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MISFITTING ELLIPTIC ELASTIC INHOMOGENEITY PROBLEM IN PERFECTLY ANISOTROPIC MEDIA

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The problem considered in this paper is that of a misfitting elliptic inclusion in an infinite elastic region. The stresses develop because of the misfit. The inclusion and the outside material, called the matrix, are both of homogeneous and perfectly anisotropic materials. Further, the elastic properties of the two materials may differ. The complex variable technique is employed to evaluate two sets of complex potential functions $\{(\varphi_k): k=1, 2, 3\}$, one for the inhomogeneity and another for the outside region, which give the elastic field every where.

1. Introduction

The study of inclusion problems was initiated by the works of Mott and Nabarro [1], Frankel [2], Eshelby [3], Jaswon and Bhargava [4] and others. In these papers the materials were supposed to be elastically isotropic and homogeneous. In some papers the materials were cubic [7] and orthotropic [5, 6]. Chen [8] considered the problem of elliptical inhomogeneity for materials possessing one plane of symmetry, normal to the generators of an infinite cylinder. Bhargava and Saxena [9] studied the circular elastic inclusion problem in a completely anisotropic media. However, in most of these papers either the inhomogeneity was of the same size as the hole in the matrix or, if it was oversize, the elastic properties of the inhomogeneity were the same as those of the matrix.

In this paper the oversize elliptic inhomogeneity problem in an infinite elastic medium is considered. The two media are perfectly anisotropic.

2. Statement of the problem

An elliptic region with x and y along the semi-axes a and b of a perfectly anisotropic material tends to undergo a spontaneous deformation characterized by the displacements $u_x=\delta_a x+\gamma_a y$, $u_y=\gamma_b x+\delta_b y$, $u_z=0$. The elliptic region is, however, embedded in another anisotropic material. The elastic properties of the two materials may differ from each other. The problem is treated as a generalized plane strain problem. Because of the constraints of the outer material, the stresses develop everywhere. The problem is to evaluate the elastic field for the matrix and the inhomogeneity.

3. General theory

As it is well known in generalized plane strain, the displacement components u_i $(i=x, y, z)$ are functions of (x, y) only. This may be compared with the plane strain problem of the elasticity theory when u_x and u_y are functions of x, y and $u_z=0$. Thus all the strain components (except e_{zz}) and the stress components P_{ij} $(i, j=x, y, z)$ are non-zero. Note that in this case $e_{yz}=\dfrac{\partial u_x}{\partial y}=2\omega_x$, $e_{zx}=\dfrac{\partial u_z}{\partial x}=-2\omega_y$. However, the rotation components ω_z is different from e_{xy}. For ease of notation we shall write $P_{xx}=P_1$, $P_{yy}=P_2$, $P_{zz}=P_3$, $P_{yz}=P_4$, $P_{zx}=P_5$, $P_{xy}=P_6$ and, similarly, for strains $e_{xx}=e_1$, $e_{yy}=e_2$, $e_{zz}=e_3$, $e_{yz}=e_4$, $e_{zx}=e_5$, $e_{xy}=e_6$ and the generated Hooke's law as

$$P_i=\alpha_{ij}\, e_j \quad (i, j=1 \text{ to } 6), \tag{3.1}$$

where α_{ij} are the elastic constants of the material.

As it is well known [10] in anisotropic elasticity, the stresses and displacements are known if three complex potential functions $\{\varphi_k(z_k);\ k=1, 2, 3\}$ can be determined. Here $z_k=x+\mu_k y$ and μ_k are the roots with positive imaginary parts of the characteristic equation

$$l_2(\mu)\, l_4(\mu)-l_3^2(\mu)=0, \tag{3.2}$$

where

$$l_2(\mu)=\beta_{55}\,\mu^2-2\beta_{45}\,\mu+\beta_{44},$$

$$l_3(\mu)=\beta_{15}\,\mu^3-(\beta_{14}+\beta_{56})\,\mu^2+(\beta_{25}+\beta_{46})\,\mu-\beta_{24},$$

$$l_4(\mu)=\beta_{11}\,\mu^4-2\beta_{16}\,\mu^3+(2\beta_{12}+\beta_{66})\,\mu^2-2\beta_{26}\,\mu+\beta_{22},$$

$$\beta_{ij}=\alpha_{ij}=\frac{\alpha_{i3}\,\alpha_{j3}}{\alpha_{33}} \quad (i, j=1, 2, 4, 5, 6).$$

It may be shown that μ_k's cannot be purely real [10].

The stresses and displacements are given by the following relations:

The stresses are given by

$$\begin{aligned}
P_{xx}&=2\mathrm{Re}\,[\mu_1^2\,\varphi_1'(z_1)+\mu_2^2\,\varphi_2'(z_2)+\mu_3^2\,\lambda_3\,\varphi_3'(z_3)],\\
P_{yy}&=2\mathrm{Re}\,[\varphi_1'(z_1)+\varphi_2'(z_2)+\lambda_3\,\varphi_3'(z_3)],\\
P_{yz}&=-2\mathrm{Re}\,[\lambda_1\,\varphi_1'(z_1)+\lambda_2\,\mu_2\,\varphi_2'(z_2)+\varphi_3'(z_3)],\\
P_{zx}&=2\mathrm{Re}\,[\mu_1\,\lambda_1\,\varphi_1'(z_1)+\lambda_2\,\mu_2\,\varphi_2'(z_2)+\mu_3\,\varphi_3'(z_3)],\\
P_{xy}&=-2\mathrm{Re}\,[\mu_1\,\varphi_1'(z_1)+\mu_2\,\varphi_2'(z_2)+\mu_3\,\lambda_3\,\varphi_3'(z_3)],\\
P_{zz}&=\frac{-1}{\alpha_{33}}\,[\alpha_{31}\,P_{xx}+\alpha_{32}\,P_{yy}+\alpha_{34}\,P_{yz}+\alpha_{35}\,P_{zx}+\alpha_{36}\,P_{xy}],
\end{aligned} \tag{3.3}$$

the displacements by

$$(3.4)\qquad \begin{aligned} &u_x=2\mathrm{Re}\sum_{k=1}^{3} p_k\,\varphi_k(z_k), \qquad u_y=2\mathrm{Re}\sum_{k=1}^{3} q_k\,\varphi_k(z_k),\\ &u_z=2\mathrm{Re}\sum_{k=1}^{3} r_k\,\varphi_k(z_k). \end{aligned}$$

