ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSACTIONS • 25, 4, 559—570, 1977
Polska Akademia Nauk • Instytut Podstawowch Problemów Techniki

NONLINEAR BENDING OF SYMMETRICALLY LAMINATED AND HOMOGENEOUS ANISOTROPIC PLATES

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A large deflection analysis of symmetrically laminated anisotropic rectangular plates under transverse load is presented with homogeneous anisotropic plates treated as a special case. Solutions to the von Kármán-type large deflection equations expressed in terms of trnasverse deflection and force function are formulated in the form of double series for clamped and simply supported plates respectively. Numerical results are graphically presented for symmetric angle-ply and homogeneous graphite-epoxy square plates. In the case of small deflections the present results are in good agreement with the existing solutions.

1. Introduction

As it is well known, the governing equations of symmetrically laminated anisotropic plates are different from those of homogeneous anisotropic plates by constant coefficients. Thus the techniques developed for the latter can be applied to the former. Basing on the linear plate theory some investigators have considered the elastic problem of anisotropic plates. Green and Hearmon [1] have considered the buckling of the plates with clamped and simply supported edges. The latter boundary conditions have been satisfied by virtue of a procedure developed by Green [2] for clamped isotropic plates. Using the same procedure Whitney [3, 4] has investigated the bending, stability, and vibration of clamped and simply supported plates and Sun [5] has considered the bending of simply supported plates. Utilizing the Ritz method associated with a double series of beam eigenfunctions for the transverse deflection, Ashton and Waddoups [6] have studied the bending, stability, and vibration of the plates with several sets of boundary conditions. Except for clamped edges the expressions for the deflection do not satisfy all the boundary conditions. The other works on the subject are not mentioned herein. In addition, the large deflection of rigidly clamped anisotropic rectangular plates has been considered by Chia [7] using the perturbation technique and by Turvey and Wittrick [8] using the dynamic relaxation method.

In this paper large deflections of symmetrically laminated anisotropic plates under uniform transverse load are presented. The edges of both clamped and simply supported plates under consideration are assumed to be free from the applied in-plane forces. The von Kármán-type large deflection equations are adopted in this work. In the case of clamped edges, a solution is formulated on the basis of the double series solution [9] involving characteristic beam functions and satisfying all the boundary conditions. In the case of simply supported edges the deflection is expressed as a double sine series and the force function as a double series in terms of beam functions. The solution satisfies the boundary conditions for the vanishing of deflection and stresses but not those for the vanishing of normal bending moments. The latter is satisfied by means of the procedure suggested by GREEN [2]. In both cases the coupled nonlinear partial differential equations are reduced to a system of nonlinear algebraic equations which are solved by an iterative process for the deflection and force function coefficients.

2. ANALYSIS

Consider a rectangular plate of the thickness h in the z direction which lies in the region $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ and is subjected to uniform transverse load of the intensity q. The plate consists of n layers of orthotropic sheets perfectly bonded together. Each layer has an arbitrary thickness, elastic properties and orientation of orthotropic axes with respect to the plate axes. However, the layers are so arranged that a mid-plane symmetry exists. That is, for each layer above the mid-plane there is a corresponding layer which is identical in thickness, elastic properties and orientation and is located at the same distance below the mid-plane. The differential equations governing the large deflection of the plate can be written in the nondimensional form [9]

$$(2.1)\qquad A^*_{22} F_{,\zeta\zeta\zeta\zeta} - 2\lambda A^*_{26} F_{,\zeta\zeta\zeta\eta} + \lambda^2 (2A^*_{12} + A^*_{66}) F_{,\zeta\zeta\eta\eta} - \\ - 2\lambda_3 A^*_{16} F_{,\zeta\eta\eta\eta} + \lambda^4 A^*_{11} F_{,\eta\eta\eta\eta} = \lambda^2 [(W_{,\zeta\eta})^2 - W_{,\zeta\zeta} W_{,\eta\eta}],$$

$$(2.2)\qquad D^*_{11} W_{,\zeta\zeta\zeta\zeta} + 4\lambda D^*_{16} W_{\zeta\zeta\zeta\eta} + 2\lambda^2 (D^*_{12} + 2D^*_{66}) W_{,\zeta\zeta\eta\eta} + \\ + 4\lambda^3 D^*_{26} W_{,\zeta\eta\eta\eta} + \lambda^4 D^*_{22} W_{,\eta\eta\eta\eta} = \lambda^4 Q + \lambda^2 [W_{,\zeta\zeta} F_{,\eta\eta} + W_{,\eta\eta} F_{,\zeta\zeta} - 2W_{,\zeta\eta} F_{,\zeta\eta}],$$

