ENGINEERING TRANSACTIONS • *Engng. Trans.* • **55**, 2, 129–153, 2007
Polish Academy of Sciences • Institute of Fundamental Technological Research

WAVE POLYNOMIALS IN ELASTICITY PROBLEMS

A. M a c i ą g

Department of Mathematics
Faculty of Management and Computer Modelling
Kielce University of Technology
Al. 1000–lecia P.P. 7, Poland
e-mail: matam@tu.kielce.pl

The paper demonstrates a new technique of obtaining the approximate solution of the two- and three-dimensional elasticity problems. The system of equations of elasticity can be converted to the system of wave equation. In this case, as solving functions (Trefftz functions), the so-called wave polynomials can be used. The presented method is useful for a finite body of a certain shape. Then the obtained solutions are coupled through initial and boundary conditions. Recurrent formulas for the two- and three-dimensional wave polynomials and their derivatives are obtained. The methodology for solution of systems of partial differential equations with common initial and boundary conditions by means of solving functions is presented. The advantage of using the method of solving functions is that the solution exactly satisfies the given equation (or system of equations). Some examples are included.

Key words: elasticity, Trefftz method, wave equation, wave polynomials.

1. Introduction

The method of solving functions applied for linear partial differential equations has been developed recently. The key idea of the method is to determine functions (polynomials) satisfying a given differential equation and fitted to the governing initial and boundary conditions. In this sense it is a variant of the Trefftz method [1, 2].

The method was first described in the paper [3] where it was applied to one-dimensional heat conduction problems. Heat polynomials were used for solving unsteady heat conduction problems in [4]. The method is continued in the Cartesian coordinate system in [5, 6], describing heat polynomials for the two- and three-dimensional case. Application of heat polynomials in polar and cylindrical coordinates is shown in the papers [7–9]. Application of this method to inverse heat-conduction problems is described in [5–11]. Reference [12] contains the highly interesting idea of using heat polynomials as a new type of finite-element base functions.

The work [13] deals with numerous cases involving other differential equations such as the Laplace, Poisson, Helmholtz and one-dimensional wave equations. Dimensionless wave polynomials for solving the two-dimensional wave equation are presented in [14, 15] and three-dimensional wave equation in [16, 17]. The wave functions can be obtained by using symbolic operations (for example in Maple or Mathematica) or inverse operations. These techniques are described correspondingly in [19, 20] and [21, 22]. Basically a linear differential equations (or system of equations) can be solved by means of various methods. Some of them are better for infinite bodies and others are suitable for finite bodies but of simple shape. The method presented here is useful for finite bodies but the shape of the body can be more complicated.

In Sec. 2 two- and three-dimensional wave polynomials and their properties in the Cartesian coordinate system are considered. Section 3 contains equations of elasticity. The method applied for a system of equations is presented in Sec. 4. In Sec. 5 some examples are discussed. Concluding remarks are given in Sec. 6.

2. Wave polynomials

The elasticity problems will be solved by means of wave polynomials. The papers [14, 16] show the way to obtain two- and three-dimensional wave equation for dimensionless wave equation. In engineering practice it is often convenient to use the dimensional wave equation. Analogously to the papers mentioned above, the dimensional wave polynomials can be obtained. There are two methods to obtain the wave polynomials. The first one is using a "generating function". The second one (giving the error estimator) is the expansion of the function satisfying the wave equation in Taylor series. Both methods lead to equivalent wave polynomials.

2.1. Two-dimensional wave polynomials

2.1.1. Generating function. Let us consider the wave equation

$$\frac{1}{v^2}\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}. \tag{2.1}$$

By using the separating variables method, we get a function called a generating function for wave polynomials

$$g = \mathrm{e}^{\mathrm{i}(ax+by+cvt)} \tag{2.2}$$

satisfying Eq. (2.1) when $c^2 = a^2 + b^2$. The power series expansion for (2.2) is

$$\mathrm{e}^{\mathrm{i}(ax+by+cvt)} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{n-k} S_{(n-k-l)kl}(x,y,t)\,a^{n-k-l}b^k c^l, \tag{2.3}$$

where $S_{(n-k-l)kl}(x,y,t)$ are polynomials of variables x, y, t containing v.

Substituting $c^2 = a^2 + b^2$ in (2.3), we obtain

$$\text{(2.4)} \qquad e^{i(ax+by+ct)} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{\substack{l=0\\ l<2}}^{n-k} R_{(n-k-l)kl}(x,y,t)a^{n-k-l}b^k c^l.$$

The real and imaginary parts of polynomials R satisfy Eq. (2.1) and are called wave polynomials:

$$\text{(2.5)} \qquad \begin{aligned} P_{(n-k-l)kl}(x,y,t) &= \Re(R_{(n-k-l)kl}(x,y,t)),\\ Q_{(n-k-l)kl}(x,y,t) &= \Im(R_{(n-k-l)kl}(x,y,t)), \end{aligned}$$

e.g.

$$\text{(2.6)} \qquad \begin{aligned} &P_{000}(x,y,t) = 1, && Q_{000}(x,y,t) = 0,\\ &P_{100}(x,y,t) = 0, && Q_{100}(x,y,t) = x,\\ &P_{010}(x,y,t) = 0, && Q_{010}(x,y,t) = y,\\ &P_{001}(x,y,t) = 0, && Q_{001}(x,y,t) = vt,\\ &P_{200}(x,y,t) = -\frac{x^2}{2} - \frac{v^2t^2}{2}, && Q_{200}(x,y,t) = 0,\\ &P_{110}(x,y,t) = -xy, && Q_{110}(x,y,t) = 0,\\ &P_{101}(x,y,t) = -vxt, && Q_{101}(x,y,t) = 0,\\ &P_{011}(x,y,t) = -vyt, && Q_{011}(x,y,t) = 0,\\ &P_{020}(x,y,t) = -\frac{y^2}{2} - \frac{v^2t^2}{2}, && Q_{020}(x,y,t) = 0, \quad \ldots \end{aligned}$$

Notice that here R_{002} does not appear, because $l < 2$ (see Eq. (2.4)).

2.1.2. Partial derivatives of wave polynomials. To obtain the recurrent formulas of partial derivatives for wave polynomials we follow analogously as in [14]. Because function (2.4) is analytical, the Taylor series on the right-hand side of (2.4) is convergent. Therefore we can differentiate consecutive terms

$$\frac{\partial g}{\partial x} = iag = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{\substack{l=0\\ l<2}}^{n-k} \frac{\partial R_{(n-k-l)kl}}{\partial x} a^{n-k-l}b^k c^l,$$

hence

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{\substack{l=0\\ l<2}}^{n-k} iR_{(n-k-l)kl}a^{n-k-l+1}b^k c^l = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{\substack{l=0\\ l<2}}^{n-k} \frac{\partial R_{(n-k-l)kl}}{\partial x} a^{n-k-l}b^k c^l,$$

and

$$\frac{\partial R_{(n-k-l)kl}}{\partial x} = \mathrm{i}R_{(n-k-l-1)kl},$$

so that finally

$$\begin{aligned} \frac{\partial P_{(n-k-l)kl}}{\partial x} &= -Q_{(n-k-l-1)kl}, \\ \frac{\partial Q_{(n-k-l)kl}}{\partial x} &= P_{(n-k-l-1)kl}. \end{aligned} \tag{2.7}$$

Similarly we get

$$\begin{aligned} \frac{\partial P_{(n-k-l)kl}}{\partial y} &= -Q_{(n-k-l)(k-1)l}, \\ \frac{\partial Q_{(n-k-l)kl}}{\partial y} &= P_{(n-k-l)(k-1)l}, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \frac{\partial P_{(n-k)k0}}{\partial t} &= -vQ_{(n-k-2)k1} - vQ_{(n-k)(k-2)1}, \\ \frac{\partial P_{(n-k-1)k1}}{\partial t} &= -vQ_{(n-k-1)k0}, \\ \frac{\partial Q_{(n-k)k0}}{\partial t} &= vP_{(n-k-2)k1} + vP_{(n-k)(k-2)1}, \\ \frac{\partial Q_{(n-k-1)k1}}{\partial t} &= vP_{(n-k-1)k0}. \end{aligned} \tag{2.9}$$

The starting values for the derivatives (2.7), (2.8) and (2.9) are obtained either from (2.6) or directly by putting zero instead of the polynomial, in which any of its subscripts takes a negative value.