In the above relations

$$\begin{aligned} &\lambda_k=-\frac{l_3(\mu_k)}{l_2(\mu_k)}\,(k=1,\,2), \qquad \lambda_3=-\frac{l_3(\mu_3)}{l_4(\mu_3)},\\ &p_k=\beta_{11}\,\mu_k^2+\beta_{12}-\beta_{16}\,\mu_k+\lambda_k(\beta_{15}\,\mu_k-\beta_{14}),\\ &q_k=\beta_{12}\,\mu_k+\frac{\beta_{22}}{\mu_k}-\beta_{26}+\lambda_k(\beta_{25}-\beta_{24}/\mu_k),\\ &r_k=\beta_{14}\,\mu_k+\frac{\beta_{24}}{\mu_k}-\beta_{46}+\lambda_k(\beta_{45}-\beta_{44}/\mu_k) \qquad (k=1,\,2), \end{aligned}$$

and

$$\begin{aligned} &p_3=\lambda_3(\beta_{11}\,\mu_3^2+\beta_{12}-\beta_{16}\,\mu_3)+(\beta_{15}\,\mu_3-\beta_{14}),\\ &q_3=\lambda_3\left(\beta_{12}\,\mu_3+\frac{\beta_{22}}{\mu_3}-\beta_{26}\right)+(\beta_{25}-\beta_{24}/\mu_3),\\ &r_3=\lambda_3(\beta_{14}\,\mu_3+\beta_{24}/\mu_3-\beta_{46})+(\beta_{45}-\beta_{44}/\mu_3). \end{aligned}$$

Some important expressions for tractions and displacements are

$$(3.5)\qquad \begin{aligned} &P=X+iY=\int_0^s (P_{nx}+iP_{ny})\,ds=-i\sum_{k=1}^{3}[a_k\,\varphi_k(z_k)+b_k\,\bar\varphi_k(\bar z_k)],\\ &\psi=\int_0^s P_{nz}\,ds=2\mathrm{Re}\sum_{k=1}^{3} c_k\,\varphi_k(z_k),\\ &D=u_k+iu_y=\sum_{k=1}^{3}[g_k\,\varphi_k(z_k)+h_k\,\bar\varphi_k(\bar z_k)],\\ &u_z=2\mathrm{Re}\sum_{k-1}^{3} r_k\,\varphi_k(z_k), \end{aligned}$$

where P_{nj} is the stress component in the j-th direction on the element whose normal is the n-th direction and

$$\begin{aligned} &a_k=1+i\mu_k, \quad b_k=1+i\bar\mu_k, \quad c_k=\lambda_k \qquad (k=1,2),\\ &a_3=\lambda_3(1+i\mu_3), \quad b_3=\bar\lambda_3(1+i\bar\mu_3), \quad c_3=1,\\ &g_k=p_k+iq_k, \quad h_k=\bar p_k+i\bar q_k, \quad (k=1,2,3). \end{aligned}$$

4. Boundary conditions

The boundary conditions to be satisfied by final the solution are as follows:

(i) At the interface, the tractions P, ψ and the displacement u_z must be continuous, i.e. at the interface

$$(P)_{(m)}=(P)_{(i)}, \quad (\psi)_{(m)}=(\psi)_{(i)} \quad \text{and} \quad (u_z)_{(m)}=(u_z)_{(i)},$$

where the subscripts (m) and (i) denote the matrix and inclusion respectively.

(ii) If the inclusion undergoes a spontaneous deformation characterized by the displacements $(\delta_a x+\gamma_a y, \quad \gamma_b x+\delta_b y)$, which is the free state of inclusion,

$$(D)_{(m)}-(D)_{(i)}=\frac{1}{2}[\{(a\delta_a+b\delta_b)-i(b\gamma_a-a\gamma_b)\}e^{i\theta}+ \\ +\{(a\delta_a-b\delta_b)+i(b\gamma_a+a\gamma_b)\}e^{-i\theta}].$$

Heie θ is the eccentric angle at a point on the elliptic boundary.

5. Analysis

For an infinite region bounded internally by an ellipse, a suitable transformation is

$$z=\frac{1}{2}[(a+b)Z+(a-b)Z^{-1}], \tag{5.1}$$

which transforms the ellipse of semi-axes (a, b) in the z-plane onto a unit circle $|Z|=1$ in the Z-plane. Also, the outside region in the z-plane is transformed onto the region exterior to the unit circle $|Z|=1$ in the Z-plane. Three other complex variables are defined in the following way:

$$z_k=\frac{1}{2}[(a-i\mu_k b)Z_k+(a+i\mu_k b)Z_k^{-1}] \qquad (k=1, 2, 3), \tag{5.2}$$

where the Z_k-planes are chosen such that the circles $|Z_k|=1$ correspond to the circle $|Z|=1$, and $|Z_k|\to\infty$ as $|z|\to\infty$. As stated earlier, the problem is solved if we can determine the set of functions $\{\varphi_k(z_k);\ k=1, 2, 3\}$ for the outside region and the inclusion.