where a comma denotes partial differentiation with respect to the corresponding coordinates and where

$$(2.3)\qquad \begin{gathered} \zeta = x/a, \quad \eta = y/b, \quad \lambda = a/b, \quad W = w/h, \\ F = \Phi/A_{22} h^2, \quad Q = qb^4/A_{22} h^3, \quad \bar{A} = A^{-1}, \\ A^*_{ij} = A_{22} \bar{A}_{ij}, \quad D^*_{ij} = D_{ij}/A_{22} h^2 \quad (i, j = 1, 2, 6), \\ (A_{ij}, D_{ij}) = \int_{-h/2}^{h/2} C^{(k)}_{ij} (l, z^2)\, dz \quad (i, j = 1, 2, 6). \end{gathered}$$

In Eqs. (2.3), Φ is the force function, $C^{(k)}_{ij}$ are the anisotropic stiffness coefficients of the k^{th} layer of the plate.

If the membrane forces are denoted by N_x, N_y, N_{xy}, the nondimensional membrane forces N_ζ, N_η, $N_{\zeta\eta}$ are related to the nondimensional force function F by

$$(2.4)\qquad N_\zeta = F_{,\eta\eta}, \quad N_\eta = F_{,\zeta\zeta}/\lambda^2, \quad N_{\zeta\eta} = -F_{,\zeta\eta}/\lambda$$

in which

$$\{N_\zeta, N_\eta, N_{\zeta\eta}\}=(b^2/A_{22}h^2)\{N_x, N_y, N_{xy}\}, \tag{2.5}$$

The bending moments, M_x, M_y and M_{xy} are also written in the nondimensional form

$$\begin{bmatrix} M_\zeta \\ M_\eta \\ M_{\zeta\eta} \end{bmatrix}=\begin{bmatrix} D_{11}^*/\lambda^2 & D_{12}^* & D_{16}^*/\lambda \\ D_{12}^*/\lambda^2 & D_{22}^* & D_{26}^*/\lambda \\ D_{16}^*/\lambda^2 & D_{26}^* & D_{66}^*/\lambda \end{bmatrix}\begin{bmatrix} -W_{,\zeta\zeta} \\ -W_{,\eta\eta} \\ -2W_{,\zeta\eta} \end{bmatrix}, \tag{2.6}$$

where

$$\{M_\zeta, M_\eta, M_{\zeta\eta}\}=(b^2/A_{22}h^3)\{M_x, M_y, M_{xy}\}, \tag{2.7}$$

Equations (2.1) to (2.7) governing the nonlinear bending of a symmetrically laminated rectangular ainsotropic plate under transverse load are reduced to those of a homogeneous rectangular anisotropic plate when the thickness, elastic properties and orientation of all the layers are taken to be identical with one another.

If the edges of the plate are free from in-plane forces, the boundary conditions are, for a clamped plate,

$$\begin{aligned} &W=W_{,\zeta}=F_{,\eta\eta}=F_{,\zeta\eta}=0 \quad \text{at} \quad \zeta=0,1, \\ &W=W_{,\eta}=F_{,\zeta\zeta}=F_{,\zeta\eta}=0 \quad \text{at} \quad \eta=0,1, \end{aligned} \tag{2.8}$$

and for a simply supported plate

$$\begin{aligned} &W=M_\zeta=F_{,\eta\eta}=F_{,\zeta\eta}=0 \quad \text{at} \quad \zeta=0,1, \\ &W=M_\eta=F_{,\zeta\zeta}=F_{,\zeta\eta}=0 \quad \text{at} \quad \eta=0,1. \end{aligned} \tag{2.9}$$

Equations (2.1) and (2.2) are to be solved in conjunction with the appropriate boundary conditions given by Eqs. (2.8) or (2.9). The solutions are respectively formulated for these two sets of boundary conditions.