2.1.3. Recurrent formulas for wave polynomials. In numerical practice the recurrent formulas are very useful. Theorem 1 enables us to obtain the two-dimensional wave polynomials.

THEOREM 1: *Let* $P_{000} = 1$, $Q_{000} = 0$ *and* $P_{(n-k-l)kl} = Q_{(n-k-l)kl} = 0$ *when any subscript is negative. Then, the polynomials*

$$\begin{aligned} P_{(n-k)k0} = \frac{1}{n}(-xQ_{(n-k-1)k0} &- yQ_{(n-k)(k-1)0} \\ &- vtQ_{(n-k-2)k1} - vtQ_{(n-k)(k-2)1}), \end{aligned} \tag{2.10}$$

$$P_{(n-k-1)k1} = \frac{1}{n}(-xQ_{(n-k-2)k1} - yQ_{(n-k-1)(k-1)1} - vtQ_{(n-k-1)k0}), \tag{2.11}$$

$$Q_{(n-k)k0} = \frac{1}{n}(xP_{(n-k-1)k0} + yP_{(n-k)(k-1)0} + vtP_{(n-k-2)k1} + vtP_{(n-k)(k-2)1}), \tag{2.12}$$

$$Q_{(n-k-1)k1} = \frac{1}{n}(xP_{(n-k-2)k1} + yP_{(n-k-1)(k-1)1} + vtP_{(n-k-1)k0}), \tag{2.13}$$

satisfy the wave equation (2.1).

Proof. For relation (2.10) we assume that all polynomials on the right-hand side either satisfy Eq. (2.1) or equal zero. Substituting (2.10) in (2.1) we get

$$x\left(\underbrace{\frac{1}{v^2}\frac{\partial^2 Q_{(n-k-1)k0}}{\partial t^2} - \frac{\partial^2 Q_{(n-k-1)k0}}{\partial x^2} - \frac{\partial^2 Q_{(n-k-1)k0}}{\partial y^2}}_{=0}\right)$$

$$+ y\left(\underbrace{\frac{1}{v^2}\frac{\partial^2 Q_{(n-k)(k-1)0}}{\partial t^2} - \frac{\partial^2 Q_{(n-k)(k-1)0}}{\partial x^2} - \frac{\partial^2 Q_{(n-k)(k-1)0}}{\partial y^2}}_{=0}\right)$$

$$+ vt\left(\underbrace{\frac{1}{v^2}\frac{\partial^2 Q_{(n-k-2)k1}}{\partial t^2} - \frac{\partial^2 Q_{(n-k-2)k1}}{\partial x^2} - \frac{\partial^2 Q_{(n-k-2)k1}}{\partial y^2}}_{=0}\right)$$

$$+ vt\left(\underbrace{\frac{1}{v^2}\frac{\partial^2 Q_{(n-k)(k-2)1}}{\partial t^2} - \frac{\partial^2 Q_{(n-k)(k-2)1}}{\partial x^2} - \frac{\partial^2 Q_{(n-k)(k-2)1}}{\partial y^2}}_{=0}\right)$$

$$+ \frac{2}{v}\left(\frac{\partial Q_{(n-k-2)k1}}{\partial t} + \frac{\partial Q_{(n-k)(k-2)1}}{\partial t}\right) = 2\frac{\partial Q_{(n-k-1)k0}}{\partial x} + 2\frac{\partial Q_{(n-k)(k-1)0}}{\partial y},$$

hence

$$\frac{\partial Q_{(n-k-2)k1}}{\partial t} + \frac{\partial Q_{(n-k)(k-2)1}}{\partial t} = v\left(\frac{\partial Q_{(n-k-1)k0}}{\partial x} + \frac{\partial Q_{(n-k)(k-1)0}}{\partial y}\right)$$

and according to (2.7), (2.8) and (2.9), we obtain

$$vP_{(n-k-2)k0} + vP_{(n-k)(k-2)0} = v(P_{(n-k-2)k0} + P_{(n-k)(k-2)0}).$$

□

This proves the theorem. The proof for (2.11), (2.12), and (2.13) is similar.

Starting values of the polynomials (2.10)–(2.13) can be obtained either from (2.6) or directly by putting zero instead of the polynomial in which any of its subscripts takes a negative value.

2.1.4. Expansion of the function satisfying wave equation in Taylor series. Another way to obtain the wave polynomials is to expand the function satisfying the wave equation in Taylor series. Similarly as for other equations [13] and for dimensionless wave equation [14], the wave polynomials can be obtained by means of Taylor series for the function w. Let the function $w(x,y,t)$ satisfy the wave equation (2.1). We assume that $w \in C^{N+1}$ in the neighborhood of (x_0, y_0, t_0). Let $\hat{x} = x - x_0, \hat{y} = y - y_0, \hat{t} = t - t_0$. Then, the Taylor series for function w and for $N = 2$ is

$$w(x,y,t) = w(x_0,y_0,t_0) + \frac{\partial w}{\partial x}\hat{x} + \frac{\partial w}{\partial y}\hat{y} + \frac{\partial w}{\partial t}\hat{t} + \frac{\partial^2 w}{\partial x^2}\frac{\hat{x}^2}{2} + \frac{\partial^2 w}{\partial y^2}\frac{\hat{y}^2}{2} + \frac{\partial^2 w}{\partial t^2}\frac{\hat{t}^2}{2} + \frac{\partial^2 w}{\partial x \partial y}\hat{x}\hat{y} + \frac{\partial^2 w}{\partial x \partial t}\hat{x}\hat{t} + \frac{\partial^2 w}{\partial y \partial t}\hat{y}\hat{t} + R_3. \tag{2.14}$$

Eliminating the derivative $\dfrac{\partial^2 w}{\partial t^2}$ by Eq. (2.1) we obtain

$$w(x,y,t) = w(x_0,y_0,t_0) + \frac{\partial w}{\partial x}\hat{x} + \frac{\partial w}{\partial y}\hat{y} + \frac{\partial w}{\partial t}\hat{t} + \frac{\partial^2 w}{\partial x^2}\left(\frac{\hat{x}^2}{2} + \frac{v^2\hat{t}^2}{2}\right) + \frac{\partial^2 w}{\partial y^2}\left(\frac{\hat{y}^2}{2} + \frac{v^2\hat{t}^2}{2}\right) + \frac{\partial^2 w}{\partial x \partial y}\hat{x}\hat{y} + \frac{\partial^2 w}{\partial x \partial t}\hat{x}\hat{t} + \frac{\partial^2 w}{\partial y \partial t}\hat{y}\hat{t} + R_3. \tag{2.15}$$

The coefficients following the derivative terms on the right-hand side represent, within the accuracy of a constant, the non-zero wave polynomials (2.6). As a solution of (2.1) we take a linear combination of wave polynomials. Therefore the constants in the polynomials are insignificant. Similarly, we get polynomials for $N = 3, 4, \ldots$.

The procedure described above is important. If w is the solution of the problem described by Eq. (2.1) and the corresponding initial and boundary conditions and if w is analytical, then we can control the accuracy of approximation by the properties of Taylor series. Moreover, using this procedure we can separate the stationary and nonstationary parts of expansion. For example, substituting $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial^2 w}{v^2 \partial t^2} - \dfrac{\partial^2 w}{\partial y^2}$ into (2.14) we obtain

$$w(x,y,t) = w(x_0,y_0,t_0) + \frac{\partial w}{\partial x}\hat{x} + \frac{\partial w}{\partial y}\hat{y} + \frac{\partial^2 w}{\partial y^2}\left(\frac{\hat{y}^2}{2} - \frac{\hat{x}^2}{2}\right) + \frac{\partial^2 w}{\partial x \partial y}\hat{x}\hat{y} + \frac{\partial w}{\partial t}\hat{t} + \frac{\partial^2 w}{\partial t^2}\left(\frac{\hat{x}^2}{2v^2} + \frac{\hat{t}^2}{2}\right) + \frac{\partial^2 w}{\partial x \partial t}\hat{x}\hat{t} + \frac{\partial^2 w}{\partial y \partial t}\hat{y}\hat{t} + R_3.$$

The coefficients $1,\ \hat{x},\ \hat{y},\ \dfrac{\hat{y}^2}{2} - \dfrac{\hat{x}^2}{2},\ \hat{x}\hat{y}$ are harmonic polynomials and satisfy the Laplace equation (stationary part) and coefficients $\hat{t},\ \dfrac{\hat{x}^2}{2v^2} + \dfrac{\hat{t}^2}{2},\ \hat{x}\hat{t},\ \hat{y}\hat{t}$ satisfy the wave equation (nonstationary part).