Using the transformation given in Eq. (5.2), one may write

$$\{\varphi_k(z_k)\}_{(m)}=\{\varphi_k(Z_k)\}_{(m)}.$$

The form of the functions $\varphi_k(Z_k)$, for the matrix, are chosen as follows:

$$\{\varphi_k(Z_k)\}_{(m)}=\sum_{n=1}^{\infty} G_{nk} Z_k^{-n} \qquad (k=1, 2, 3). \tag{5.3}$$

To determine G_{nk}'s we first find the tractions P, ψ and the displacements D, u_z on the inner boundary of the matrix. Using Eqs. (3.5) and (5.3) we have

(5.4)
$$\begin{aligned} P_{(m)} &= -i \sum_{k=1}^{3} \sum_{n=1}^{\infty} (b_{k\,(m)}\, \bar{G}_{nk}\, e^{in\theta} + a_{k\,(m)}\, G_{nk}\, e^{-in\theta}), \\ \psi_{(m)} &= 2\mathrm{Re} \sum_{k=1}^{3} \sum_{n=1}^{\infty} c_{k\,(m)}\, G_{nk}\, e^{-in\theta}, \\ D_{(m)} &= \sum_{k=1}^{3} \sum_{n=1}^{\infty} (g_{k\,(m)}\, G_{nk}\, e^{-in\theta} + h_{k\,(m)}\, \bar{G}_{nk}\, e^{in\theta}), \\ u_{z\,(m)} &= 2\mathrm{Re} \sum_{k=1}^{3} \sum_{n=1}^{\infty} r_{k\,(m)}\, G_{nk}\, e^{-in\theta}. \end{aligned}$$

For inclusion the functions $\varphi_k(z_k)$ may be assumed as

(5.5)
$$\{\varphi_k(z_k)\}\ \{\varphi_k(z_k)\}_{(i)} = \sum_{n=1}^{\infty} A_{nk}\, z_k^n,$$

where at the interface

(5.6)
$$z_k = \frac{1}{2}\left[(a - i\mu_{k\,(i)}\, b)\, e^{i\theta} + (a + i\mu_{k\,(i)}\, b)\, e^{-i\theta}\right],$$

and therefore on the boundary of the inclusion

(5.7)
$$\begin{aligned} P_{(i)} &= -\sum_{k=1}^{3} \sum_{n=1}^{\infty} 2^{-n} [a_{k\,(i)}\, A_{nk} \{(a - i\mu_{k\,(i)}\, b)\, e^{i\theta} + (a + i\mu_{k\,(i)}\, b)\, e^{-i\theta}\}^n + \\ &\qquad + b_{k\,(i)}\, \bar{A}_{nk} \{a - i\bar{\mu}_{k\,(i)}\, b)\, e^{i\theta} + (a + i\bar{\mu}_{k\,(i)}\, b)\, e^{-i\theta}\}^n], \\ \psi_{(i)} &= 2\mathrm{Re} \sum_{k=1}^{3} \sum_{n=1}^{\infty} 2^{-n} [c_{k\,(i)}\, A_{nk} \{(a - i\mu_{k\,(i)}\, b)\, e^{e\theta} + (a + i\mu_{k\,(i)}\, b)\, e^{-i\theta}\}^n], \\ D_{(i)} &= \sum_{k=1}^{3} \sum_{n=1}^{\infty} 2^{-n} [g_{k\,(i)}\, A_{nk} \{a - i\mu_{k\,(i)}\, b)\, e^{i\theta} + (a + i\mu_{k\,(i)}\, b)\, e^{-e\theta}\}^n + \\ &\qquad + h_{k\,(i)}\, \bar{A}_{nk} \{(a - i\bar{\mu}_{k\,(i)}\, b)\, e^{i\theta} + (a + i\bar{\mu}_{k\,(i)}\, b)\, e^{-i\theta}\}^n], \\ u_{z\,(i)} &= 2\mathrm{Re} \sum_{k=1}^{3} \sum_{n=1}^{\infty} 2^{-n} [r_{k\,(i)}\, A_{nk} \{a - i\mu_{k\,(i)}\, b)\, e^{-i\theta} + (a + i\mu_{k\,(i)}\, b)\, e^{-i\theta}\}^n]. \end{aligned}$$

At the interface, now, we use the conditions (i) and (ii); we then equate the coefficients of like powers of $e^{i\theta}$ from both sides. After solving the equations, we get

(5.8)
$$G_{nk} = 0 = A_{nk} \qquad \text{for} \qquad n \geqslant 2$$

and

$$G_{1k} = G_k, \qquad A_{1k} = A_k \qquad (k = 1, 2, 3),$$

where G_k's and A_k's are given by the following set of equations:

$$\sum_{k=1}^{3} [(a-i\mu_{k(i)}b)\, a_{k(i)}\, A_k+(a-i\bar{\mu}_{k(i)}b)\, b_{k(i)}\, \bar{A}_k-2b_{k(m)}\, \bar{G}_k]=0\,,$$

$$\sum_{k=1}^{3} [(a-i\mu_{k(i)}b)\, \bar{b}_{k(i)}\, A_k+(a-i\bar{\mu}_{k(i)}b)\, \bar{a}_{k(i)}\, \bar{A}_k-2\bar{a}_{k(m)}\, \bar{G}_k]=0\,,$$

$$\sum_{k=1}^{3} [(a-i\mu_{k(i)}b)\, c_{k(i)}\, A_k+(a-i\bar{\mu}_{k(i)}b)\, \bar{c}_{k(i)}\, \bar{A}_k-2\bar{c}_{k(m)}\, \bar{G}_k]=0\,,$$

(5.9)
$$\sum_{k=1}^{3} [(a-i\mu_{k(i)}b)\, g_{k(i)}\, A_k+(a-i\bar{\mu}_{k(i)}b)\, h_{k(i)}\, \bar{A}_k-2h_{k(m)}\, \bar{G}_k]+ +(a\delta_a+b\delta_b)-i(b\gamma_a-a\gamma_b)=0\,,$$

$$\sum_{k=1}^{3} [(a-i\mu_{k(i)}b)\, \bar{h}_{k(i)}\, A_k+(a-i\bar{\mu}_{k(i)}b)\, \bar{g}_{k(i)}\, \bar{A}_k-2\bar{g}_{k(m)}\, \bar{G}_k]+ +(a\delta_a-b\delta_b)-i(b\gamma_a+a\gamma_b)=0\,,$$

$$\sum_{k=1}^{3} [(a-i\mu_{k(i)}b)\, r_{k(i)}\, A_k+(a-i\bar{\mu}_{k(i)}b)\, \bar{r}_{k(i)}\, \bar{A}_k-2\bar{r}_{k(m)}\, \bar{G}_k]=0\,.$$

