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NONSTATIONARY TEMPERATURE DISTRIBUTION AND THERMAL STRESSES IN A LAYERED ELASTIC OR VISCOELASTIC MEDIUM*)

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A formal solution based on integral transforms and matrix analysis is given for the nonstationary fields of temperature, heat flux, stresses and displacements of a multilayered elastic medium. Non--linearities, inertia- and coupling effects are not taken into account. The applicability of the theory is demonstrated by means of numerically worked out examples. Extension to viscoelastic media is indicated.

1. Introduction

Heat conduction problems are extensively treated in the monographs by Carslaw and Jaeger [1] and by Luikov [2]. Especially Vodička [3, 4] considers the two- and three-dimensional stationary temperature fields in layered systems by means of Fourier transforms, while Carslaw and Jaeger [1] solve the one-dimensional nonstationary heat conduction problem of a layered medium using the Laplace transform.

The theory and applications of thermoelasticity are laid down for instance in the books of Boley and Weiner [5], Nowacki [6] and Parkus [7]. The thermoelastic problem of a single layer using integral transforms is treated e.g. by the following authors: McDowell [8], Sneddon and Lockett [9], Martin and Payton [10] consider the two and three-dimensional stationary case; Heisler [11] and Derski [12] consider the one-dimensional nonstationary case; Mossakowska and Nowacki [13], Jarvis and Hardy [14], Goshima, Koizumi and Nakahara [15] consider the two- and three-dimensional nonstationary case. Solutions for a two-layer system in the one-dimensional nonstationary case are given by Pohle and Oliver [16] and Pankowski [17]. Sandwich systems are analysed under the usual approximations for the stationary case by Ebcioglu [18], Huang and Ebcioglu [19] and Kovařik [20]. Ryabov [21], also for the stationary case, solves the thermoelastic problem of multilayer plates.

In this paper, basing on Meier's dissertation [22], an exact solution is developed for the nonstationary fields of temperature, heat flux, stresses and displacements of a multilayered elastic or viscoelastic medium. Starting with the heat conduction equations and the equations of the linear, noncoupled and quasi-static theory of isotropic thermoelasticity we consider at first the problem of an infinite single layer. The governing relations are derived with respect to Cartesian coordinates by using matrix formulation from the beginning as has been demonstrated for the

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isothermal case by BUFLER [23, 24]. After having performed the two-dimensional Fourier transforms with respect to the coordinates x and y (Fig. 1), the integration in z-direction gives the transfer matrices which contain the time derivatives. These ordinary differential equations in the time are integrated by means of the Laplace transform. Now, the continuity and boundary conditions for a layered medium can easily be formulated. Generally, the inverse Laplace transform and the inverse Fourier transforms have to be carried out numerically; for the first one we use the method of PAPOULIS [25], while the second ones may be performed by means of numerical integrations. The temperature and heat flux distributions of the layered medium are determined in the first step, while the stresses and displacements are evaluated in the second one. As special cases we consider the plane and the axisymmetrical ones respectively.

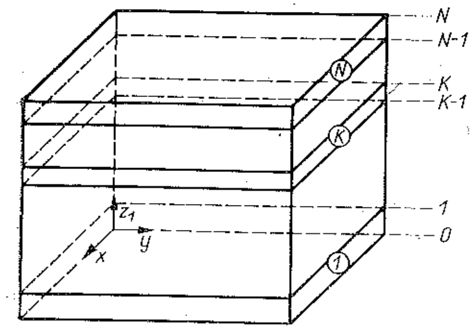


Fig. 1

As an example an elastic four-layer system under nonstationary conditions is analysed numerically.