2.1. Clamped plate

In this case the two variables F and W are assumed to be of the form

$$F=\sum_m \sum_n F_{mn} X_m(\zeta) Y_n(\eta), \tag{2.10}$$

$$W=\sum_p \sum_q W_{pq} X_p(\zeta) Y_q(\eta). \tag{2.11}$$

In Eqs. (2.10) and (2.11) the beam eigenfunctions, X_i and Y_j, and their properties are given in Ref. 9. It can be shown that F and W satisfy all the boundary conditions given by Eqs. (2.8). Substituting Eqs. (2.10) and (2.11) into Eqs. (2.1) and (2.2), multiplying the resulting equations by $X_i(\zeta) Y_j(\eta)$, integrating from 0 to 1 with respect to ζ and η, and using the properties of beam eigenfunctions, the system of coupled nonlinear algebraic equations is obtained as follows:

$$\tag{2.12} F_{ij}[\alpha_i^4 A_{22}^* + \lambda^4 \beta_j^4 A_{11}^*] - \sum_m \sum_n F_{mn}[2\lambda\alpha_m^3 \beta_n A_{26}^* K_3^{im} L_1^{jn} - \lambda^2 \alpha_m^2 \beta_n^2 K_2^{im} L_2^{jn}(2A_{12}^* + A_{66}^*) + 2\lambda^3 \alpha_m \beta_n^3 A_{16}^* K_1^{im} L_3^{jn}] = \lambda^2 \sum_r \sum_s \sum_k \sum_l W_{rs} W_{kl}[\alpha_r \alpha_k \beta_s \beta_l K_4^{irk} L_4^{jsl} - \alpha_r^2 \beta_l^2 K_5^{irk} L_5^{jls}],$$

$$\tag{2.13} W_{ij}[\alpha_i^4 D_{11}^* + \lambda^4 \beta_j^4 D_{22}^*] + \sum_p \sum_q W_{pq}[4\lambda\alpha_p^3 \beta_q D_{16}^* K_3^{ip} L_1^{jq} + 2\lambda^2 \alpha_p^2 \beta_q^2 K_2^{ip} L_2^{jq}(D_{12}^* + 2D_{66}^*) + 4\lambda^3 \alpha_p \beta_q^3 D_{26}^* K_1^{ip} L_3^{jq}] - \lambda^2 \sum_r \sum_s \sum_k \sum_l W_{rs} F_{kl}[\alpha_r^2 \beta_l^2 K_5^{irk} L_5^{jls} + \alpha_k^2 \beta_s^2 K_5^{ikr} L_5^{jsl} - 2\alpha_r \alpha_k \beta_l \beta_s K_4^{irk} L_4^{jsl}] = \lambda^4 Q_{ij},$$

where α_m, β_n, K_p and L_p $(p=1, 2, 3, 4, 5)$ are given in [9] and where

$$\tag{2.14} Q = \sum_p \sum_q Q_{pq} X_p(\zeta) Y_q(\eta).$$

2.2. *Simply supported plate*

The two dependent functions F and W in this case are assumed to be of the form,

$$\tag{2.15} F = \sum_m \sum_n F_{mn} X_m(\zeta) Y_n(\eta),$$

$$\tag{2.16} W = \sum_p \sum_q W_{pq} \sin p\pi\zeta \sin q\pi\eta,$$

in which $X_m(\zeta)$ and $Y_n(\eta)$ are the beam eigenfunctions as those in Eq. (2.10) satisfying all the prescribed in-plane boundary conditions in Eqs. (2.9). Equation (2.16) for W satisfies the edge condition for the vanishing of the transverse deflection but not the edge condition for the vanishing of the normal bending moment. In order to satisfy the latter boundary condition the method suggested by GREEN [1, 2] is used herein. In the present case Eq. (2.16) cannot be differentiated term by term beyond $W_{,\zeta\zeta}$ with respect to ζ, and $W_{,\eta\eta}$ with respect to η. Assume that $W_{,\zeta\zeta\zeta}$ can be represented by a cosine-sine series. The Fourier cofficients in the series can be determined by integrating by parts and then using Eq. (2.16). Thus the series can be written as

$$\tag{2.17} W_{,\zeta\zeta\zeta} = \frac{1}{2}\sum_q a_q \sin q\pi\eta + \sum_{p=2,4} \sum_q a_q \cos p\pi\zeta \sin q\pi\eta + \sum_{p=1,3} \sum_q b_q \cos p\pi\zeta \sin q\pi\eta - \sum_p \sum_q p^3 \pi^3 W_{pq} \cos p\pi\zeta \sin q\pi\eta,$$

where

$$(2.18)\qquad \begin{aligned} a_q &= 4\int_0^1 [W_{,\zeta\zeta}(1,\eta)-W_{,\zeta\zeta}(0,\eta)]\sin q\pi\eta d\eta\,,\\ b_q &= -4\int_0^1 [W_{,\zeta\zeta}(1,\eta)+W_{,\zeta\zeta}(0,\eta)]\sin q\pi\eta\, d\eta\,. \end{aligned}$$

