}(-1)^{i-1}\alpha_i - \delta_l\alpha_b + \delta_k\alpha\right) + 0.25\left(\sum_{l+1}^{k}(-1)^{i-1}\sin 2\alpha_i - \delta_l\sin 2\alpha_b + \delta_k\sin 2\alpha\right) - \cos\alpha\left(\sum_{l+1}^{k}(-1)^{i-1}\sin\alpha_i - \delta_l\sin\alpha_b + \delta_k\sin\alpha\right),$$

(3.2)

$$Y_2(\alpha,\alpha_b) = (1+\cos^2\alpha)\left(\sum_{l+1}^{k}(-1)^{i-1}\sin\alpha_i - \delta_l\sin\alpha_b + \delta_k\sin\alpha\right) - \frac{1}{3}\left(\sum_{l+1}^{k}(-1)^{i-1}\sin^3\alpha_i - \delta_l\sin^3\alpha_b + \delta_k\sin^3\alpha\right) - \cos\alpha\left[\sum_{l+1}^{k}(-1)^{i-1}\alpha_i - \delta_l\alpha_b + \delta_k\alpha + 0.5\left(\sum_{l+1}^{k}(-1)^{i-1}\sin 2\alpha_i - \delta_l\sin 2\alpha_b + \delta_k\sin 2\alpha\right)\right],$$

(3.2) [cont.]

$$Y_3(\alpha,\alpha_{a1},\alpha_{a2}) = \rho\Bigg\{\rho\Bigg[0.5\Bigg(\sum_{k1+1}^{k2}(-1)^{i-1}\alpha_i - \delta_{k1}\alpha_{a1} + \delta_{k2}\alpha_{a2}\Bigg)$$
$$+ 0.25\Bigg(\sum_{k1+1}^{k2}(-1)^{i-1}\sin 2\alpha_i - \delta_{k1}\sin 2\alpha_{a1} + \delta_{k2}\sin 2\alpha_{a2}\Bigg)\Bigg]$$
$$- \cos\alpha\Bigg(\sum_{k1+1}^{k2}(-1)^{i-1}\sin\alpha_i - \delta_{k1}\sin\alpha_{a1} + \delta_{k2}\sin\alpha_{a2}\Bigg)\Bigg\},$$

$$Y_4(\alpha,\alpha_b) = 0.5\Bigg(\sum_{i=1}^{l}(-1)^{i-1}\alpha_i + \delta_l\alpha_b\Bigg)$$
$$+ 0.25\Bigg(\sum_{i=1}^{l}(-1)^{i-1}\sin 2\alpha_i + \delta_l\sin 2\alpha_b\Bigg)$$
$$- \cos\alpha\Bigg(\sum_{i=1}^{l}(-1)^{i-1}\sin\alpha_i + \delta_l\sin\alpha_b\Bigg),$$

$$Y_5(\alpha,\alpha_{a1}) = \rho\Bigg[0.5\Bigg(\sum_{i=1}^{k1}(-1)^{i-1}\alpha_i + \delta_{k1}\alpha_{a1}\Bigg)$$
$$+ \Bigg(0.25\sum_{i=1}^{k1}(-1)^{i-1}\sin 2\alpha_i + \delta_{k1}\sin 2\alpha_{a1}\Bigg)\Bigg]$$
$$- \cos\alpha\Bigg(\sum_{i=1}^{k1}(-1)^{i-1}\sin\alpha_i + \delta_{k1}\sin\alpha_{a1}\Bigg),$$

$$Y_6(\alpha,\alpha_{a2}) = \rho\Bigg[0.5\Bigg(\sum_{k2+1}^{m}(-1)^{i-1}\alpha_i + \delta_{k2}\alpha_{a2}\Bigg)$$
$$+ 0.25\Bigg(\sum_{k2+1}^{m}(-1)^{i-1}\sin 2\alpha_i + \delta_{k2}\sin 2\alpha_{a2}\Bigg)\Bigg]$$
$$- \cos\alpha\Bigg(\sum_{k2+1}^{m}(-1)^{i-1}\sin\alpha_i + \delta_{k2}\sin\alpha_{a2}\Bigg),$$

$$Y_7(\alpha_b) = \sum_{i=1}^{l}(-1)^{i-1}\sin\alpha_i + \delta_l\sin\alpha_b,$$

$$Y_8(\alpha_{a1}) = \sum_{i=1}^{k1}(-1)^{i-1}\sin\alpha_i + \delta_{k1}\sin\alpha_{a1},$$

$$Y_9(\alpha_{a2}) = \sum_{k2+1}^{m}(-1)^{i-1}\sin\alpha_i + \delta_{k2}\sin\alpha_{a2}.$$