2. MATRIX FORMULATION OF BASIC EQUATIONS

2.1. Heat conduction

It is essential to write the basic equations for a thermal homogeneous and isotropic medium with temperature-independent thermal properties with respect to Cartesian coordinates $x_1=x$, $x_2=y$, $x_3=z$, namely

$$q_i=-\kappa\vartheta_{,i}, \tag{2.1}$$

$$\frac{\partial\vartheta}{\partial t}=\frac{1}{\rho c}(-q_{i,i}+Q) \tag{2.2}$$

(q_i heat flux, Q specific heat source, ϑ temperature relative to a reference configuration at time $t=0$, ρ mass density, κ thermal conductivity, c specific heat capacity; summation convention) in the matrix form

$$\frac{\partial}{\partial z}\mathbf{r}=\mathbf{R}\mathbf{r}+\mathbf{s}, \tag{2.3}$$

$$\mathbf{p}=\mathbf{P}\mathbf{r}, \tag{2.24}$$

where

$$\mathbf{r}=\{\vartheta,\ q_z^*\}^T \tag{2.5}$$

and

$$\mathbf{p}=\{q_x^*,\ q_y^*\}^T \tag{2.6}$$

mean the (unknown) "thermal state vectors"[1] and

$$\mathbf{s}=\left\{0, \frac{h^*}{\kappa^*} Q\right\}^T \tag{2.7}$$

a (known) vector. Further, **R** and **P** represent the differential matrices

$$\mathbf{R}=\left[\begin{array}{c|c} 0 & -\dfrac{\kappa^*}{\kappa h^*} \\ \hline \dfrac{\kappa}{\kappa^*} h^*\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}-\dfrac{1}{a}\dfrac{\partial}{\partial t}\right) & 0 \end{array}\right] \tag{2.8}$$

and

$$\mathbf{P}=\left[\begin{array}{c|c} -\dfrac{\kappa h^*}{\kappa^*}\dfrac{\partial}{\partial x} & 0 \\ \hline -\dfrac{\kappa h^*}{\kappa^*}\dfrac{\partial}{\partial y} & 0 \end{array}\right] \tag{2.9}$$

and a and q_i^* the abbreviations

$$a=\frac{\kappa}{\rho c}, \tag{2.10}$$

$$q_i^*=\frac{h^*}{\kappa^*} q_i, \tag{2.11}$$

where h^* and κ^* are a reference length and a reference thermal conductivity, respectively. In the stationary case, of course, the time operator in (2.8) disappears.

2.2. *Thermoelasticity and thermoviscoelasticity*

The displacement-stress relations for a homogeneous and isotropic linear elastic medium and the equilibrium equations read

$$\frac{1}{2}(u_{i,j}+u_{j,i})=\frac{1+\nu}{E}\left(\sigma_{ij}-\delta_{ij}\frac{\nu}{1+\nu}\sigma_{kk}\right)+\delta_{ij}\gamma\vartheta, \tag{2.12}$$

$$\sigma_{ij,j}+f_i=0, \quad \sigma_{ij}=\sigma_{ji} \tag{2.13}$$

(u_i displacement, σ_{ij} stress, f_i body force, E Young's modulus, $1/\nu$ Poisson's ratio, γ coefficient of linear thermal expansion). By means of suitable eliminations they may be put into the following form:

$$\frac{\partial \mathbf{a}}{\partial z}=\mathbf{A}\mathbf{a}+\mathbf{d}, \tag{2.14}$$

$$\mathbf{b}=\mathbf{B}\mathbf{a}+\mathbf{e}, \tag{2.15}$$

[1] We designate **r** as the essential and **p** as the residual thermal state vector

with the (unknown) "mechanical state vectors"(2)

$$\mathbf{a}=\{\sigma_{zz}, \sigma_{zx}, \sigma_{zy}, u_y^*, u_x^*, u_z^*\}^T \tag{2.16}$$

and

$$\mathbf{b}=\{\sigma_{xx}+\sigma_{yy}, \sigma_{xx}-\sigma_{yy}, 2\sigma_{xy}\}^T \tag{2.17}$$

and the (after having solved the heat conduction problem to be considered as known) vectors

$$\mathbf{d}=\left\{-f_z, -f_x+\frac{E\gamma}{1-\nu}\frac{\partial\vartheta}{\partial x}, -f_y+\frac{E\gamma}{1-\nu}\frac{\partial\vartheta}{\partial y}, 0, 0, \frac{1+\nu}{1-\nu}\frac{E^*}{h^*}\gamma\vartheta\right\}^T, \tag{2.18}$$

$$\mathbf{e}=\left\{-\frac{2E}{1-\nu}\gamma\vartheta, 0, 0\right\}^T. \tag{2.19}$$

In (2.14) and (2.15) **A** and **B** mean differential matrices according to (2.20) and (2.21) respectively which had already been given by BUFLER [23]. (The Eqs. (2.20) and (2.21) are on page 103).