A similar procedure is applied to $W_{,\eta\eta}$ which leads to two sets of constants c_p and d_p. Substituting $W_{,\zeta\eta}$ from Eq. (2.16) into the condition that $M_\zeta=-D^*_{11}\,W_{,\zeta\zeta}/\lambda^2-$ $-2D^*_{16}\,W_{,\zeta\eta}/\lambda=0$ at $\zeta=0, 1$, adding these equations, multiplying the resulting equation by $\sin i\pi\eta$, integrating from 0 to 1 and using Eqs. (2.18), the coefficients b_i can be expressed in terms of W_{pq}. If subtraction is carried out instead of addition in the calculation, a_i can be determined. Similarly, c_i and d_i can be obtained. The result is

$$(2.19)\qquad \begin{aligned} a_i &= (8\lambda D^*_{16}/D^*_{11})\sum_p\sum_q pq\pi^2\,W_{pq}\,H^{iq}_1\,, && p=\text{odd},\\ b_i &= (8\lambda D^*_{16}/D^*_{11})\sum_p\sum_q pq\pi^2\,W_{pq}\,H^{iq}_1\,, && p=\text{even},\\ c_i &= (8D^*_{26}/\lambda D^*_{22})\sum_p\sum_q pq\pi^2\,W_{pq}\,H^{ip}_1\,, && q=\text{odd},\\ d_i &= (8D^*_{26}/\lambda D^*_{22})\sum_p\sum_q pq\pi^2\,W_{pq}\,H^{ip}_1\,, && q=\text{even}, \end{aligned}$$

in which

$$(2.20)\qquad \begin{aligned} H^{mn}_1 &= 0 && \text{if}\quad m+n=\text{even},\\ H^{mn}_3 &= 4m/\pi(m^2-n^2) && \text{if}\quad m+n=\text{odd}. \end{aligned}$$

By virtue of Eqs. (2.19) it can be shown that

$$(2.21)\qquad D^*_{11}\,[(ia_j)_{i\ \text{even}}+(ib_j)_{i\ \text{odd}}]+D^*_{22}\,\lambda^4\,[(jc_i)_{j\ \text{even}}+(id_i)_{j\ \text{odd}}]+$$
$$+4\pi^3\lambda\sum_p\sum_q W_{pq}pqH^j_1\;q(D^*_{16}\,p^2+D^*_{26}\,\lambda^2\,q^2)=$$
$$=2\pi^3\lambda\sum_p\sum_q W_{pq}pqH^{ip}_1H^{jq}_1\{D^*_{16}\,(D^*_{16}\,i^2+p^2)+D^*_{26}\,\lambda^2\,(j^2+q^2)\}.$$

Now the fourth derivatives of the deflection function W in Eq. (2.2) except for $W_{,\zeta\zeta\eta\eta}$ can be obtained through term-by-term differentiation of $W_{,\zeta\zeta\zeta}$ and $W_{,\eta\eta\eta}$ as given by Eq. (2.17). Substituting these derivatives and Eqs. (2.15) and (2.16) into Eqs. (2.1) and (2.2), multiplying the first of the resulting equations by $X_m(\zeta)\,Y_n(\eta)$ and the second by $\sin i\pi\zeta\,\sin j\pi\eta$, integrating from 0 to 1 with respect to ζ and η

and using the properties of beam eigenfunctions [9] and Eqs. (2.19) and (2.21), the following system of equations is obtained after some manipulation:

$$(2.22)\quad F_{ij}[\alpha_i^4 A_{22}^* + \lambda^4 \beta_j^4 A_{11}^*] - \sum_m \sum_n F_{mn}[2\alpha_m^3 \beta_n A_{26}^* K_3^{im} L_1^{jn} - \lambda^2 \alpha_m^2 \beta_n^2 K_2^{im} L_2^{jn}(2A_{12}^* + A_{66}^*) + 2\lambda^3 \alpha_m \beta_n^3 A_{16}^* K_1^{im} L_3^{jn}] = \lambda^2 \pi^4 \sum_r \sum_s \sum_k \sum_l W_{rs} W_{kl} [rsklR_3^{ikr} S_3^{jls} - k^2 s^2 R_6^{ikr} S_5^{jsl}],$$