$k = 0, 1, ..., m$ – the index of the interval satisfying the condition $\alpha \in \langle \alpha_k, \alpha_{k+1} \rangle$, where α is the angle describing the location of the neutral axis of the section, l denotes the index of the interval satisfying the condition $\alpha_b \in \langle \alpha_l, \alpha_{l+1} \rangle$, $l = 0, 1, ..., k$, $k1$ – the index of the interval satisfying the condition $\alpha_{\alpha 1} \in \langle \alpha_{k1}, \alpha_{k1+1} \rangle$, $k1 = 0, ..., k$, $k2$ – the index of the interval satisfying the condition $\alpha_{a2} \in \langle \alpha_{k2}, \alpha_{k2+1} \rangle$, $k2 = k + 1, ..., m$.

In the above formulae it is assumed that if $m1 > m2$ then $\sum_{m1}^{m2} (\,) = 0$.

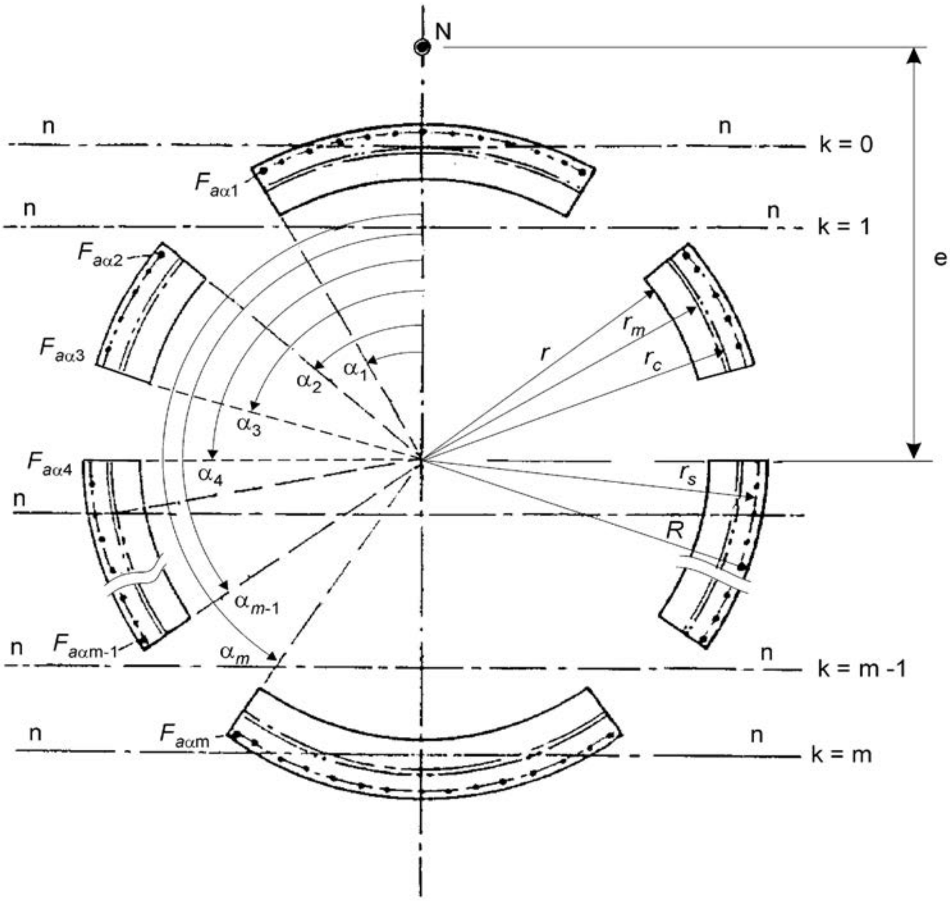

FIG. 2. The cross-section weakened by m openings located symmetrically with respect to the bending direction.