If instead of an elastic medium we have a linear viscoelastic one obeying the law

$$\begin{gathered} Q(D)\left(\varepsilon_{ij}-\frac{1}{3}\delta_{ij}\varepsilon_{kk}\right)=P(D)\left(\sigma_{ij}-\frac{1}{3}\delta_{ij}\sigma_{kk}\right), \\ Q^*(D)(\varepsilon_{kk}-3\gamma\vartheta)=P^*(D)\sigma_{kk} \end{gathered} \tag{2.22}$$

with $Q(D)$, $P(D)$, $Q^*(D)$, $P^*(D)$ as polynomials in the time derivatives $D=d/dt$, the elastic constants E and ν in (2.12) have formally to be substituted by the operators

$$E(D)=\frac{3QQ^*}{2PQ^*+P^*Q}, \qquad \nu(D)=\frac{PQ^*-P^*Q}{2PQ^*+P^*Q}. \tag{2.23}$$

(Alfrey's analogy, see e.g. [26]). As a consequence, the time operator — hitherto only in (2.8) — now also appears in (2.18), (2.19), (2.20) and (2.21).

3. General integration procedure for a single layer

3.1. Spatial integration: Method of Lur'e [27]

Comparison of the Eq. (2.3) with the Eq. (2.14) and of the Eq. (2.4) with the Eq. (2.15) shows the same structure of the respective relations. For the moment therefore it is enough to discuss the problem (2.3), (2.4) (heat conduction).

Formal integration of (2.3) with respect to the z-coordinate gives

$$\mathbf{r}=\mathbf{W}\mathbf{r}_0+\mathbf{w} \tag{3.1}$$

with

$$\mathbf{r}_0=\mathbf{r}(x, y, 0; t)$$

(2) We designate **a** as the essential and **b** as the residual mechanical state vector.

$$
\mathbf{A}=\begin{bmatrix}
0 & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & 0 & 0 & 0 \\
-\frac{\nu}{1-\nu}\frac{\partial}{\partial x} & 0 & 0 & -\frac{h^*}{2(1-\nu)}\frac{E}{E^*}\frac{\partial^2}{\partial x\,\partial y} & \frac{-h^*}{1-\nu^2}\frac{E}{E^*}\left(\frac{\partial^2}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2}{\partial y^2}\right) & 0 \\
-\frac{\nu}{1-\nu}\frac{\partial}{\partial y} & 0 & 0 & -\frac{h^*}{1-\nu^2}\frac{E}{E^*}\left(\frac{\partial^2}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial^2}{\partial x^2}\right) & -\frac{h^*}{2(1-\nu)}\frac{E}{E^*}\frac{\partial^2}{\partial x\partial y} & 0 \\
0 & 0 & \frac{2(1+\nu)}{h^*}\frac{E^*}{E} & 0 & 0 & -\frac{\partial}{\partial y} \\
0 & \frac{2(1+\nu)}{h^*}\frac{E^*}{E} & 0 & 0 & 0 & -\frac{\partial}{\partial x} \\
\frac{(1+\nu)(1-2\nu)}{(1-\nu)h^*}\frac{E^*}{E} & 0 & 0 & -\frac{\nu}{1-\nu}\frac{\partial}{\partial y} & -\frac{\nu}{1-\nu}\frac{\partial}{\partial x} & 0
\end{bmatrix}, \tag{2.20}
$$

$$
\mathbf{B}=\begin{bmatrix}
\frac{2\nu}{1-\nu} & 0 & 0 & \frac{h^*}{1-\nu}\frac{E}{E^*}\frac{\partial}{\partial y} & \frac{h^*}{1-\nu}\frac{E}{E^*}\frac{\partial}{\partial x} & 0 \\
0 & 0 & 0 & \frac{-h^*}{1+\nu}\frac{E}{E^*}\frac{\partial}{\partial y} & \frac{h^*}{1+\nu}\frac{E}{E^*}\frac{\partial}{\partial x} & 0 \\
0 & 0 & 0 & \frac{h^*}{1+\nu}\frac{E}{E^*}\frac{\partial}{\partial x} & \frac{h^*}{1+\nu}\frac{E}{E^*}\frac{\partial}{\partial y} & 0
\end{bmatrix}, \tag{2.21}
$$