It can be shown that the coefficient matrix of the above set of linear simultaneous algebraic equations is non-singular. These are six equations with six unknowns G_k's and A_k's ($k=1, 2, 3$). The values of these constants A_k's and G_k's can uniquely be obtained. These values are then substituted in Eqs (5.3) and (5.5) to obtain the values of the functions φ_k's. The stress field in the inclusion as well as in the matrix is determined with the aid of Eqs. (3.3) (5.3) and (5.5).

6. Results

It can easily be verified that the normal stress P_{nn} and the shear stresses P_{ns} and P_{nz} are continuous across the interface. Here n denotes the direction of the normal to the ellipse at the interface, s the tangential direction and z is the direction of the z-axis. It is observed that

$$(P_{nn}+iP_{ns})_{(i)}=(P_{nn}+iP_{ns})_{(m)}=\left(\frac{e^{-i\beta}}{2ab}\right)\sum_{k=1}^{3} [b_{k(m)}\, \bar{G}_k\times \times\{(a+b)\, e^{i\beta}+(a-b)\, e^{-i\beta}\}-a_{k(m)}\, G_k\, \{(a-b)\, e^{i\beta}+(a+b)\, ^{-i\beta}\}]\,,$$

$$(P_{nz})_{(i)}=(P_{nz})_{(m)}=\left(\frac{i}{2ab}\right)\sum_{k=1}^{3} [\bar{c}_{k(m)}\, \bar{G}_k\, \{(a+b)\, e^{i\beta}+(a-b)\, e^{-i\beta}\}- -c_{k(m)}\, G_k\, \{(a-b)\, e^{i\beta}+(a+b)\, e^{-i\beta}\}]\,.$$

Here β is the angle which the outward drawn normal at ($a\cos\theta$, $b\sin\theta$) on the inner boundary makes with the x-axis.

The hoop stresses P_{ss} and P_{zz} are discontinuous on the elliptical boundary

$$(P_{ss})_{(m)} = 2\mathrm{Re} \sum_{k=1}^{3} \frac{d_{k(m)} \{(a+b)(G_k e^{-i\beta}) + (a-b)(G_k e^{-i\beta})\}}{ab \{(1+i\mu_{k(m)}) e^{-i\beta} - (1-i\mu_{k(m)}) e^{i\beta}\}},$$

$$(P_{ss})_{(i)} = 2\mathrm{Re} \sum_{k=1}^{3} d_{k(i)} A_k,$$

$$(P_{zz})_{(m)} = 2\mathrm{Re} \sum_{k=1}^{3} \frac{t_{k(m)} \{(a+b)(G_k e^{-i\beta}) + (a-b)(G_k e^{i\beta})\}}{ab \{(1+i\mu_{k(m)}) e^{i\beta} - (1-i\mu_{k(m)}) e^{i\beta}\}},$$

$$(P_{zz})_{(i)} = 2\mathrm{Re} \sum_{k=1}^{3} (t_{k(i)} A_k),$$

where

$$d_k = (\cos\beta + \mu_k \sin\beta)^2, \qquad t_k \frac{1}{\alpha_{33}} (\alpha_{31} \mu_k^2 + \alpha_{32} - \alpha_{34} \lambda_k + \alpha_{35} \mu_k \lambda_k - \alpha_{36} \mu_k) \qquad (k=1, 2),$$

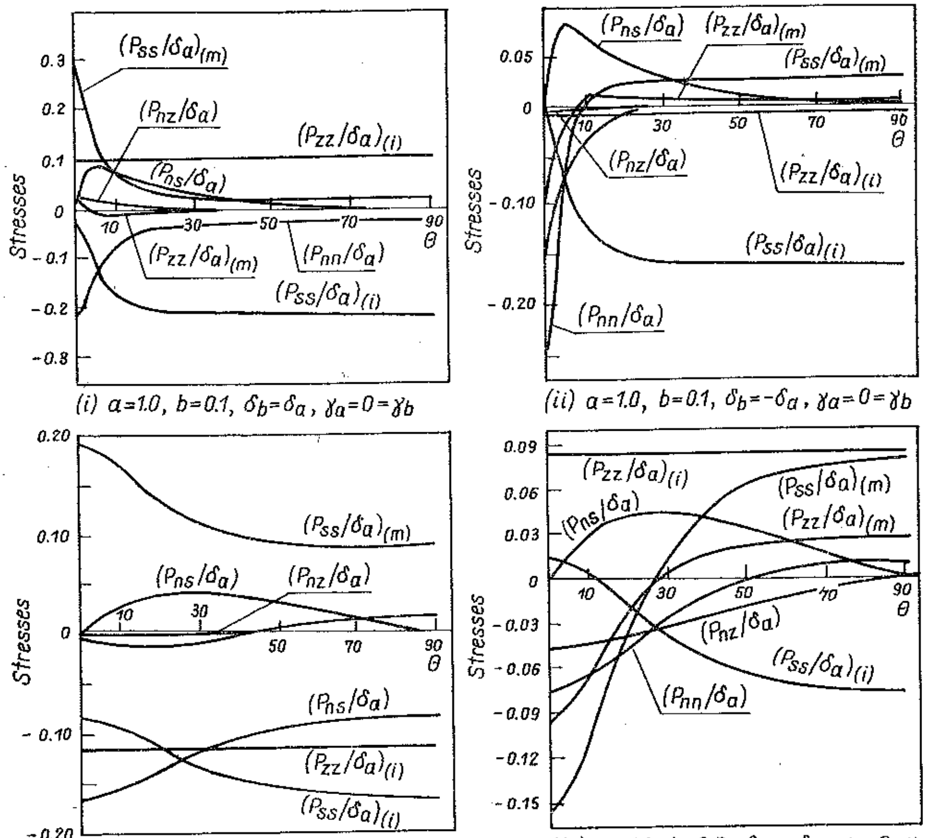
and

$$d_3 = \lambda_3 (\cos\beta + \mu_3 \sin\beta)^2, \qquad t_3 = \frac{1}{\alpha_{33}} (\alpha_{31} \mu_3^2 \lambda_3 + \alpha_{32} \lambda_3 - \alpha_{34} + \alpha_{35} \mu_3 - \alpha_{36} \mu_3 \lambda_3).$$