2.2. Three-dimensional wave polynomials

The wave polynomials for three-dimensional wave equation

$$\frac{1}{v^2}\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \tag{2.16}$$

we obtain in a similar manner. Recurrent formulas for partial derivatives of wave polynomial are

$$\begin{aligned} \frac{\partial P_{(n-k-l-m)klm}}{\partial x} &= -Q_{(n-k-l-m-1)klm}, \\ \frac{\partial Q_{(n-k-l-m)klm}}{\partial x} &= P_{(n-k-l-m-1)klm}. \end{aligned} \tag{2.17}$$

$$\begin{aligned} \frac{\partial P_{(n-k-l-m)klm}}{\partial y} &= -Q_{(n-k-l-m)(k-1)lm}, \\ \frac{\partial Q_{(n-k-l-m)klm}}{\partial y} &= P_{(n-k-l-m)(k-1)lm}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \frac{\partial P_{(n-k-l-m)klm}}{\partial z} &= -Q_{(n-k-l-m)k(l-1)m}, \\ \frac{\partial Q_{(n-k-l-m)klm}}{\partial z} &= P_{(n-k-l-m)k(l-1)m}, \end{aligned} \tag{2.19}$$

$$(2.20)\qquad \begin{aligned} \frac{\partial P_{(n-k-l)kl0}}{\partial t} &= -vQ_{(n-k-l-2)kl1} - vQ_{(n-k-l)(k-2)l1} - vQ_{(n-k-l)k(l-2)1},\\ \frac{\partial P_{(n-k-l-1)kl1}}{\partial t} &= -vQ_{(n-k-l-1)k0},\\ \frac{\partial Q_{(n-k-l)kl0}}{\partial t} &= vP_{(n-k-l-2)kl1} + vP_{(n-k-l)(k-2)l1} + vP_{(n-k-l)k(l-2)1},\\ \frac{\partial Q_{(n-k-l-1)kl1}}{\partial t} &= vP_{(n-k-l-1)kl0}. \end{aligned}$$

Theorem 2 enables us to obtain the three-dimensional wave polynomials $P_{(n-k-l-m)klm)}$ and $(Q_{(n-k-l-m)klm}$.

THEOREM 2: *Let* $(P_{0000} = 1)$ *and* $(Q_{0000} = 0)$. *Let* $(P_{(n-k-l-m)klm} = Q_{(n-k-l-m)klm} = 0)$ *when any subscript is negative. Then, the polynomials*

$$(2.21)\qquad P_{(n-k-l)kl0} = -\frac{1}{n}(xQ_{(n-k-l-1)kl0} + yQ_{(n-k-l)(k-1)l0} + zQ_{(n-k-l)k(l-1)0} + vtQ_{(n-k-l-2)kl1} + vtQ_{(n-k-l)(k-2)l1} + vtQ_{(n-k-l)k(l-2)1}),$$

$$(2.22)\qquad P_{(n-k-l-1)kl1} = -\frac{1}{n}(xQ_{(n-k-l-2)kl1} + yQ_{(n-k-l-1)(k-1)l1} + zQ_{(n-k-l-1)k(l-1)1} + vtQ_{(n-k-l-1)kl0}),$$

$$(2.23)\qquad Q_{(n-k-l)kl0} = \frac{1}{n}(xP_{(n-k-l-1)kl0} + yP_{(n-k-l)(k-1)l0} + zP_{(n-k-l)k(l-1)0} + vtP_{(n-k-l-2)kl1} + vtP_{(n-k-l)(k-2)l1} + vtP_{(n-k-l)k(l-2)1}),$$

$$(2.24)\qquad Q_{(n-k-l-1)kl1} = \frac{1}{n}(xP_{(n-k-l-2)kl1} + yP_{(n-k-l-1)(k-1)l1} + zP_{(n-k-l-1)k(l-1)1} + vtP_{(n-k-l-1)kl0})$$

satisfy the wave equation (2.16).

We prove Theorem 2 similarly to Theorem 1.

For example, from formulas (2.21)–(2.24) we get

$$(2.25)\qquad \begin{aligned} &P_{0000} = 1,\\ &Q_{1000} = x, \qquad Q_{0100} = y, \qquad Q_{0010} = z, \qquad Q_{0001} = vt, \end{aligned}$$

(2.25) [cont.]

$$P_{2000} = -\frac{x^2}{2} - \frac{v^2t^2}{2}, \qquad P_{1100} = -xy, \qquad P_{1010} = -xz,$$
$$P_{1001} = -vxt, \qquad P_{0200} = -\frac{y^2}{2} - \frac{v^2t^2}{2}, \qquad P_{0110} = -yz,$$
$$P_{0101} = -vyt, \qquad P_{0020} = -\frac{z^2}{2} - \frac{v^2t^2}{2}, \qquad P_{0011} = -vzt, \ldots$$

and

$$Q_{0000} = P_{1000} = P_{0100} = P_{0010} = P_{0001} = Q_{2000} = Q_{1100} = Q_{1010} = Q_{1001} = \cdots = 0$$

Notice that there is no R_{0002} because $m < 2$.

3. Equations of elasticity

In general, elasticity problems are described by the following system of equations [18]:

(3.1) $$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\text{grad div }\mathbf{u} + \mathbf{X} = \rho\ddot{\mathbf{u}}$$

where $\mathbf{u}$ – displacement vector, $\mathbf{X}$ – body force vector, ∇ – nabla operator, μ, λ, ρ – constants. If we omit the body forces, we obtain

(3.2) $$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\text{grad div }\mathbf{u} = \rho\ddot{\mathbf{u}}.$$

Equations (3.2) are completed by initial and boundary conditions for displacements and/or stresses. The relationship between the displacements and stresses is given by Hooke's law:

(3.3) $$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}$$

where $\varepsilon_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)$ – strain tensor. The system of equations (3.2) can be simplified by substitution:

(3.4) $$\mathbf{u} = \text{grad }\phi + \text{rot }\mathbf{\Psi}$$

Then we obtain

(3.5) $$\left(\nabla^2 - \frac{1}{v_1^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0,$$

(3.6) $$\left(\nabla^2 - \frac{1}{v_2^2}\frac{\partial^2}{\partial t^2}\right)\psi_i = 0, \qquad i = 1,2,3$$

where $v_1^2 = \dfrac{\lambda + 2\mu}{\rho}$, $v_2^2 = \dfrac{\mu}{\rho}$. The Eqs. (3.5) and (3.6) are wave equations, but for a finite domain they are still coupled by initial and boundary conditions. The main purpose of this work is to solve the system of (3.5), (3.6) by means of to solve functions' method.

4. Method of solving functions

The wave-polynomial method discussed below belongs to the class of Trefftz methods. As a solution of each wave equation (3.5), (3.6) we take a linear combination of the corresponding wave polynomials. The succeeding non-zero polynomials satisfying Eqs. (3.5) and (3.6) we denote correspondingly by V_n^0 and V_n^i, $i = 1, 2, 3$

As approximations for the solution of Eqs. (3.5) and (3.6) we take correspondingly

$$\phi \approx \widehat{\phi} = \sum_{n=1}^{N} c_n^0 V_n^0 \tag{4.1}$$

and

$$\psi_i \approx \widehat{\psi}_i = \sum_{n=1}^{N} c_n^i V_n^i, \quad i = 1, 2, 3. \tag{4.2}$$

Then

$$\mathbf{u} \approx \widehat{\mathbf{u}} = \operatorname{grad} \widehat{\phi} + \operatorname{rot} \widehat{\mathbf{\Psi}}. \tag{4.3}$$

Because polynomials V_n satisfy the corresponding wave equation, also the linear combination satisfies this equation. The coefficients c_n in (4.1) and (4.2) are chosen so that the error of fulfilling the given boundary and initial conditions corresponding to Eqs. (3.5) and (3.6) is minimized (see examples).

5. Examples

Some examples presented in this section show the application of the method of solving functions in elasticity. The first two concern the two-dimensional and the next two the three-dimensional elasticity problems. In all examples presented here the constants are established as follows: $\lambda = 10^{11}$ [Pa], $\mu = 8 \cdot 10^{10}$ [Pa], $\rho = 8000$ [kg/m^3].