as "initial state vector",

$$\mathbf{W}=e^{\mathbf{R}z}=\sum_{k=0}^{\infty}\frac{1}{k!}(\mathbf{R}z)^k \tag{3.2}$$

as differential matrix of infinite order containing the derivatives $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial t$, and

$$\mathbf{w}=\int_0^z e^{(z-\zeta)\mathbf{R}}\,\mathbf{s}(\zeta)\,d\zeta \tag{3.3}$$

as a particular integral(3) of (2.3). The matrix (3.2) may be evaluated in closed form, see later. Now the residual state vector $\mathbf{p}$ becomes

$$\mathbf{p}=\mathbf{PWr}_0+\mathbf{Pw}. \tag{3.4}$$

Let us consider a layer with the constant height h and the boundary conditions at the faces (e.g.)

$$\vartheta(x,y,0;t)=\vartheta_0(x,y;t), \qquad q_z(x,y,h;t)=q_{z1}(x,y;t). \tag{3.5}$$

Then, from (3.1) we receive with $z=h$ the following differential equation for $q_{z0}(x,y;t)=q_z(x,y,0;t)$

$$W_{22}\,q^x_{z0}=q^x_{z1}-W_{21}\,\vartheta_0-w_2, \tag{3.6}$$

where W_{ij} and w_i designate the elements of $\mathbf{W}$ and $\mathbf{w}$, respectively. Generally, the boundary conditions at the faces yield the differential equations for the not-given components of the initial state vector, as has been shown for the stress problem by Vlasov and Leont'ev [28].

The symbolic method of Lur'e is advantageous especially for layers with finite lateral extensions. Retaining only a few members of the series (3.2), the Eq. (3.6) will be of finite order. According to this order and using (3.1) and (3.4), one calculates the "resultants"

$$\vartheta^{(\mu)}=\int_{-h/2}^{h/2} z'^{\mu}\,\vartheta\,dz', \qquad q_n^{(\mu)}=\int_{-h/2}^{h/2} z'^{\mu}\,q_n\,dz', \qquad z'=-h/2+z, \quad \mu=0,1,2,\ldots, \tag{3.7}$$

which are needed at the cylindrical boundary $S=S_\vartheta+S_q$, where the boundary conditions read:

$$\vartheta^{(\mu)}=\hat{\vartheta}^{(\mu)} \quad \text{on} \quad S_\vartheta, \qquad q_n^{(\mu)}=\hat{q}_n^{(\mu)} \quad \text{on} \quad S_q, \tag{3.8}$$

with $q_n=q_x n_x+q_y n_y$ (n outward normal).

3.2. *Spatial integration: Sneddon's method* [29]

The method of Fourier transforms is suitable for infinite layers, where only the boundary conditions at the faces are relevant. In this case the application of the (two-dimensional) Fourier transforms with respect to the coordinates x and

(3) The relation $\mathbf{s}=\mathbf{s}(x,y,z,t)$ is here abbreviated by $\mathbf{s}=\mathbf{s}(z)$ or $\mathbf{s}=\mathbf{s}(\zeta)$, respectively

y transforms the Eq. (2.3) into a matrix differential equation in z and t, and the Eq. (2.4) into an algebraic matrix equation according to

$$\frac{\partial \bar{\mathbf{r}}}{\partial z} = \bar{\mathbf{R}}\bar{\mathbf{r}} + \bar{\mathbf{s}}, \tag{3.9}$$

$$\bar{\mathbf{p}} = \bar{\mathbf{P}}\bar{\mathbf{r}}, \tag{3.10}$$

where

$$\left.\begin{matrix}\bar{\mathbf{r}}\\ \bar{\mathbf{s}}\end{matrix}\right\} = \mathbf{G}\left\{\begin{matrix}\mathbf{r}\\ \mathbf{s}\end{matrix}\right., \tag{3.11}$$