To obtain the plane strain case of CHEN [8], one puts $\alpha_{ij} = 0$ ($i = 1, 2, 3, 6$, $j = 4, 5$) and $u_z = 0$. The results for the cases of elliptic inclusion for different types of anisotropic symmetries may also be obtained as particular cases by putting the suitable values of α_{ij} and $u_z = 0$.

NUMERICAL CALCULATIONS

Numerical calculations are for three cases when (i) $a = 1.0$, $b = 0.1$, (ii) $a = 1.0$, $b = 0.5$, (iii) $a = 1.0$, $b = 1.0$. In each of the cases three different combinations are taken for the size of the misfit, i.e. when (i) $\delta_a = \delta_b$, $\gamma_a = \gamma_b = 0$ (ii) $\delta_a = -\delta_b$, $\gamma_a = \gamma_b = 0$, (iii) $\delta_a = \delta_b = 0$, $\gamma_a = \gamma_b$. The first set of (a, b) represents roughly the case of a thin inclusion. The last case gives the case of the circle. For the sake of illustration dipotassium tartrate and sodium thiosulphate are taken as materials for matrix and inclusion respectively. The elastic properties of dipotassium tartrate [12] are given by the following elastic constants: $\alpha_{11} = 4.75$, $\alpha_{22} = 3.53$, $\alpha_{33} = 2.40$, $\alpha_{44} = 11.4$, $\alpha_{55} = 10.2$, $\alpha_{66} = 12.3$, $\alpha_{12} = -1.74$, $\alpha_{13} = -0.8$, $\alpha_{23} = 0.62$, $\alpha_{15} = -0.75$, $\alpha_{25} = 0.8$, $\alpha_{35} = -1.40$, $\alpha_{46} = -0.68$. For sodium thiosulphate [13] the elastic constants are $\alpha_{11} = 5.02$, $\alpha_{22} = 15.6$, $\alpha_{33} = 6.74$, $\alpha_{44} = 22.3$, $\alpha_{55} = 32.7$, $\alpha_{66} = 21.2$, $\alpha_{12} = -3.23$, $\alpha_{13} = -0.621$, $\alpha_{23} = -7.19$, $\alpha_{15} = -1.2$, $\alpha_{35} = 11.0$ and $\alpha_{46} = 10.0$, $\alpha_{25} = -18.2$.



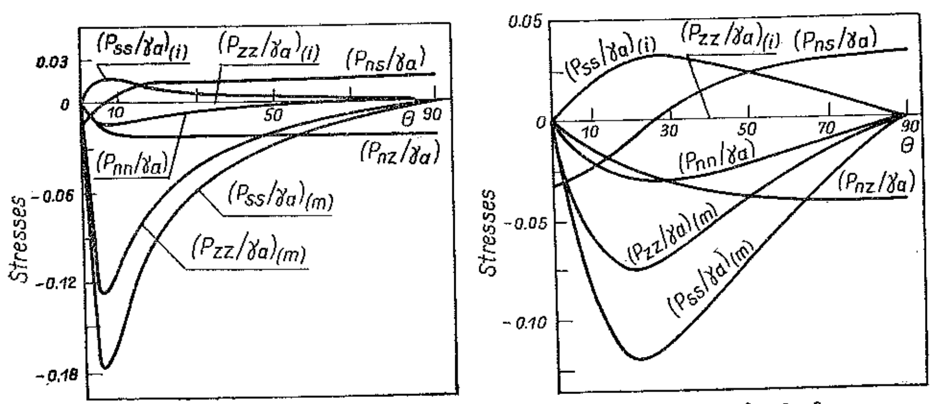
(i) $a=1.0,\ b=0.1,\ \delta_b=\delta_a,\ \gamma_a=0=\gamma_b$