5.1. *Example 1 – two-dimensional problem in a square*

First we consider the plane state of strain when the strain tensor depends on time and two variables $\varepsilon_{ij} = \varepsilon_{ij}(x, y, t), (i, j = 1, 2)$ and $\varepsilon_{i3} = 0, (i = 1, 2, 3)$. Then

$$\mathbf{u} = [u_x(x, y, t), u_y(x, y, t)]$$

$$= \left[\frac{\partial\phi(x, y, t)}{\partial x} + \frac{\partial\psi(x, y, t)}{\partial y}, \frac{\partial\phi(x, y, t)}{\partial y} - \frac{\partial\psi(x, y, t)}{\partial x}\right],$$

(5.1)
$$\sigma_{xx} = (2\mu + \lambda)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_y}{\partial y}, \qquad \sigma_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right),$$
$$\sigma_{yy} = \lambda\frac{\partial u_x}{\partial x} + (2\mu + \lambda)\frac{\partial u_y}{\partial y}, \qquad \sigma_{zz} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right).$$

The system of equations (3.5) and (3.6) has the form:

(5.2)
$$\frac{1}{v_1^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2},$$

(5.3)
$$\frac{1}{v_2^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}.$$

Let us consider the two-dimensional elasticity problem in a square $(x, y) \in [0, 1] \times [0, 1]$, described by the system of Eqs. (5.2)–(5.3) and conditions:

(5.4)
$$u_x(x, y, 0) = u_{x0}(x, y) = -\frac{x}{10000}, \qquad u_y(x, y, 0) = 0,$$

(5.5)
$$\frac{\partial u_x(x, y, 0)}{\partial t} = \frac{\partial u_y(x, y, 0)}{\partial t} = 0,$$

(5.6)
$$u_x(0, y, t) = u_y(0, y, t) = 0,$$

(5.7)
$$\sigma_{xx} = \sigma_{xy} = \sigma_{yy} = 0,$$
$$\text{for} \quad x = 1, \quad y = 0, \quad y = 1.$$

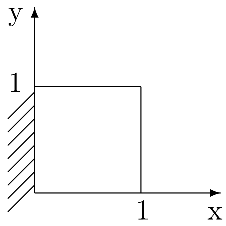

FIG. 1. Fixation of the square.

The problem described by relationships (5.2)–(5.7) can be solved approximately by means of the method of solving functions. Let us denote the non-zero two-dimensional wave polynomials (2.6) by

$$V_1^0 = 1, \quad V_2^0 = x, \quad V_3^0 = y, \quad V_4^0 = v_1 t, \quad V_5^0 = -\frac{x^2}{2} - \frac{v_1^2 t^2}{2}, \quad V_6^0 = -xy,$$
$$V_7^0 = -v_1 xt, \quad V_8^0 = -v_1 yt, \quad V_9^0 = -\frac{y^2}{2} - \frac{v_1^2 t^2}{2}, \quad \ldots$$

$$V_1^1 = 1, \quad V_2^1 = x, \quad V_3^1 = y, \quad V_4^0 = v_2 t, \quad V_5^1 = -\frac{x^2}{2} - \frac{v_2^2 t^2}{2}, \quad V_6^1 = -xy,$$
$$V_7^1 = -v_2 xt, \quad V_8^1 = -v_2 yt, \quad V_9^1 = -\frac{y^2}{2} - \frac{v_2^2 t^2}{2}, \quad \ldots.$$

As the approximations for the solutions of Eqs. (5.2)–(5.3), we take correspondingly

$$\phi \approx \widehat{\phi} = \sum_{n=1}^{N} c_n^0 V_n^0. \tag{5.8}$$

and

$$\psi \approx \widehat{\psi} = \sum_{n=1}^{N} c_n^1 V_n^1. \tag{5.9}$$

The coefficients c_n in (5.8) and (5.9) are chosen so that the error of fulfilling the boundary and initial conditions (5.4)–(5.7) is minimized. After applyting the least squares method, the functional describing this error can be written in the time interval $(0, \Delta t)$ as:

$$
\begin{aligned}
(5.10)\quad I \;=\;& w_u \int_0^1 \int_0^1 \Big\{ \underbrace{\big[\widehat{u}_x(x,y,0) - u_{x0}(x,y)\big]^2 + \big[\widehat{u}_y(x,y,0)\big]^2}_{\text{cond.}(5.4)} \Big\} dydx \\
&+ w_u \int_0^1 \int_0^1 \Big\{ \underbrace{\Big[\frac{\partial \widehat{u}_x(x,y,0)}{\partial t}\Big]^2 + \Big[\frac{\partial \widehat{u}_y(x,y,0)}{\partial t}\Big]^2}_{\text{cond.}(5.5)} \Big\} dydx \\
&+ w_u \int_0^{\Delta t} \int_0^1 \Big\{ \underbrace{\big[\widehat{u}_x(0,y,t)\big]^2 + \big[\widehat{u}_y(0,y,t)\big]^2}_{\text{cond.}(5.6)} \Big\} dydt \\
&+ w_\sigma \int_0^{\Delta t} \int_0^1 \Big\{ \underbrace{\big[\widehat{\sigma}_{xx}(1,y,t)\big]^2 + \big[\widehat{\sigma}_{xy}(1,y,t)\big]^2}_{\text{cond.}(5.7)} \Big\} dydt \\
&+ w_\sigma \int_0^{\Delta t} \int_0^1 \Big\{ \underbrace{\big[\widehat{\sigma}_{yx}(x,0,t)\big]^2 + \big[\widehat{\sigma}_{yy}(x,0,t)\big]^2}_{\text{cond.}(5.7)} \Big\} dxdt \\
&+ w_\sigma \int_0^{\Delta t} \int_0^1 \Big\{ \underbrace{\big[\widehat{\sigma}_{yx}(x,1,t)\big]^2 + \big[\widehat{\sigma}_{yy}(x,1,t)\big]^2}_{\text{cond.}(5.7)} \Big\} dxdt.
\end{aligned}
$$

The constants μ, λ are large. They appear in the second power by conditions connected with stresses. Therefore in functional I we have to introduce weights by each condition. The sum of all weights equals one. Because in functional (5.10) there are six conditions connected with stresses and six conditions connected with displacements, the weight $w_\sigma = 1/(6 \cdot 10^{24})$.

The necessary condition to minimize the functional I is

$$
(5.11)\qquad \frac{\partial I}{\partial c_1^0} = \cdots = \frac{\partial I}{\partial c_N^0} = \frac{\partial I}{\partial c_1^1} = \cdots = \frac{\partial I}{\partial c_N^1} = 0.
$$

The linear system of equations (5.11) can be written as

$$
(5.12)\qquad AC = B
$$

where $C = [c_1^0, \ldots, c_N^0, c_1^1, \ldots, c_N^1]^T$ and $A = \begin{array}{c} \left[\begin{array}{cc} A^1 & A^2 \\ A^3 & A^4 \end{array}\right] \\ \underbrace{}_{c_j^0}\ \underbrace{}_{c_j^1} \end{array} \begin{array}{l} \}\frac{\partial I}{\partial c_i^0} \\ \}\frac{\partial I}{\partial c_i^1} \end{array}$.