The effects of the additional reinforcements at the openings are determined by the following conditions:

– with respect to n_u:

$$\begin{aligned}(3.3)\quad \frac{f_{yk}}{f_{ck}}\Bigg\{-\frac{1}{\gamma_s}\sum_{i=1}^{k1}\mu_{ai}\left[1-\frac{c_{sh}}{\varepsilon_{su}-\varepsilon_{sy}}[\varepsilon'_\alpha(\rho\cos\alpha_i-\cos\alpha)+\varepsilon_{sy}]\right]+ \\ +\frac{\varepsilon'_\alpha}{\varepsilon_{ss}}\sum_{i=k1+1}^{k2}\mu_{ai}(\rho\cos\alpha_i-\cos\alpha) \\ +\frac{1}{\gamma_s}\sum_{i=k2+1}^{m}\mu_{ai}\left[1+\frac{c_{sh}}{\varepsilon_{su}-\varepsilon_{sy}}[\varepsilon'_\alpha(\rho\cos\alpha_i-\cos\alpha)-\varepsilon_{sy}]\right]\Bigg\}\end{aligned}$$

– with respect to m_u:

$$\begin{aligned}(3.4)\qquad 0.5\frac{f_{yk}}{f_{ck}}\rho\Bigg\{-\frac{1}{\gamma_s}\sum_{i=1}^{k1}\mu_{ai}\left[1-\frac{c_{sh}}{\varepsilon_{su}-\varepsilon_{sy}}[\varepsilon'_\alpha(\rho\cos\alpha_i-\cos\alpha)+\varepsilon_{sy}]\right]\cos\alpha_i \\ +\frac{\varepsilon'_\alpha}{\varepsilon_{ss}}\sum_{i=k1+1}^{k2}\mu_{ai}(\rho\cos\alpha_i-\cos\alpha)\cos\alpha_i \\ +\frac{1}{\gamma_s}\sum_{i=k2+1}^{m}\mu_{ai}\left[1+\frac{c_{sh}}{\varepsilon_{su}-\varepsilon_{sy}}[\varepsilon'_\alpha(\rho\cos\alpha_i-\cos\alpha)-\varepsilon_{sy}]\right]\cos\alpha_i\Bigg\}.\end{aligned}$$

To prove validity of the formulae (3.1)–(3.2) in a general case, the mathematical induction is employed. Substituting $m = 4$, $k = 2$, $l = 2$ in the formulae (3.1) and (3.2), the relationships given by (2.20)–(2.21) for the section weakened by two openings are obtained (Fig. 1a):

$$\begin{aligned}X_1(\alpha,\alpha_b) &= \sum_{3}^{2}(-1)^{i-1}\sin\alpha_i-\delta_2\sin\alpha_b+\delta_2\sin\alpha \\ &-\cos\alpha\left(\sum_{3}^{2}(-1)^{i-1}\alpha_i-\delta_2\alpha_b+\delta_2\alpha\right) = -\sin\alpha_b+\sin\alpha-\cos\alpha(-\alpha_b+\alpha), \\ Y_1(\alpha,\alpha_b) &= 0.5\left(\sum_{3}^{2}(-1)^{i-1}\alpha_i-\delta_2\alpha_b+\delta_2\alpha\right) \\ &+0.25\left(\sum_{3}^{2}(-1)^{i-1}\sin 2\alpha_i-\delta_2\sin 2\alpha_b+\delta_2\sin 2\alpha\right) \\ &-\cos\alpha\left(\sum_{3}^{2}(-1)^{i-1}\sin\alpha_i-\delta_2\sin\alpha_b+\delta_2\sin\alpha\right) \\ &= 0.5(-\alpha_b+\alpha)+0.25(-\sin 2\alpha_b+\sin 2\alpha)-\cos\alpha(-\sin\alpha_b+\sin\alpha),\end{aligned}$$

due to $\sum_{3}^{2}(\,)=0$.

In a similar way the functions X_2–X_9 and Y_2–Y_9 given by (2.20) and (2.21) can be obtained.

Let us in turn assume validity of the formulae (3.1)–(3.2) for the section weakened by m openings, $k = 1, 2, ..., m$, and let us consider the section weakened by two additional, symmetrically situated openings, the locations of which are determined by a couple of angular coordinates (α_{m+1}, α_{m+2}), $0 \le \alpha_1 \le \alpha_2 \le ... \le \alpha_m \le \alpha_{m+1} \le \alpha_{m+2} \le \pi$ (Fig. 2).