(ii) $a=1.0,\ b=0.1,\ \delta_b=-\delta_a,\ \gamma_a=0=\gamma_b$

(iii) $a=1.0,\ b=0.5,\ \delta_b=\delta_a,\ \gamma_a=0=\gamma_b$

(iv) $a=1.0,\ b=0.5,\ \delta_b=-\delta_a,\ \gamma_a=0=\gamma_b$

I, II, III, IV

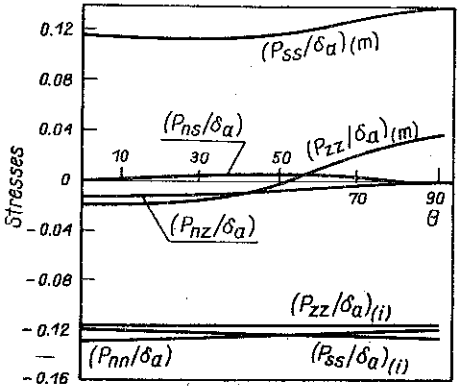


(v) $a=1.0,\ b=0.1,\ \delta_b=0=\delta_a,\ \gamma_b=\gamma_a$

(vi) $a=1.0,\ b=0.5,\ \delta_b=0=\delta_a,\ \gamma_b=\gamma_a$

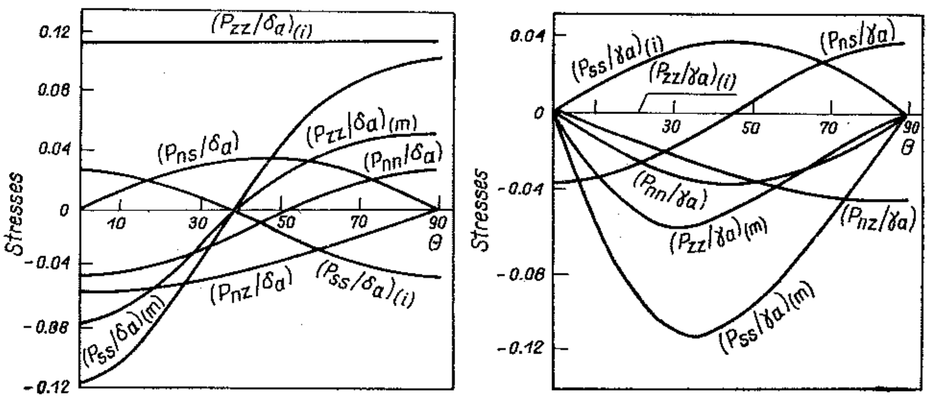
V, VI

FIG. 1



(vii) $a=b=1.0,\ \delta_b=\delta_a,\ \gamma_b=\gamma_a=0$

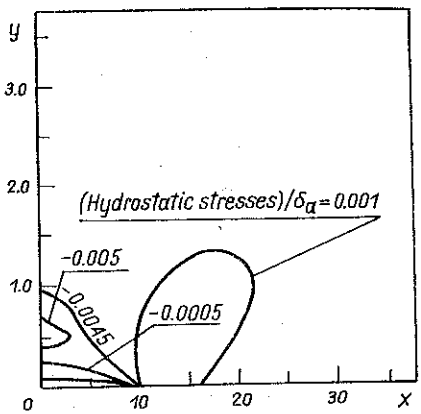
VII



(viii) $a=b=1.0,\ \delta_b=-\delta_a,\ \gamma_b=\gamma_a=0$ (ix) $a=b=1.0,\ \delta_b=\delta_a=0,\ \gamma_b=\gamma_a$

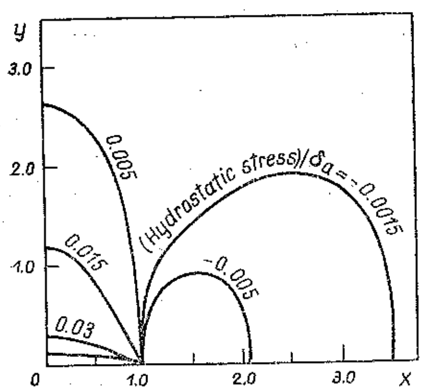
VIII, IX

FIG. 1.

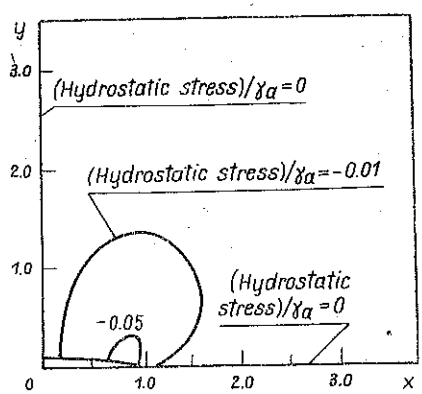


(i) $a = 1.0$, $b = 0.1$, $\delta_b = \delta_a$, $\gamma_b = \gamma_a = 0$

I



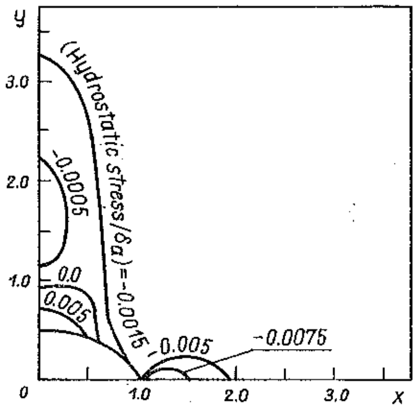
(ii) $a = 1.0$, $b = 0.1$, $\delta_b = -\delta_a$, $\gamma_b = \gamma_a = 0$



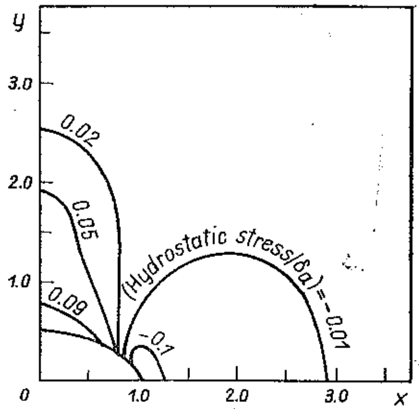
(iii) $a = 1.0$, $b = 0.1$, $\delta_a = \delta_b = 0$, $\gamma_b = \gamma_a$

II. III

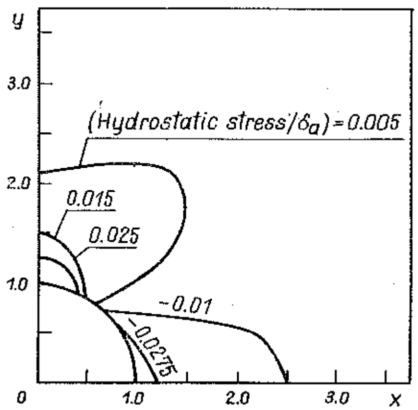
Fig. 2



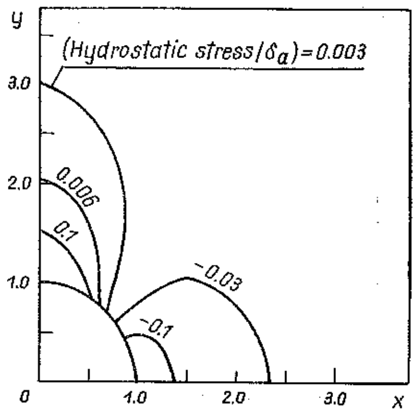
(iv) $a=1.0,\ b=0.5,\ \delta_b=\delta_a,\ \gamma_b=\gamma_a=0$