For example, the elements of matrix A^2 are

$$
\begin{aligned}
a_{i,j}^2 &= w_u \int_0^1\int_0^1 \left\{ \frac{\partial V_i^0(x,y,0)}{\partial x}\frac{\partial V_j^1(x,y,0)}{\partial x} - \frac{\partial V_i^0(x,y,0)}{\partial y}\frac{\partial V_j^1(x,y,0)}{\partial x} \right. \\
&+ \left. \frac{\partial^2 V_i^0(x,y,0)}{\partial x\partial t}\frac{\partial^2 V_j^1(x,y,0)}{\partial y\partial t} - \frac{\partial^2 V_i^0(x,y,0)}{\partial y\partial t}\frac{\partial^2 V_j^1(x,y,0)}{\partial x\partial t} \right\} dydx \\
&+ \int_0^1\int_0^{\Delta t} \left\{ w_u \left(\frac{\partial V_i^0(0,y,t)}{\partial x}\frac{\partial V_j^1(0,y,t)}{\partial y} - \frac{\partial V_i^0(0,y,t)}{\partial y}\frac{\partial V_j^1(0,y,t)}{\partial x} \right) \right. \\
&+ w_\sigma \left(2\mu \left((2\mu+\lambda)\frac{\partial^2 V_i^0(1,y,t)}{\partial x^2} + \lambda\frac{\partial^2 V_i^0(1,y,t)}{\partial y^2} \right) \frac{\partial^2 V_j^1(1,y,t)}{\partial x\partial y} \right. \\
&+ \left. \left. 2\mu^2 \frac{\partial^2 V_i^0(1,y,t)}{\partial x\partial y} \left(\frac{\partial^2 V_j^1(1,y,t)}{\partial y^2} - \frac{\partial^2 V_j^1(1,y,t)}{\partial x^2} \right) \right) \right\} dtdy \\
&+ w_\sigma 2\mu \int_0^1\int_0^{\Delta t} \left\{ \mu \frac{\partial^2 V_i^0(x,0,t)}{\partial x\partial y} \left(\frac{\partial^2 V_j^1(x,0,t)}{\partial y^2} - \frac{\partial^2 V_j^1(x,0,t)}{\partial x^2} \right) \right. \\
&- \left((2\mu+\lambda)\frac{\partial^2 V_i^0(x,0,t)}{\partial y^2} + \lambda\frac{\partial^2 V_i^0(x,0,t)}{\partial x^2} \right) \frac{\partial^2 V_j^1(x,0,t)}{\partial x\partial y} \\
&+ \mu \frac{\partial^2 V_i^0(x,1,t)}{\partial x\partial y} \left(\frac{\partial^2 V_j^1(x,1,t)}{\partial y^2} - \frac{\partial^2 V_j^1(x,1,t)}{\partial x^2} \right) \\
&- \left. \left((2\mu+\lambda)\frac{\partial^2 V_i^0(x,1,t)}{\partial y^2} + \lambda\frac{\partial^2 V_i^0(x,1,t)}{\partial x^2} \right) \frac{\partial^2 V_j^1(x,1,t)}{\partial x\partial y} \right\} dtdx.
\end{aligned}
$$

From Eq. (5.12) we obtain the coefficients c_n. In practice it turns out that this system of linear equations is indeterminate. Nevertheless, for different values of the parameter we get the same solution. In the time intervals $(\Delta t, 2\Delta t)$, $(2\Delta t, 3\Delta t)$,..., we proceed analogously. Here, the initial condition for time interval $((m-1)\Delta t, m\Delta t)$ is the value of function u at the end of interval $((m-2)\Delta t, (m-1)\Delta t)$. All results below have been obtained for $\Delta t = 0.00016$. Then $v_1\Delta t = 0.91214034, v_2\Delta t = .5059644256$. We obtain an approximation in the entire time interval $(0, \Delta t)$. For example, Fig. 2 shows an approximation of displacement u_x by polynomials from order 0 to 9 for times a) $t = 0$,

b) $t = 0.00008$, c) $t = 0.00014$. Figures 2 show that the initial and boundary conditions for displacement u_x are well approximated

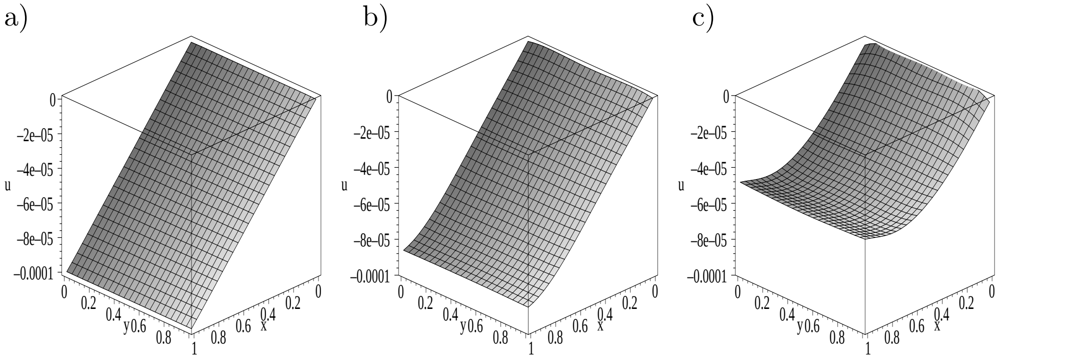

FIG. 2. Approximation of displacement u_x for time a) $t = 0$, b) $t = 0.00008$, c) $t = 0.00014$.

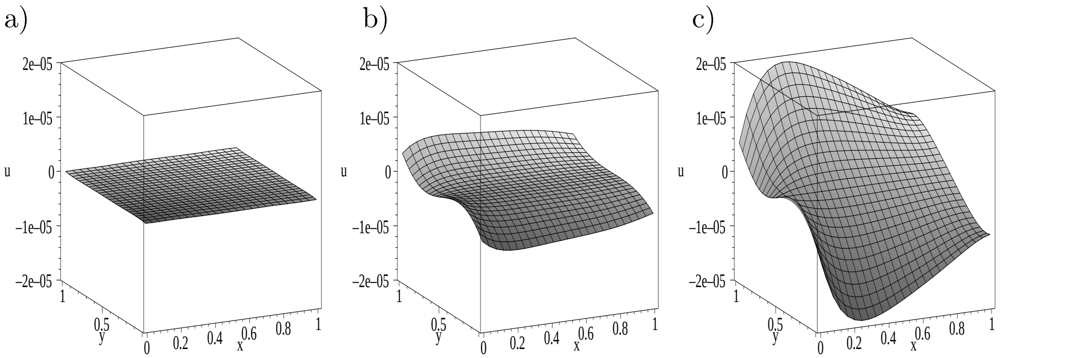

FIG. 3. Approximation of displacement u_y for time a) $t = 0$, b) $t = 0.00008$, c) $t = 0.00014$.

Figure 3 show an approximation of displacement u_y by polynomials from order 0 to 9 for times a) $t = 0$, b) $t = 0.00008$, c) $t = 0.00014$. Figures 2 and 3 shows that the the physical character of the displacement is preserved.

In approximations (5.8) and (5.9) we take all wave polynomials of orders from 0 to K. Table 1 shows the value of functional I which depends on the order K. The error decreases when the number of polynomials in the approximation increases.

Table 1. ***I*** **dependence of the polynomial order.**

Order K	1	2	3	4	5
I	$0.139 \cdot 10^{-9}$	$0.234 \cdot 10^{-13}$	$0.233 \cdot 10^{-13}$	$0.211 \cdot 10^{-13}$	$0.166 \cdot 10^{-13}$

5.2. *Example 2 – two-dimensional problem in a triangle*

The majority of analytical methods used for solving partial differential equations are effective for simple shapes of the body (square, circle, cube or sphere). Solving functions' method can be applied for more complicated domains. The only difficulty for such a shape may be the calculation of the integrals determining the coefficients c_n – for most shapes this does not create any problem.

Similarly as in Sec. 5.1, we consider a plane state of strain when the strain tensor depends on time and two variables. Let us consider the two-dimensional elasticity problem in a triangle $0 \le x \le 1$, $0 \le y \le 1 - x$, described by the system of Eqs. (5.2)–(5.3) and conditions:

$$u_x(x,y,0) = u_{x0}(x,y) = \frac{y(1-y)(1-x-y)}{1000}, \qquad u_y(x,y,0) = 0, \tag{5.13}$$

$$\frac{\partial u_x(x,y,0)}{\partial t} = \frac{\partial u_y(x,y,0)}{\partial t} = 0, \tag{5.14}$$

$$u_x(x,0,t) = u_y(x,0,t) = u_x(x,1-x,t) = u_y(x,1-x,t) = 0, \tag{5.15}$$

$$\sigma_{xx}(0,y,t) = \sigma_{xy}(0,y,t) = 0. \tag{5.16}$$

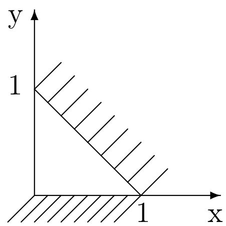

Fig. 4. Fixation of the triangle.