$$(2.23)\quad W_{ij}[i^4 D_{11}^* + 2\lambda^2 i^2 j^2 (D_{12}^* + 2D_{66}^*) + \lambda^4 j^4 D_{22}^*] - 2\lambda \sum_p \sum_q W_{pq}[pqH_1^{ip} H_1^{jq}\{D_{16}^*(i^2+p^2) + \lambda^2 D_{26}^*(j^2+q^2)\}] + (16\lambda^2 D_{16}^{*2}/\pi D_{11}^*) \sum_r \sum_s W_{rs} rs \Big[\sum_q qH_2^i H_3^r H_1^{jq} H_1^{qs} + \sum_p \sum_q 2qH_1^{ip} H_1^{jq} H_1^{qs} H_4^{pr}\Big] + (16\lambda^2 D_{26}^{*2}/\pi D_{22}^*) \sum_r \sum_s W_{rs} rs \times \Big[\sum_p pH_2^j H_3^s H_1^{ip} H_1^{pr} + \sum_p \sum_q 2pH_1^{ip} H_1^{jq} H_1^{pr} H_4^{qs} + (4\lambda^2/\pi^2) \sum_m \sum_n \sum_r \sum_s W_{rs} F_{mn}[\alpha_m^2 s^2 R_7^{mir} S_4^{njs} + \beta_n^2 r^2 R_4^{mir} S_7^{njs} + 2\alpha_m \beta_n rsR_8^{mir} S_8^{njs}\Big] = \lambda^4 Q_{ij}/\pi^4,$$

where α_i, β_i, K_j, L_j $(j=1, 2, 3)$ R_k and S_k $(k=3, 4, 7, 8)$ are given in Ref. 9 and where

$$(2.24)\quad H_2^n = \begin{cases} 0, & n \text{ even}, \\ 4/n\pi, & n \text{ odd}, \end{cases} \quad H_3^n = \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd}, \end{cases} \quad H_4^{mn} = \begin{cases} 0, & m+n \text{ even}, \\ 1, & m+n \text{ odd}, \end{cases}$$
$$Q = \sum_p \sum_q Q_{pq} \sin p\pi\zeta \sin \pi\eta .$$

Equations (2.12) and (2.13) for a clamped plate or Eqs. (2.22) and (2.23) for a simply supported plate constitute an infinite system of coupled nonlinear algebraic equations which are to be solved for coefficients W_{pq} and F_{mn} for a given set of material properties, orientation angle and aspect ratio. As soon as these coefficients are determined, the deflection, membrane forces and bending moments can be obtained from Eqs. (2.11) [or (2.16)], (2.4) and (2.6) respectively.

3. Results and discussions

Numerical results are presented for square symmetric angle-ply and homogeneous anisotropic graphite-epoxy plates. The elastic constants with respect to the material orthotropic axes are taken to be $E_L/E_T=40$, $G_{LT}/E_T=0.5$, $\nu_{LT}=0.25$, in which E_L is the tensile modulus in the filament direction, E_T the tensile modulus in the direction, perpendicular to the filaments, ν_{LT} the Poisson's ratio and G_{LT} the shear modulus. In the case of a symmetric angle-ply plate the number of layers considered are $n=1$, 3, 5 and 7. The orthotropic axes of the layers are alternately oriented at $+45°$ and $-45°$ with respect to the plate axes and all the layers are of equal thickness. For homogeneous anisotropic plates the angle of orientation, θ, between the material axes of symmetry and the plate axes is taken as 0°, 15°, 30° and 45°. In these cases the nondimensional constants A^*_{ij} and D^*_{ij} $(i, j=1, 2, 6)$ in Eqs. (2.3), (2.5) and (2.7) can be simplified by replacing A_{22} in these equations by $E_T h$. The nondimensional load parameter also simplifies to $Q=qb^4/E_T h^4$. This change will not affect the form of other equations. The largest value of the transverse load parameter is so selected that the maximum deflection does not exceed three times the thickness of the plate.

The system of nonlinear algebraic equations (2.12) and (2.13) or (2.22) and (2.23) is solved by an iterative procedure. The value of the deflection coefficient W_{11} is prescribed and the values of transverse load parameter Q, other deflection coefficients W_{ij} and force function coefficients F_{ij} are then determined from the system of equations. Once the value of W_{11} is prescribed and an initial guess for W_{ij} $(i\neq 1, j\neq 1)$ is made, Eqs. (2.12) or (2.22) become linear and are solved for the F_{ij} coefficients. These F_{ij} coefficients are then substituted into Eqs. (2.13) or (2.23) and the resulting set of linear equations are solved for Q and W_{ij} $(i\neq 1, j\neq 1)$. These values of W_{ij} and the prescribed value of W_{11} are now used in Eqs. (2.12) or (2.22) to determine the new values of the F_{ij} coefficients. These F_{ij} coefficients are now used in Eqs. (2.13) or (2.23) to determine new values of Q and W_{ij} $(i\neq 1, j\neq 1)$ and the process is continued until the desired accuracy is achieved. The criterion for the convergence of the iterative process is that the difference between the final value of the central deflection and the average of the values in the previous five iterations is less than one percent.