The task leads to two additional cases of location of the neutral axis α to be considered: $\alpha_{m+1} \le \alpha \le \alpha_{m+2}$ or $\alpha_{m+2} \le \alpha \le \pi$.

Integrating the equilibrium equation (2.9) in the first case, the function X_1 (α, α_b) takes the following form ($k = m + 1$):

$$X_1(\alpha, \alpha_b) = \sum_{l+1}^{m-1} (-1)^{i-1} \sin\alpha_i - \delta_l \sin\alpha_b + \sin\alpha_{m+1} - \sin\alpha_m$$
$$- \cos\alpha \left(\sum_{l+1}^{m-1} (-1)^{i-1} \alpha_i - \delta_l \alpha_b + \alpha_{m+1} - \alpha_m \right)$$
$$= \sum_{l+1}^{m+1} (-1)^{i-1} \sin\alpha_i - \delta_l \sin\alpha_b - \cos\alpha \left(\sum_{l+1}^{m+1} (-1)^{i-1} \alpha_i - \delta_l \alpha_b \right), \qquad (\delta_{m+1} = 0).$$

For the case $\alpha_{m+2} \le \alpha \le \pi$ ($k = m + 1$):

$$X_1(\alpha, \alpha_b) = \sum_{l+1}^{m+1} (-1)^{i-1} \sin\alpha_i - \delta_l \sin\alpha_b + \sin\alpha - \sin\alpha_{m+2}$$
$$- \cos\alpha \left(\sum_{l+1}^{m+1} (-1)^{i-1} \alpha_i - \delta_l \alpha_b + \alpha - \alpha_{m+2} \right)$$
$$= \sum_{l+1}^{m+2} (-1)^{i-1} \sin\alpha_i - \delta_l \sin\alpha_b + \delta_{m+2} \sin\alpha$$
$$- \cos\alpha \left(\sum_{l+1}^{m+2} (-1)^{i-1} \alpha_i - \delta_l \alpha_b + \delta_{m+2} \alpha \right), \qquad (\delta_{m+2} = 1).$$

The remaining functions X_2–X_9 and Y_1–Y_9 can be checked in a similar way. Thus, the general formulae (3.1) and (3.2) are proved.

4. Numerical examples

The presented approach enables the determination of the resistance of the sections under consideration. Using the derived formulae (2.11)–(2.17) and

(3.1)–(3.4), the interaction curves with the design values of the normalized, cross-sectional forces n_u and m_u have been obtained for the section weakened by two or four openings (Fig. 3, Fig. 4). Each curve refers to the corresponding value of reinforcement ratio $\mu f_{yk}/f_{ck}$. The maximum compressive strain in concrete is calculated at the extreme fibre in the compression zone of the section. The two numbers $\varepsilon_c/\varepsilon_s$ at each indication point are concrete strain and steel strain in ‰. The points located on the n_u axis are related to pure compression and on the m_u axis – to pure bending. The points denoted by $\varepsilon_c/0$ can be interpreted as transition from the state $\varepsilon_c/(\varepsilon_s < 0)$ described as whole compression (uncracked) to the one $\varepsilon_c/(\varepsilon_s > 0)$ characterized by the occurrence of the tensile strains which cause the crack formation in concrete (cracked).