Similarly as in Sec. 5.1, approximations for the solution of Eqs. (5.2)–(5.3) we take correspondingly (5.8) and (5.9). The coefficients c_n in the linear combinations are chosen such that the error in fulfilling the boundary and initial conditions (5.13)–(5.16) is minimized. The functional describing this error is similar to (5.10). Here we have other domains for integrals:

$$
\begin{aligned}
(5.17)\quad I \;=\; & w_u \int_0^1 \int_0^{1-x} \Big\{ \underbrace{[\widehat{u}_x(x,y,0) - u_{x0}(x,y)]^2 + [\widehat{u}_y(x,y,0)]^2}_{\text{cond.(5.13)}} \Big\}\, dydx \\
+ \; & w_u \int_0^1 \int_0^{1-x} \Big\{ \underbrace{\left[\frac{\partial \widehat{u}_x(x,y,0)}{\partial t}\right]^2 + \left[\frac{\partial \widehat{u}_y(x,y,0)}{\partial t}\right]^2}_{\text{cond.(5.14)}} \Big\}\, dydx \\
+ \; & w_u \int_0^{\Delta t} \int_0^1 \{\underbrace{[\widehat{u}_x(x,0,t)]^2 + [\widehat{u}_y(x,0,t)]^2}_{\text{cond.(5.15)}}\, dxdt \\
+ \; & w_u \sqrt{2} \int_0^{\Delta t} \int_0^1 \underbrace{[\widehat{u}_x(x,1-x,t)]^2 + [\widehat{u}_x(x,1-x,t)]^2}_{\text{cond.(5.15)}} \} dxdt \\
+ \; & w_\sigma \int_0^{\Delta t} \int_0^1 \{\underbrace{[\widehat{\sigma}_{xx}(0,y,t)]^2 + [\widehat{\sigma}_{xy}(0,y,t)]^2}_{\text{cond.(5.16)}} \} dydt
\end{aligned}
$$

There are two conditions connected with stresses and eight conditions connected with displacements. Therefore in functional I the weight $w_\sigma = 2/10^{23}$. The sum of all weights equals one. We obtain the coefficients c_n in the same manner as in Sec. 5.1. All results below have been obtained for $\Delta t = 0.00016[s]$. Figure 5 shows the initial condition for displacement u_x a) the exact solution, b) an approximation by polynomials from order 0 to 9, c) the difference between a) and b). Figure 5 shows that the initial condition for displacement u_x is well approximated. Figure 6 shows an approximation of displacement $u_y(0,y,t)$ in time by polynomials from order 0 to 9. Let u_K denote the approximation of $u_y(0,y,t)$ by polynomials of order from 0 to K. We define the average, relative difference between solutions u_K and u_{K-1}:

$$
D(K) = \sqrt{\frac{\int_0^{\Delta t} \int_0^1 (u_K(0,y,t) - u_{K-1}(0,y,t))^2 \mathrm{d}y\mathrm{d}t}{\int_0^{\Delta t} \int_0^1 (u_{K-1}(0,y,t))^2 \mathrm{d}y\mathrm{d}t}}
$$

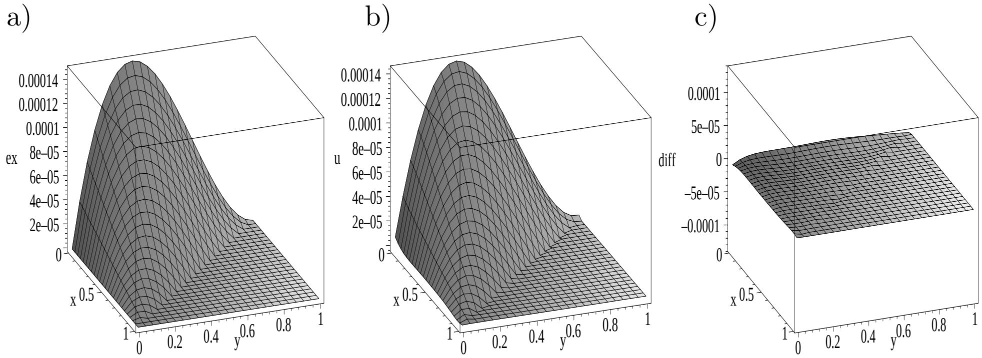

FIG. 5. Initial condition for displacement u_x: a) exact, b) approximation, c) difference.

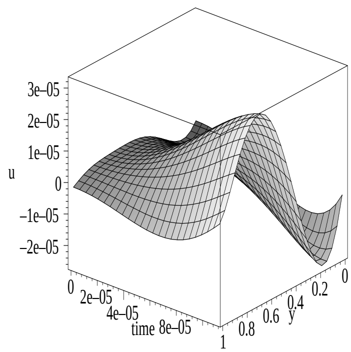

FIG. 6. Approximation in time of displacement $u_y(0, y, t)$.

Table 2 shows the difference $D(K)$ which depends on the order K. The error decreases when the number of polynomials in the approximation u increases. It suggests that the method is convergent.

Table 2. Error dependence of the polynomial order.

Order K	3	5	7	9
$D(K)$	201.2	1.235	1.256	0.558

Stresses can be calculated by means of formula (3.3). For example, Fig. 7 shows the approximation of stress σ_{xx} by polynomials of order from 0 to 9 for times a) $t = 0.00001$, b) $t = 0.00008$, c) $t = 0.00011$. Figures 6 and 7 show that the physical character of the displacements and stresses is preserved.

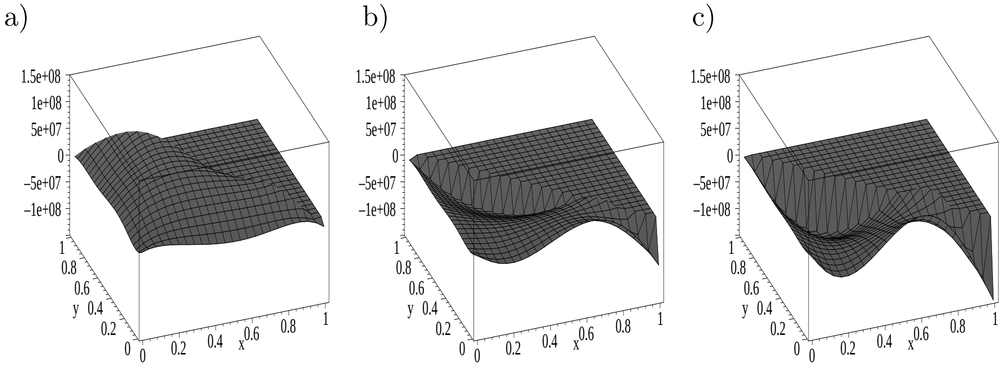

FIG. 7. Approximation of stress σ_{xx} for times a) $t = 0.00001$, b) $t = 0.00008$, c) $t = 0.00011$.

5.3. Example 3 – three-dimensional problem in a cube

The method presented here can be also applied to three-dimensional elasticity problems. Let us consider the elasticity problem described for a cube $(x, y, z) \in [0,1] \times [0,1] \times [0,1]$ by the system of Eqs. (3.5)–(3.6) and conditions:

$$\begin{aligned} &u_x(x,y,z,0) = 0, \qquad u_y(x,y,z,0) = 0, \\ &u_z(x,y,z,0) = u_{x0}(x,y,z) = -\frac{z}{10000}, \end{aligned} \tag{5.18}$$

$$\frac{\partial u_x(x,y,z,0)}{\partial t} = \frac{\partial u_y(x,y,z,0)}{\partial t} = \frac{\partial u_z(x,y,z,0)}{\partial t} = 0, \tag{5.19}$$

$$u_x(x,y,0,t) = u_y(x,y,0,t) = u_z(x,y,0,t) = 0, \tag{5.20}$$

$$\begin{aligned} &\sigma_{xx}(0,y,z,t) = \sigma_{xy}(0,y,z,t) = \sigma_{xz}(0,y,z,t) = 0, \\ &\sigma_{xx}(1,y,z,t) = \sigma_{xy}(1,y,z,t) = \sigma_{xz}(1,y,z,t) = 0, \\ &\sigma_{yx}(x,0,z,t) = \sigma_{yy}(x,0,z,t) = \sigma_{yz}(x,0,z,t) = 0, \\ &\sigma_{yx}(x,1,z,t) = \sigma_{yy}(x,1,z,t) = \sigma_{yz}(x,1,z,t) = 0, \\ &\sigma_{zx}(x,y,1,t) = \sigma_{zy}(x,y,1,t) = \sigma_{zz}(x,y,1,t) = 0. \end{aligned} \tag{5.21}$$