Table 1. Comparison of 25-term solution with 49-term solution

Clamped homogeneous graphite-epoxy plate, $\theta=30°$, $\lambda=1.0$			Simply supported five-layer graphite-epoxy laminate, $\theta=\pm45°$, $\lambda=1.0$		
W_A	Q		W_A	Q	
	25 terms	49 terms		25 terms	49 terms
0.60	469.0	466.0	0.49	173.0	174.0
1.20	968.0	961.0	0.97	354.0	357.0
1.75	1520.0	1503.0	1.45	553.0	557.0
2.30	2133.0	2106.0	2.38	1026.0	1030.0
2.83	2815.0	2779.0	2.82	1303.0	1310.0

In order to investigate the convergence of the present series solution, calculations for two typical cases were made by the use of 25-terms $(i, j=1, 2, ..., 5)$ and 49-terms $(i, j=1, 2, ..., 7)$ in the series for F and W. The results are presented in Table 1 in which W_A is the nondimensional deflection W at the center of the plate. These results indicate that the two series both converge rapidly and that the 49-term solution should be reasonably accurate for engineering purposes.

In the case of small deflections the numerical results obtained from the present solution for the central deflection of an anisotropic plate with various values of θ agree very well with those of WHITNEY [3] and ASHTON and WADOUPS [6] for clamped edges and of SUN [5] for simply supported edges but are different from those presented by WHITNEY [4] for simply supported edges by the amount of five per cent.

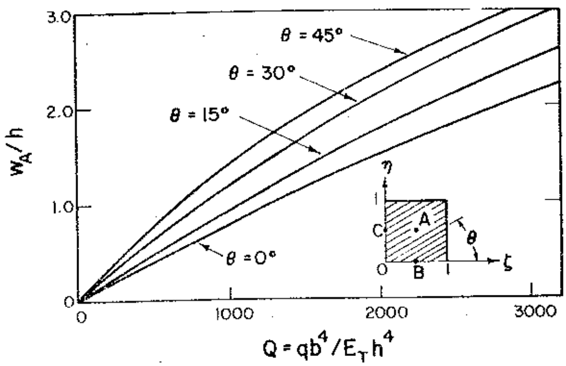

FIG. 1. Load deflection relations for clamped square anisotropic plate with different fiber orientations.

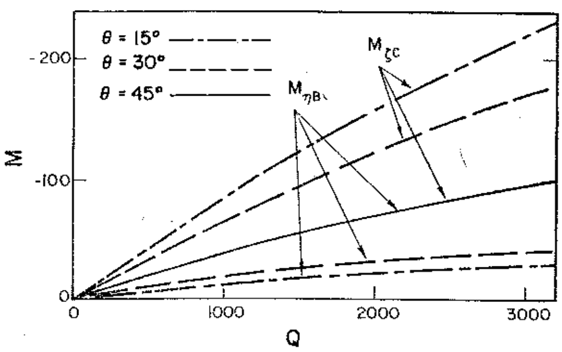

FIG. 2. Bending moments at the mid-points of the edges ($\zeta=\eta=0$) for clamped square anisotropic plate with different fiber orientations.

Calculations were carried out by using a CDC-6400 computer. Numerical results are graphically presented in Figs. 1 to 3 for clamped anisotropic plates, in Figs. 4 to 7 for simply supported anisotrpiç plates and in Fig. 8 for a simply supported symmetric angle-ply plate. The relation between load and central deflection w_A is shown in Fig. 1 for a clamped anisotropic plate with various values of θ. It may be observed that for a fixed load the central deflection increases as θ increases with the orthotropic plate having the smallest deflection. In Fig. 2 bending moments at the mid-points of the sides are presented for various values of θ. The largest bending moment M_ζ in magnitude occurs at the mid-points of sides, $\zeta=0, 1$. The relation between load and membrane forces at the center of a clamped plate is shown in Fig. 3. As θ decreases, the membrane force N_ζ increases whereas N_η decreases. At $\theta=45°$, N_ζ and N_η are equal to each other. In Fig. 4 the relation between the load and central deflection of a simply supported anisotropic plate is presented