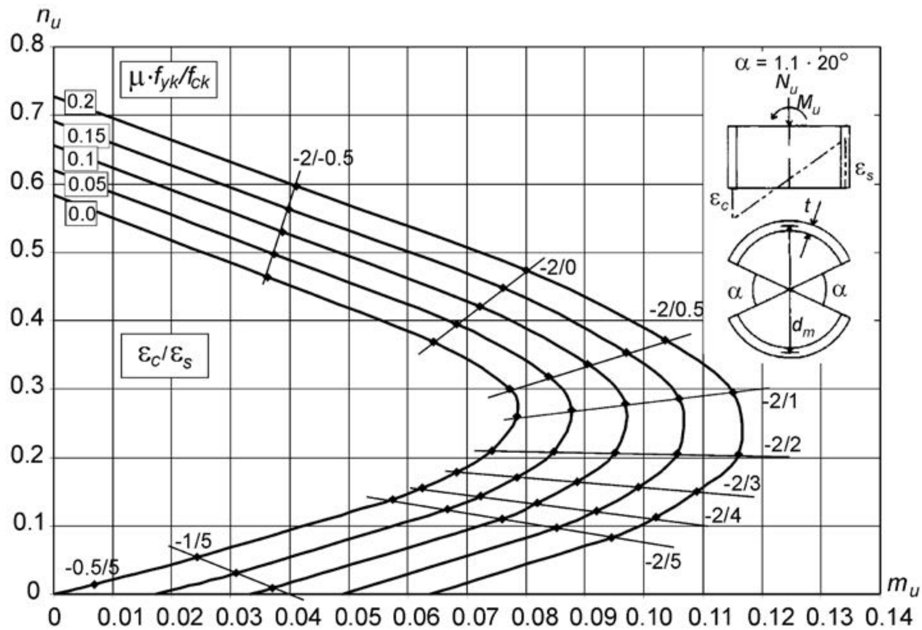

FIG. 3. Interaction diagram with the design values of the normalized cross-sectional forces n_u and m_u for the section weakened by two openings: $f_{yk} = 220$ MPa; $\gamma_c = 1.5$; $\gamma_s = 1.15$, $\varepsilon_0 = 2.0$, $c_{cs} = c_{sh} = 0$.

The effect of the additional lumped reinforcement at an opening was examined under the assumption that the cross-sectional area of the additional steel bars at the sides of the opening is equal to that which would have passed through it. The comparison presented in Fig. 5. indicates that the section resistance determined by the values of n_u, m_u increases due to the additional reinforcement at a single opening by more than 10%, depending on the opening size and the ultimate values ε_{cu} and ε_{su}.

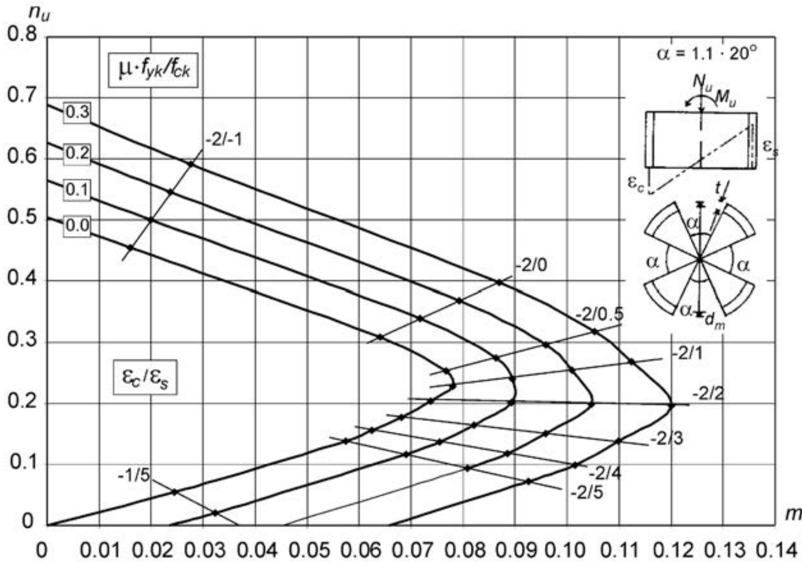

FIG. 4. Interaction diagram with the design values of the normalized cross-sectional forces n_u and m_u for the section weakened by four openings: $f_{yk} = 500$ MPa; $\gamma_c = 1.5$; $\gamma_s = 1.15$, $\varepsilon_0 = 2.0$, $c_{cs} = c_{sh} = 0$.