The problem described by relationships (5.18)–(5.21) can be solved approximately by means of the method of solving functions. Let us denote the non-zero two-dimensional wave polynomials (2.25) as

$$V_1^0 = 1, \quad V_2^0 = x, \quad V_3^0 = y, \quad V_4^0 = z, \quad V_5^0 = v_1 t, \quad V_6^0 = -\frac{x^2}{2} - \frac{v_1^2 t^2}{2},$$
$$V_7^0 = -xy, \quad V_8^0 = -xz, \quad V_9^0 = -v_1 xt, \quad V_{10}^0 = -\frac{y^2}{2} - \frac{v_1^2 t^2}{2},$$
$$V_{11}^0 = -yz, \quad V_{12}^0 = -v_1 yt, \quad V_{13}^0 = -\frac{z^2}{2} - \frac{v_1^2 t^2}{2}, \quad V_{14}^0 = -v_1 zt, \ldots$$

and

$$V_1^i = 1, \quad V_2^i = x, \quad V_3^i = y, \quad V_4^i = z, \quad V_5^i = v_2 t, \quad V_6^i = -\frac{x^2}{2} - \frac{v_2^2 t^2}{2},$$
$$V_7^i = -xy, \quad V_8^i = -xz, \quad V_9^i = -v_2 xt, \quad V_{10}^i = -\frac{y^2}{2} - \frac{v_2^2 t^2}{2},$$
$$V_{11}^i = -yz, \quad V_{12}^i = -v_2 yt, \quad V_{13}^i = -\frac{z^2}{2} - \frac{v_2^2 t^2}{2}, \quad V_{14}^i = -v_2 zt, \ldots \quad i = 1, 2, 3$$

Notice that we take the same polynomials for Eqs. (3.6). As approximations for the solution of Equations (3.5)–(3.6) we take correspondingly

$$\phi \approx \widehat{\phi} = \sum_{n=1}^{N} c_n^0 V_n^0. \tag{5.22}$$

and

$$\psi_i \approx \widehat{\psi_i} = \sum_{n=1}^{N} c_n^i V_n^i, \quad i = 1, 2, 3. \tag{5.23}$$

The coefficients c_n in (5.22) and (5.23) are chosen so that the error for fulfilling the boundary and initial conditions (5.18)–(5.21) is minimized. Further we progress as in Secs. 5.1 and 5.2. Of course, here we have more conditions. Therefore the functional I and matrices A, C and B are "bigger". In functional I we introduce weights by each condition. In this case there are fifteen conditions connected with stresses and nine conditions connected with displacements – the weight $w_\sigma = 15/10^{22}$.

All results below have been obtained for $\Delta t = 0.00016$. Figure 8 shows an approximation of displacement $u_z(x, 0.5, z, t)$ by polynomials from order 0 to 4 for time a) $t = 0$, b) $t = 0.0001$, c) $t = 0.00016$. Figures 8 show that the initial and boundary conditions are well approximated.

Figure 9 shows an approximation of displacement $u_x(x, y, 0.5, t)$ by polynomials of order from 0 to 4 for times a) $t = 0$, b) $t = 0.0001$, c) $t = 0.00016$. Figures 8 and 9 show that the physical character of the displacement is preserved.

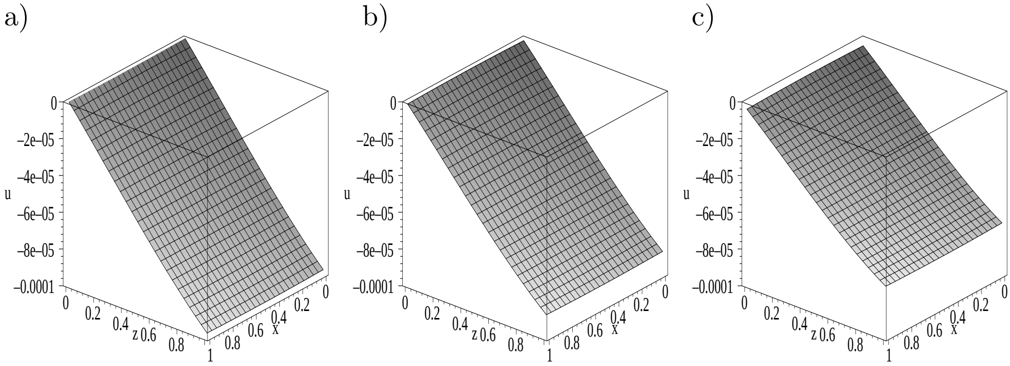

FIG. 8. Approximation of displacement $u_z(x, 0.5, z, t)$ for time a) $t = 0$, b) $t = 0.0001$, c) $t = 0.00016$.

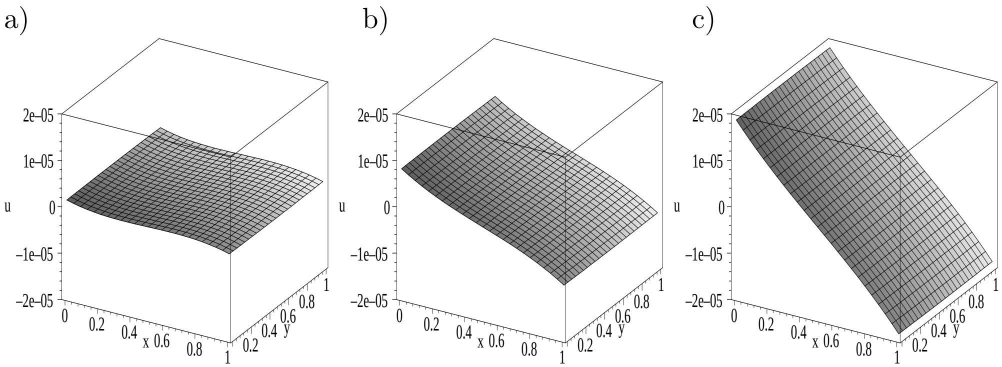

FIG. 9. Approximation of displacement $u_x(x, y, 0.5, t)$ for time a) $t = 0$, b) $t = 0.0001$, c) $t = 0.00016$.

5.4. Example 4 – three-dimensional problem in a triangular prism

Let us consider the elasticity problem described in a triangular prism $0 \le x \le 1$, $0 \le y \le 1 - x$, $0 \le z \le 1$ by the system of Eqs. (3.5)–(3.6) and conditions:

$$u_x(x, y, z, 0) = 0, \quad u_y(x, y, z, 0) = 0,$$
$$u_z(x, y, z, 0) = u_{x0}(x, y, z) = -\frac{z}{10000}, \tag{5.24}$$

$$\frac{\partial u_x(x, y, z, 0)}{\partial t} = \frac{\partial u_y(x, y, z, 0)}{\partial t} = \frac{\partial u_z(x, y, z, 0)}{\partial t} = 0, \tag{5.25}$$

$$u_x(x,y,0,t) = u_y(x,y,0,t) = u_z(x,y,0,t) = 0, \tag{5.26}$$

$$\begin{aligned}
&\sigma_{xx}(0,y,z,t) = \sigma_{xy}(0,y,z,t) = \sigma_{xz}(0,y,z,t) = 0,\\
&\sigma_{yx}(x,0,z,t) = \sigma_{yy}(x,0,z,t) = \sigma_{yz}(x,0,z,t) = 0,\\
&\sigma_{zx}(x,y,1,t) = \sigma_{zy}(x,y,1,t) = \sigma_{zz}(x,y,1,t) = 0,\\
&\sigma_{xx}(x,1-x,z,t) = \sigma_{xy}(x,1-x,z,t) = \sigma_{xz}(x,1-x,z,t) = 0,\\
&\sigma_{yy}(x,1-x,z,t) = \sigma_{yz}(x,1-x,z,t) = \sigma_{zz}(x,1-x,z,t) = 0.
\end{aligned} \tag{5.27}$$

The problem described by relationships (5.24)–(5.27) can be solved in the same manner as in Secs. 5.1, 5.2 and 5.3. In this case we have other domain for integrals in functional I where there are fourteen conditions connected with stresses and nine conditions connected with displacements – the weight $w_\sigma = 14/10^{22}$.

All results given below given have been obtained for $\Delta t = 0.00016$.

Figure 10 shows the approximation of displacement $u_x(x,y,0.5,t)$ by polynomials of order from 0 to 4 for times a) $t = 0$, b) $t = 0.0001$, c) $t = 0.00015$. Figure 10 shows that the physical character of the displacement is preserved. Figure 11 shows an approximation of displacement $u_z(x,x,z,t)$ by polynomials of order from 0 to 4 for times a) $t = 0$, b) $t = 0.0001$, c) $t = 0.00016$. Figures 11 shows that the initial and boundary conditions for u_z are well approximated.