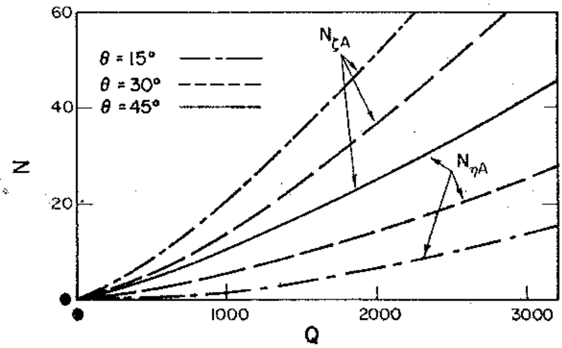

FIG. 3. Membrane forces at the center of clamped square anisotropic plate with different fiber orientation.

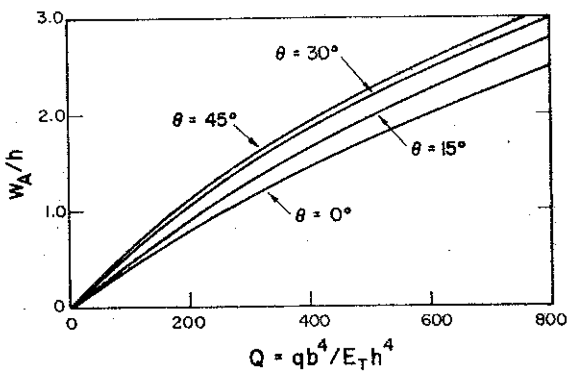

FIG. 4. Load-deflection relations for simply supported square anisotropic plate with different fiber orientations.

for various values of θ. For a given load the central deflection increases with increasing θ as in the case of clamped edges and the central deflection of a simply supported plate is larger than that of a corresponding clamped plate. Figure 5 shows the variation of central bending moments with transverse load for an anisotropic plate. As θ increases, the bending moment M_ζ decreases but M_η increases. These two bending moments are equal to each other for $\theta=45°$. The distribution of the bending moment M_ζ along the central line, $\eta=0.5$, of an anisotropic plate with $\theta=45°$ is shown in Fig. 6 for various values of load Q. For large values of Q the maximum bending moment is no longer at the center of the plate but shifts toward the edges. This indicates

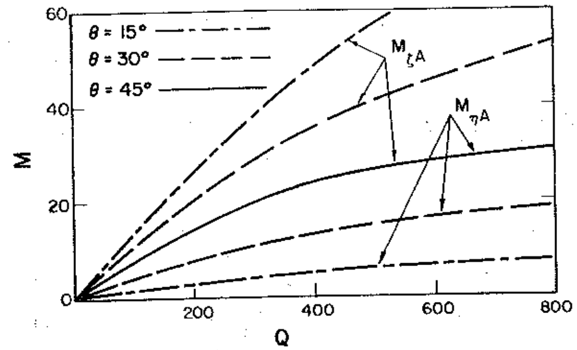

FIG. 5. Bending moments at center of simply supported anisotropic square plate with different fiber orientations.

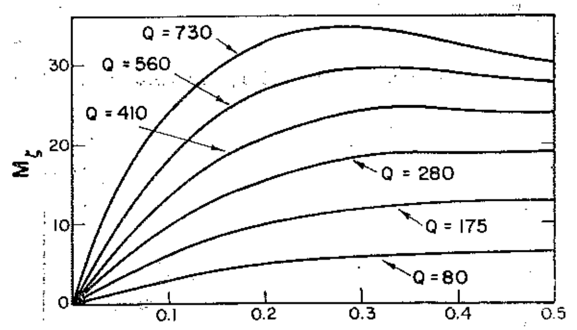

FIG. 6. Distribution of bending moment M_ζ along center line, $\eta=0.5$, of simply supported square anisotropic plate ($\theta=45°$) with various values of transverse load.