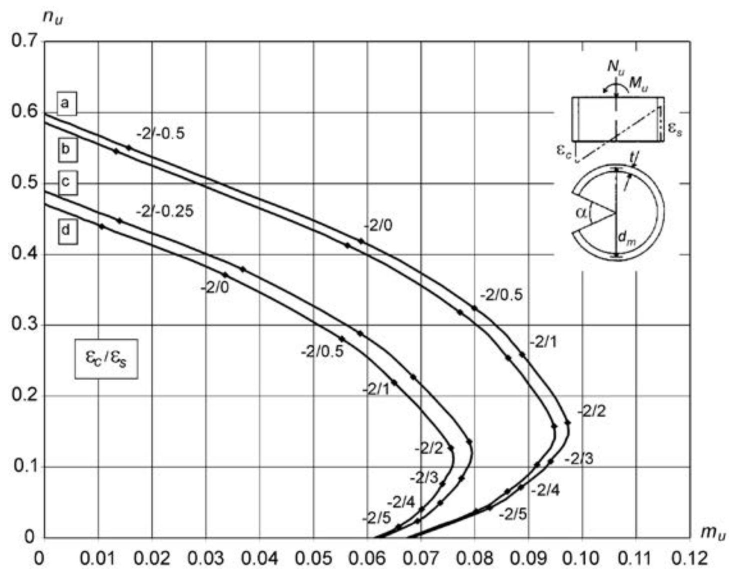

FIG. 5. The effect of the additional reinforcement at the opening on the resistance of the section with a single opening: curves a, b – $\alpha = 44°$, $F_{a\alpha 1} = 21.85$ cm^2 (a), $F_{a\alpha 1} = 0$ (b); curves c, d – $\alpha = 66°$, $F_{a\alpha 1} = 32.78$ cm^2 (c), $F_{a\alpha 1} = 0$ (d); $f_{yk} = 410$ MPa; $\gamma_c = 1.5$; $\gamma_s = 1.15$; $\mu = 1\%$, $\varepsilon_0 = 2.0$, $c_{cs} = c_{sh} = 0$.

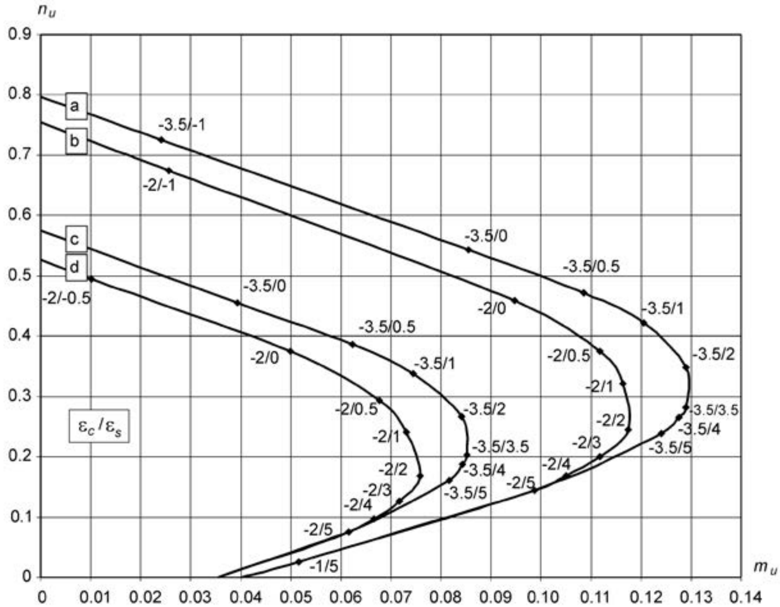

FIG. 6. The effect of a single opening of the size of 44° on the section resistance: curves a, b – the closed ring section; curves c, d – the ring section weakened by single opening; $f_{yk} = 410$ MPa; $\gamma_c = 1.5$; $\gamma_s = 1.15$; $\mu = 1\%$, $\varepsilon_0 = 2.0$, $c_{cs} = c_{sh} = 0$.