$$\bar{\mathbf{p}} = \mathbf{G}'\,\mathbf{p}, \tag{3.12}$$

with the operator matrices

$$\mathbf{G} = \mathrm{diag}\,\{\mathscr{F}, \mathscr{F}\}, \tag{3.13}$$

$$\mathbf{G}' = \mathrm{diag}\,\{\mathscr{F}_\alpha, \mathscr{F}_\beta\}, \tag{3.14}$$

the operations $\mathscr{F}$, $\mathscr{F}_\alpha$ and $\mathscr{F}_\beta$ (Fourier transforms) being defined in the Appendix (Al). Further there is because of (A4)

$$\bar{\mathbf{R}} = \left[\begin{array}{c:c} 0 & -\dfrac{1}{\kappa}\dfrac{\kappa^*}{h^*} \\ \hdashline -\dfrac{\kappa}{\kappa^* h^*}\left(\lambda^{*2} + \dfrac{h^{*2}}{a} D\right) & 0 \end{array}\right], \tag{3.15}$$

$$\bar{\mathbf{P}} = \left[\begin{array}{c:c} \dfrac{\kappa}{\kappa^*}\alpha^* & 0 \\ \hdashline \dfrac{\kappa}{\kappa^*}\beta^* & 0 \end{array}\right] \tag{3.16}$$

with the abbreviations

$$\alpha^* = |\alpha|\,h^*, \quad \beta^* = |\beta|\,h^*, \quad \lambda^* = \sqrt{\alpha^{*2} + \beta^{*2}}, \quad D = \partial/\partial t, \tag{3.17}$$

where α and β are the real transform parameters. Note that the transformed quantities have to obey the conditions given in the Appendix.

Integration of (3.9) with respect to z yields

$$\bar{\mathbf{r}} = \bar{\mathbf{W}}\bar{\mathbf{r}}_0 + \bar{\mathbf{w}} \tag{3.18}$$

with $\bar{\mathbf{r}}_0$ as the transformed thermal initial state vector,

$$\bar{\mathbf{W}} = e^{\bar{\mathbf{R}}z} = \sum_{\mu=0}^{\infty} \frac{1}{\mu!}(\bar{\mathbf{R}}z)^\mu \tag{3.19}$$

as operator matrix containing only the time derivatives and

$$\bar{\mathbf{w}} = \int_0^z e^{(z-\zeta)\bar{\mathbf{R}}}\,\bar{\mathbf{s}}(\zeta)\,d\zeta \tag{3.20}$$

as particular integral of (3.9).

It should be remarked here that the result (3.18) could be obtained also from (3.1) after performing the Fourier transforms.

For completeness we list the corresponding relations for the problem (2.14), (2.15) (thermoelasticity):

$$\frac{\partial \bar{\mathbf{a}}}{\partial z} = \bar{\mathbf{A}}\,\bar{\mathbf{a}} + \bar{\mathbf{d}}, \tag{3.21}$$

$$\bar{\mathbf{b}} = \bar{\mathbf{B}}\,\bar{\mathbf{a}} + \bar{\mathbf{e}}, \tag{3.22}$$

$$\left.\begin{matrix}\bar{\mathbf{a}}\\ \bar{\mathbf{d}}\end{matrix}\right\} = \mathbf{F}\left\{\begin{matrix}\mathbf{a}\\ \mathbf{d}\end{matrix}\right., \tag{3.23}$$

$$\left.\begin{matrix}\bar{\mathbf{b}}\\ \bar{\mathbf{e}}\end{matrix}\right\} = \mathbf{F}'\left\{\begin{matrix}\mathbf{b}\\ \mathbf{e}\end{matrix}\right., \tag{3.24}$$

$$\mathbf{F} = \operatorname{diag}\{\mathscr{F}, \mathscr{F}_\alpha, \mathscr{F}_\beta, \mathscr{F}_\beta, \mathscr{F}_\alpha, -\mathscr{F}\}, \tag{3.25}$$

$$\mathbf{F}' = \operatorname{diag}\{\mathscr{F}, -\mathscr{F}, -\mathscr{F}\}. \tag{3.26}$$

The matrices $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are given in (3.27) and (3.28), respectively; they already had been derived by BUFLER in [23]. For a viscoelastic medium, the elastic constants E and ν have to be replaced by time operators according to (2.23). The Eq. (3.27) and (3.28) are on page 107).