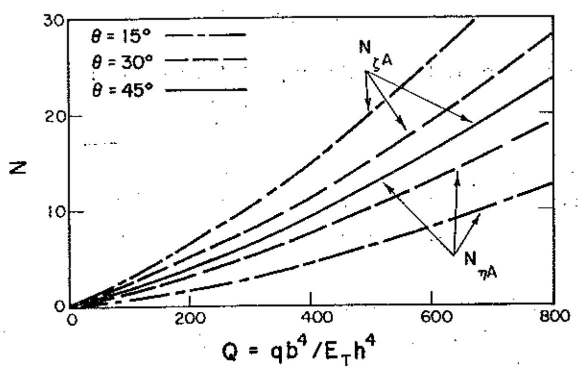

FIG. 7. Membrane forces at center of simply supported square anisotropic plate with different fiber orientations.

the development of a boundary layer in the bending moment distribution. In Fig. 7 the membrane forces at the center of an anisotropic plate are presented for various values of θ. The membrane force N_ξ increases but N_η decreases as θ decreases and N_ξ is equal to N_η for $\theta=45°$. The load deflection relations for a simply supported

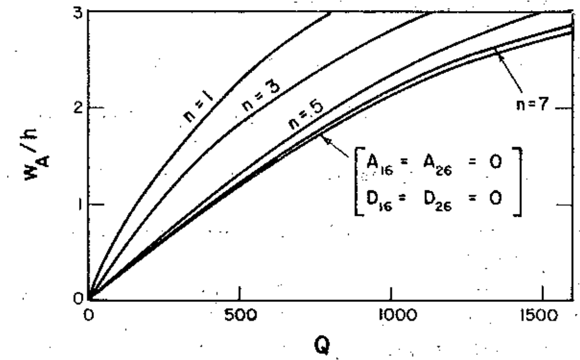

FIG. 8. Effect of number of layers on load-deflection relation for simply supported $\pm45°$ symmetric angle-ply square plate.

symmetric angle-ply anisotropic plate having different number of layers is shown in Fig. 8. It is found that the central deflection for a fixed value of the transverse load decreases with an increase in the number of layers. For $n=7$ the central deflection of the plate is very close to that of a special orthotropic plate ($\theta=\pm45°$, $A_{16}=A_{26}=D_{16}=D_{26}=0$).

4. CONCLUSION

A theoretical solution for the large deflection of homogeneous and symmetrically laminated rectangular anisotropic plates under transverse load is presented for clamped and simply supported edges.

Numerical results are presented for homogeneous graphite-epoxy plates having different orientations of the material axes of symmetry with respect to the plate axes and for a simply supported symmetric angle-ply plate with various values of the number of graphite-epoxy sheets. In the case of homogeneous anisotropic plates the central deflection, bending moment M_y and membrane force N_y increase with increasing θ, whereas M_x and N_x decrease. For a given value of transverse load the central deflection is the smallest for an orthotropic plate ($\theta=0°$) and the largest for $\theta=45°$. In the latter case M_x is equal to M_y and N_x is equal to N_y at the center of the plate. In the case of symmteric angle-ply plate, the central deflection decreases when the number of layers increases, as in the small deflection theory of unsymmetrically laminated anisotropic plates.

The results presented in this work were obtained in the course of research sponsored by the National Research Council of Canada.

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Streszczenie

NIELINIOWE ZGINANIE SYMETRYCZNIE LAMINOWANYCH I JEDNORODNYCH PŁYT ANIZOTROPOWYCH

W pracy przedstawiono analizę dużych ugięć symetrycznie laminowanych anizotropowych płyt prostokątnych, poddanych działaniu obciążenia poprzecznego. Płyty jednorodne anizotropowe są traktowane jako przypadek szczególny. Rozwiązania równań dużych ugięć typu Kármána zapisanych w funkcji ugięcia poprzecznego i siły uogólnionej przedstawiono w postaci szeregów podwójnych, odpowiednio dla płyt zamocowanych i płyt swobodnie podpartych. Wyniki numeryczne przedstawiono graficznie dla symetrycznych kątowo zwitych (angle-ply) i jednorodnych grafitowo-epoksydowych płyt kwadratowych. W przypadku małych ugięć wyniki obecne pokrywają się z dotychczas istniejącymi rozwiązaniami.

Резюме

НЕЛИНЕЙНЫЙ ИЗГИБ СИММЕТРИЧЕСКИ ЛАМИНИРОВАННЫХ И ОДНОРОДНЫХ, АНИЗОТРОПНЫХ ПЛИТ

В работе представлен анализ больших прогибов симметрически ламинированных, анизотропных, прямоугольных плит подвергнутых действию поперечной нагрузки. Однородные, анизотропные плиты трактуются как частный случай. Решения уравнений больших прогибов типа Kármána, записанных в функции поперечного прогиба и обобщенной силы, представлены в виде двойных рядов соотнветсвенно для закрепленных плит и плит свободно подпертых. Численные результаты представлены графически для симметричных угловосвернутых и однородных графит-эпоксидных квадратных плит. В случае малых прогибов настоящие результаты совпадают с существующими до сих пор решениями.

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Received February 2, 1976.