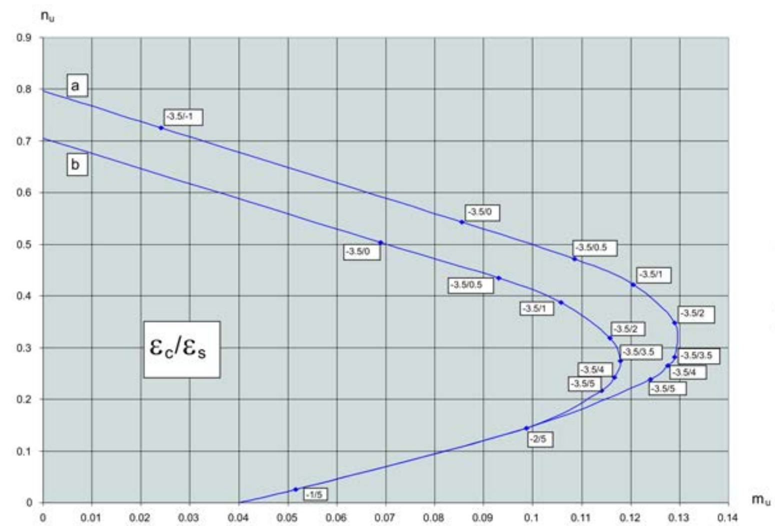

FIG. 7. Comparison of the section resistance determined by stress-strain relationship for the concrete given by (2.1) (b) with that based on the parabolic-rectagular one (a); $f_{yk} = 410$ MPa; $f_{ck} = 20$ MPa; $\gamma_c = 1.5$; $\gamma_s = 1.15$; $\mu = 0.5\%$, $\varepsilon_0 = 2.0$, $c_{cs} = 0.15$, $c_{sh} = 0$.

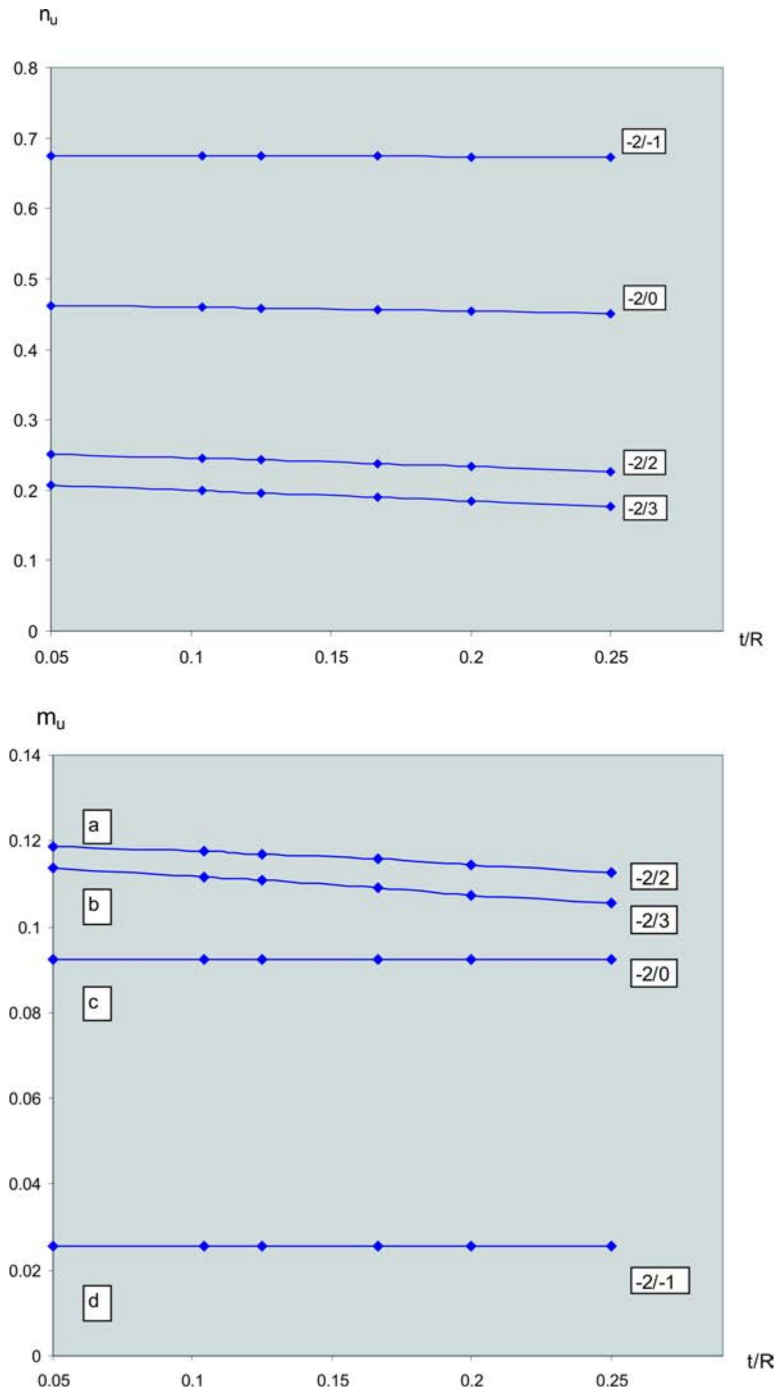

FIG. 8. Resistance of the annular cross-section as a function of the t/R ratio; $f_{yk} = 410$ MPa; $f_{ck} = 20$ MPa; $\gamma_c = 1.5$; $\gamma_s = 1.15$; $\mu = 0.5\%$, $\varepsilon_0 = 2.0$, $c_{cs} = c_{sh} = 0$.