(v) $a=1.0,\ b=0.5,\ \delta_b=-\delta_a,\ \gamma_b=\gamma_a=0$

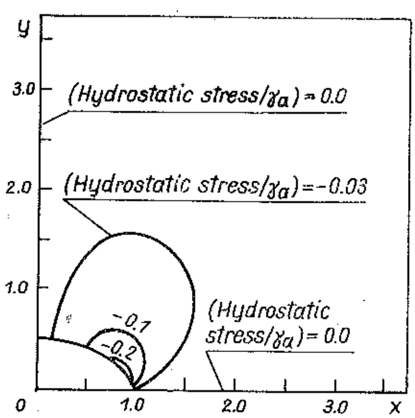


(vi) $a=b=1.0,\ \delta_b=\delta_a,\ \gamma_b=\gamma_a=0$

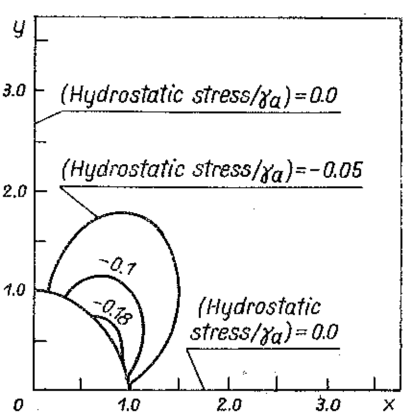


(vii) $a=b=1.0,\ \delta_b=-\delta_a,\ \gamma_b=\gamma_a=0$

IV, V, VI, VII



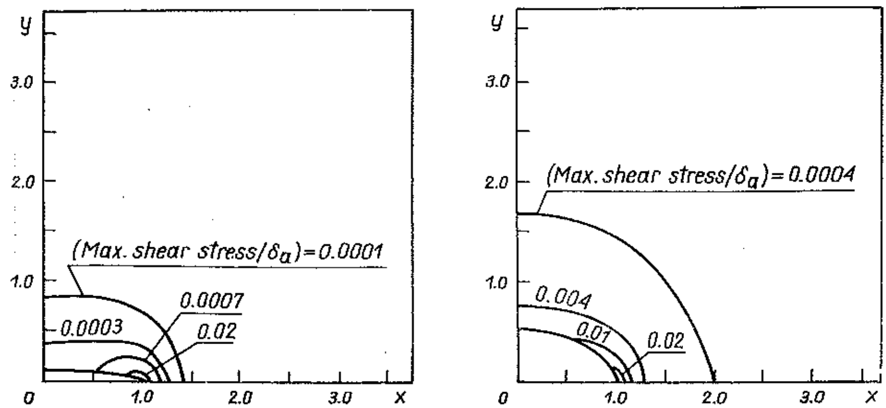
(viii) $a=1.0,\ b=0.5,\ \delta_b=\delta_a=0,\ \gamma_b=\gamma_a$



(ix) $a=b=1.0,\ \delta_b=\delta_a=0,\ \gamma_b=\gamma_a$

VIII, IX

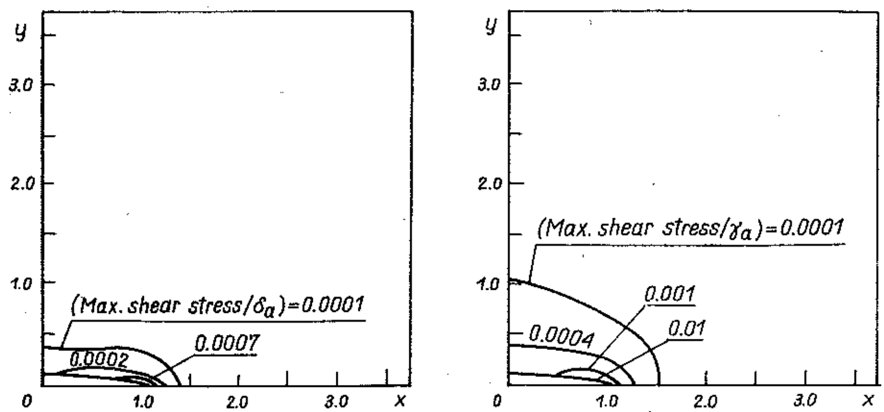
FIG. 2.



(i) $a=1.0,\ b=0.1,\ \delta_b=\delta_a,\ \gamma_b=\gamma_a=0$

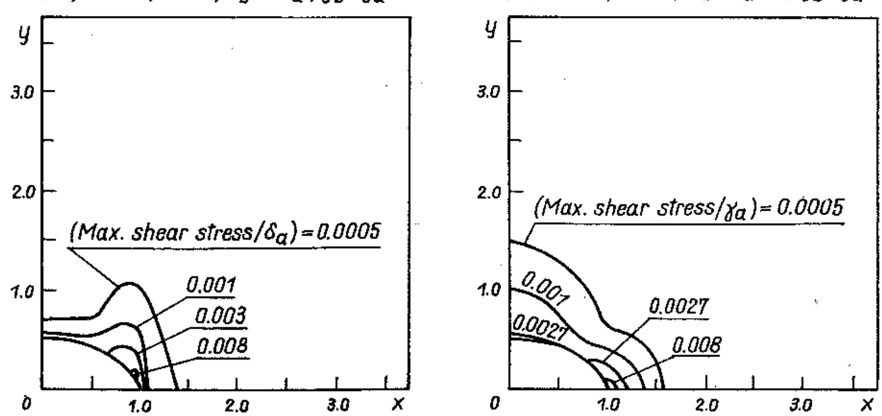
(ii) $a=1.0,\ b=0.5,\ \delta_b=\delta_a,\ \gamma_b=\gamma_a=0$