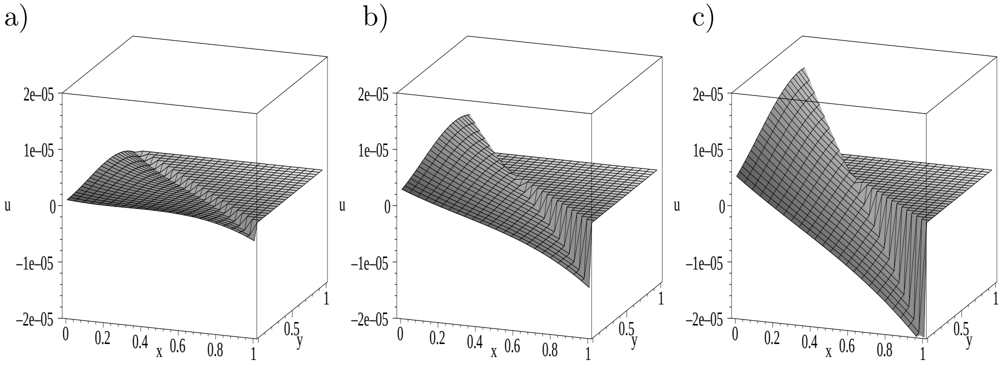

FIG. 10. Approximation of displacement $u_x(x,y,0.5,t)$ for time a) $t = 0$, b) $t = 0.0001$, c) $t = 0.00015$.

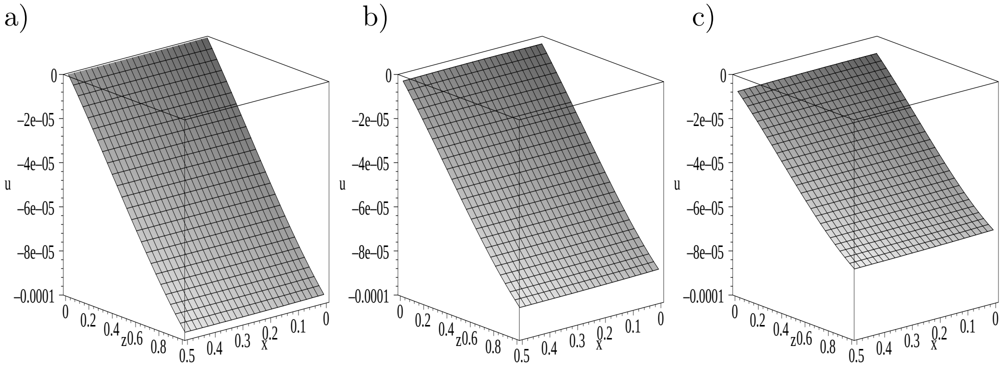

FIG. 11. Approximation of displacement $u_z(x, x, z, t)$ for time a) $t = 0$, b) $t = 0.0001$, c) $t = 0.00016$.

6. Concluding remarks

As a rule, the elasticity problems are difficult. A new simple technique for solving these two- and three- dimensional problems has been developed. The method of solving functions presented in this paper is a straightforward method for solving elasticity problems in finite bodies. This method is also useful when the shape of the body is complicated. We must calculate the integrals determining the coefficients c_n. For most shapes this does not present any problem. The simple examples presented in this paper show that in the obtained approximations, the physical character of displacements and stresses is preserved. The solution, which is a linear combination of wave polynomials, satisfies exactly the wave equation and approximately – the initial and boundary conditions. The next step of the research should be the application of this method to thermoelasticity problems. Moreover, wave polynomials can be used as finite-element base functions.

Acknowledgments

The author would like to thank the University of Karlsruhe where most of the calculations have been made.

References

1. E. Trefftz, *Ein Gegenstück zum Ritzschen Verfahren*, [in:], Proceedings of the 2-nd International Congress of Applied Mechanics. Zurich 131–137, 1926.

2. A.P. Zieliński and I.Herrera, *Trefftz method: fitting boundary conditions.* Int. J. Num. Meth. Eng., **24**, 871–891, 1987.

3. P.C. Rosenbloom and D.V. Widder, *Expansion in terms of heat polynomials and associated functions,* Trans. Am. Math. Soc., **92**, 220–266, 1956.

4. H. Yano, S. Fukutani and A. Kieda, *A boundary residual method with heat polynomials for solving unsteady heat conduction problems,* Franklin Inst, **316**, 291–298, 1983.

5. S. Futakiewicz and L. Hożejowski, *Heat polynomials method in the n-dimensional direct and inverse heat conduction problems,* [in:] A.J. Nowak, C.A. Brebbia, R. Bielecki and M. Zerroukat [Eds.], Advanced Computational Method in Heat Transfer V. Southampton, UK and Boston, USA: Computational Mechanics Publications, 103–112, 1998.

6. L. Hożejowski, *Heat polynomials and their application for solving direkt and inverse heat condutions problems* (PhD-Thesis) [in Polish], Kielce University of Technology, pp. 115, 1999.

7. S. Futakiewicz and L. Hożejowski, *Heat polynomials in solving the direct and inverse heat conduction problems in a cylindrical system of coordinates*, [in:] A.J. Nowak, C.A. Brebbia, R. Bielecki and M. Zerroukat [Eds.], Advanced Computational Method in Heat Transfer V. Southampton UK and Boston USA: Computational Mechanics Publications 71–80, 1998.

8. S. Futakiewicz, K. Grysa and L. Hożejowski, *On a problem of boundary temperature identification in a cylindrical layer,* [in:] B.T. Maruszewski, W. Muschik and A. Radowicz [Eds.], Proceedings of the International Symposium on Trends in Continuum Physics, World Scientific Publishing, Singapore, New Jersey, London, Hong Kong 119–125, 1999.

9. S. Futakiewicz, *Heat functions method for solving direct and inverse heat condutions problems* (PhD-Thesis) [in Polish], Poznań University of Technology 120pp, 1999.

10. M.J. Ciałkowski, S. Futakiewicz and L. Hożejowski, *Method of heat polynomials in solving the inverse heat conduction problems,* ZAMM, **79**, 709–710, 1999.

11. M.J. Ciałkowski, S. Futakiewicz and L. Hożejowski, *Heat polynomials applied to direct and inverse heat conduction problems,* [in:] B.T. Maruszewski, W. Muschik and A. Radowicz [Eds.], Proceedings of the International Symposium on Trends in Continuum Physics, World Scientific Publishing, Singapore, New Jersey, London, Hong Kong 79–88, 1999.

12. M.J. Ciałkowski, *Solution of inverse heat conduction problem with use of a new type of finite element base functions,* [in:] B.T. Maruszewski, W. Muschik and A.Radowicz [Eds.], Proceedings of the International Symposium on Trends in Continuum Physics, Singapore, New Jersey, London, Hong Kong: World Scientific Publishing, 64–78, 1999.

13. M.J. Ciałkowski and A. Frąckowiak, *Heat functions and their application for solving heat transfer and mechanical problems* [in Polish], University of Technology Publishers pp. 360, Poznań 2000.

14. A. Maciąg and J. Wauer, *Solution of the two-dimensional wave equation by using wave polynomials*, J. Engrg. Math., **51**, 4, 339–350, 2005,

15. A. Maciąg, J. Wauer, *Wave polynomials for solving different types of two-dimensional wave equations*, Computer Assisted Mechanics and Engineering Sciences, **12**, 87–102, 2005.

16. A. Maciąg *Solution of the three-dimensional wave polynomials*, Mathematical Problems in Engineering, **5**, 583–598, 2005.

17. A. Maciąg, *Solution of the three-dimensional wave equation by using wave polynomials*, PAMM – Proc. Math. Mech., **4**, 706–707, 2004.

18. W. Nowacki, *Thermoelascity,* Pergamon Press, Oxford–Warsaw 1962.

19. M.J. Ciałkowski, M. Jarosławski, *Application of symbolic calculations in generating the solution of the wave equation* [in Polish], Zeszyty Naukowe Politechniki Poznańskiej nr 56, Maszyny Robocze i Transport, 2003.

20. M.J. Ciałkowski, A. Frąckowiak, *Application of symbolic operations to the determination of Trefftz functions for a heat flow wave equation* [in Polish], Zeszyty Naukowe Politechniki Poznańskiej nr 57, Maszyny Robocze i Transport, 2004.

21. M.J. Ciałkowski, *Thermal and related functions used in solving certain problems of mechanics. Part I - Solution of certain differential equations by means of inverse operations* [in Polish], Studia i materiały. Technika 3. Uniwersytet Zielonogórski, 7–70, 2003.

22. M.J. Ciałkowski, A. Frąckowiak, *Thermal and related functions in solving certain problems of mechanics. Part II – Effective determination of inverse operations applied to harmonic functions* [in Polish], Studia i materiały. Technika 3. Uniwersytet Zielonogórski, 71–98, 2003.

Received February 10, 2006.