As the next problem, the effect of openings on the section resistance has been determined. The curves presented in Fig. 6 indicate that a single opening of the size of 44° results in reduction in the section resistance by 30–40% with respect to the normal force n_u and the bending moment m_u.

The comparison of the section resistance determined by the stress-strain relationship given by (2.1) with that based on the parabolic-rectagular one is presented in Fig. 7. It shows that the concrete softening in the plastic range results in decreasing the section resistance by 9–11% with respect to n_u and 8–19% with respect to m_u.

In Fig. 8 the values of n_u and m_u are shown as functions of the t/R ratio for different ultimate values ε_{cu} and ε_{su}. The increasing value of t/R ratio results in lower section resistance. It is reduced in the considered range $\langle 0.05; 0.25\rangle$ by up to 14% with respect to n_u and by up to 7% with respect to m_u.

Table 1. Comparison of the calculated values with those specified in the DIN 1056 code; $f_{yk} = 420$ MPa, $\gamma_c = 1.5$, $\gamma_c = 1.25$, α – opening size, RD – relative difference.

Type of Section	α [°]	$\mu\frac{f_{yk}}{f_{ck}}$	$\varepsilon_c/\varepsilon_s$	n_u			m_u		
				DIN	Proposed model	RD [%]	DIN	Proposed model	RD [%]
Closed		0.2	−2/2	0.260	0.244	6.6	0.14	0.138	1.7
with 1 opening	22	0.2	−2/1	0.305	0.293	4.1	0.11	0.108	2.1
with 1 opening	33	0.3	−2/1	0.30	0.286	5.1	0.111	0.109	1.4
with 2 openings	22	0.15	−2/1	0.30	0.287	4.6	0.10	0.098	1.7
with 2 openings	44	0.1	−2/4	0.100	0.105	4.7	0.059	0.0589	0.1

The calculated design values of the normalized, cross-sectional forces n_u and m_u for the sections weakened by one and two openings have been compared with those given according to DIN 1056 [2, 3] (Table 1). The resulting differences do not exceed 7%. In the author's opinion, they result from the differences in the models used and partly from the inaccuracies of reading the DIN diagrams.

5. Conclusions

Based on this study, the following conclusions can be drawn:

1. Using combinatorial approach and the method of mathematical induction, general analytical formulae have been derived for determining the resis-

tance and elasto-plastic analysis of RC annular cross-sections, weakened by an arbitrary number of openings located symmetrically with respect to the bending direction.

2. The obtained solutions are presented in the form of interaction diagrams with the design values of the normalized cross-sectional forces n_u and m_u that can be easily used in structural design.
3. The proposed section model seems to have a wider application field than the previous ones due to the assumptions of non-central layout of reinforcement, additional steel bars at openings and wall edge strains.
4. The resistance of the section increases due to the additional reinforcement at the opening by more than 10%, depending on the opening size and the ultimate values ε_{cu}, ε_{su}.
5. A single opening may result in reduction in the section resistance by 30–40% with respect to the normal force n_u and the bending moment m_u.
6. Concrete softening in the plastic range as well as increasing value of the t/R ratio result in a lower section resistance.
7. The proposed model works well in most cases encountered in engineering practice.
8. The range of validity of the obtained solutions is limited to such number, sizes and locations of openings which assure that plane sections remain plane.
9. If the assumption that plane sections remain plane is not satisfied, the method may still be used provided that the openings are treated as enlarged, as described in the CICIND 2001 Code [4] and the Eurocode EN 13084-2:2006 [7].
10. The model serves for dimensioning the cross-sections and enables to design strenghtening of RC structures by means of the external reinforcement.

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