I, II



(iii) $a=1.0,\ b=0.1,\ \delta_b=-\delta_a,\ \gamma_b=\gamma_a=0$

(iv) $a=1.0,\ b=0.1,\ \delta_b=\delta_a=0,\ \gamma_b=\gamma_a$

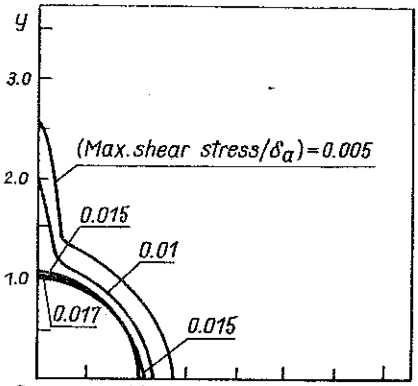


(v) $a=1.0,\ b=0.5,\ \delta_b=-\delta_a,\ \gamma_b=\gamma_a=0$

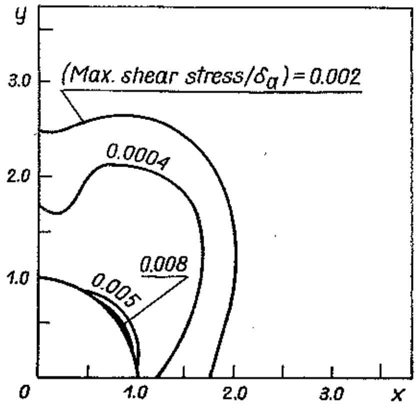
(vi) $a=1.0,\ b=0.5,\ \delta_b=\delta_a=0,\ \gamma_b=\gamma_a$

III, IV, V, VI

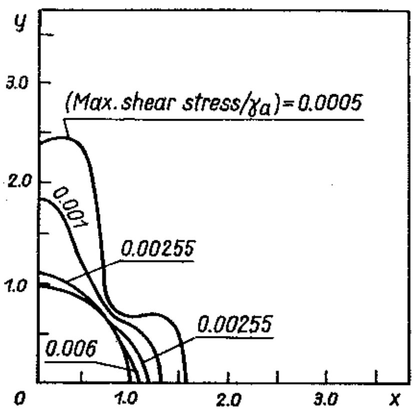
FIG. 3



(vii) $a=b=1.0,\ \delta_b=\delta_a,\ \gamma_b=\gamma_a=0$



(viii) $a=b=1.0,\ \delta_b=\delta_a,\ \gamma_b=\gamma_a=0$



(ix) $a=b=1.0,\ \delta_b=\delta_a=0,\ \gamma_b=\gamma_a$

VII, VIII, IX

FIG. 3.

In Fig. 1 curves are plotted for the stresses at the interface vs. the eccentric angle θ. The stresses P_{nn}, P_{ns}, P_{nz} are continuous at the interface while the hoop stresses P_{ss} and P_{zz} are discontinuous. In Figs. 2 and 3 the lines of hydrostatic ($H.S.=P_{xx}+$ $+P_{yy}+P_{zz}$) and maximum shear stress ($M.S.=\frac{1}{2}$ $[(P_{xx}-P_{yy})^2+(P_{yy}-P_{zz})^2+$ $+(P_{zz}-P_{xx})^2+2(P_{yz}^2+P_{zx}^2+P_{xy}^2)]$) [11] in the matrix in the (x, y) plane are drawn. It is to be remarked here that in all cases where $\delta_a=\delta_b=0$, $\gamma_a=\gamma_b$, the hydrostatic stress is zero on the x and y axes.

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Streszczenie

ZAGADNIENIE NIEDOPASOWANIA ELIPTYCZNEJ SPRĘŻYSTEJ NIEJEDNORODNOŚCI W OŚRODKACH IDEALNIE ANIZOTROPOWYCH

W pracy rozważany jest problem niedopasowania się eliptycznej inkluzji w będącym poprzednio nienaprężonym, nieskończonym ośrodku. Sprężyste własności inkluzji różnią się od własności matrycy. Matryca i inkluzja są idealnie połączone na powierzchni kontaktu. Każdy z obydwu materiałów posiada anizotropię ogólnej postaci. Do rozwiązania rozważanego zagadnienia zastosowano metodę zmiennej zespolonej. Określono dwa układy zespolonych potencjałów; jeden dla matrycy, drugi dla inkluzji. Obliczenia numeryczne przeprowadzono dla materiałów, dla których płaszczyzna x-z jest płaszczyzną symetrii sprężystej. Rozważono różne przypadki niedopasowania kształtu. Wykreślono krzywe pokazujące zachowanie się bezwymiarowych naprężeń na powierzchni kontaktu. Linie naprężenia hydrostatycznego i maksymalnych naprężeń ścinających narysowano w matrycy, w płaszczyźnie x-y.

Р е з ю м е

ЗАДАЧА НЕСОГЛАСОВАНИЯ ЭЛЛИПТИЧЕСКОЙ УПРУГОЙ НЕОДНОРОДНОСТИ В ИДЕАЛЬНО АНИЗОТРОПНЫХ СРЕДАХ

В работе рассматривается проблема несогласования эллиптического включения в будучей раньше ненапряженной, бесконечной среде. Упругие свойства включения отличаются от свойств матрицы. Матрица и включение идеально соединены на поверхности контакта. Каждый из обоих материалов обладает анизотропней общего вида. Для решения рассматриваемой задачи применен метод комплексной переменной. Определены две системы комплексных потенциалов, один для матрицы, второй для включения. Численные расчеты проведены для материалов, для которых плоскость *x-z* является плоскоьстю упругой симметрии. Рассмотрены разные случаи несогласования формы. Вычерчены кривые указывающие поведение безразмерных напряжений на поверхности контакта. Линии гидростатического напряжения и максимальных напряжений сдвига в матрицы начерчены в плоскости *x-z*.

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