Integration of (3.21) yields

$$\bar{\mathbf{a}} = \bar{\mathbf{T}}\,\bar{\mathbf{a}}_0 + \bar{\mathbf{t}} \tag{3.29}$$

with $\bar{\mathbf{a}}_0$ as the transformed mechanical initial state vector,

$$\bar{\mathbf{T}} = \mathbf{e}^{\bar{\mathbf{A}}z} = \sum_{\mu=0}^{\infty} \frac{1}{\mu!}(\bar{\mathbf{A}}z)^\mu \tag{3.30}$$

(containing the time derivatives only in the case of a viscoelastic material) and

$$\bar{\mathbf{t}} = \int_0^z e^{\bar{\mathbf{A}}(z-\zeta)}\,\bar{\mathbf{d}}(\zeta)\,d\zeta \tag{3.31}$$

as particular integral of (3.21).

4. The layered medium: Heat conduction

4.1. *The infinite single layer*

Referring to the layer k the Eqs. (3.10) and (3.18) may be written as

$$\bar{\mathbf{p}}_k(\alpha, \beta, z_k; t) = \bar{\mathbf{P}}_k(\alpha, \beta)\,\bar{\mathbf{r}}_k(\alpha, \beta, z_k; t), \tag{4.1}$$

$$\bar{\mathbf{r}}_k(\alpha, \beta, z_k; t) = \bar{\mathbf{W}}_k(\alpha, \beta, z_k; D)\,\bar{\mathbf{r}}_k(\alpha, \beta, 0; t) + \bar{\mathbf{w}}_k(\alpha, \beta, z_k; t, D), \tag{4.2}$$

$$
(3.27)\qquad \bar{\mathbf{A}}=\begin{bmatrix}
0 & -\dfrac{\alpha^*}{h^*} & -\dfrac{\beta^*}{h^*} & 0 & 0 & 0\\[2ex]
\dfrac{\nu}{1-\nu}\,\dfrac{\alpha^*}{h^*} & 0 & 0 & \dfrac{1}{2(1-\nu)}\,\dfrac{E}{E^*}\,\dfrac{\alpha^*\beta^*}{h^*} & \dfrac{1}{1-\nu^2}\,\dfrac{E}{E^*}\,\dfrac{2\alpha^{*2}+(1-\nu)\beta^{*2}}{2h^*} & \\[2ex]
\dfrac{\nu}{1-\nu}\,\dfrac{\beta^*}{h^*} & 0 & 0 & \dfrac{1}{1-\nu^2}\,\dfrac{E}{E^*}\,\dfrac{2\beta^{*2}+(1-\nu)\alpha^{*2}}{2h^*} & & \\[2ex]
0 & 0 & \dfrac{2(1+\nu)}{h^*}\,\dfrac{E^*}{E} & & & \\[2ex]
0 & \dfrac{2(1+\nu)}{h^*}\,\dfrac{E^*}{E} & & & \text{symmetrical} & \\[2ex]
-\dfrac{(1+\nu)(1-2\nu)}{(1-\nu)h^*}\,\dfrac{E^*}{E} & & & & &
\end{bmatrix},
$$

$$
(3.28)\qquad \bar{\mathbf{B}}=\begin{bmatrix}
\dfrac{2\nu}{1-\nu} & 0 & 0 & \dfrac{\beta^*}{1-\nu}\,\dfrac{E}{E^*} & \dfrac{\alpha^*}{1-\nu}\,\dfrac{E}{E^*} & 0\\[2ex]
0 & 0 & 0 & \dfrac{\beta^*}{1+\nu}\,\dfrac{E}{E^*} & -\dfrac{\alpha^*}{1+\nu}\,\dfrac{E}{E^*} & 0\\[2ex]
0 & 0 & 0 & \dfrac{\alpha^*}{1+\nu}\,\dfrac{E}{E^*}\,j_\alpha j_\beta & \dfrac{\beta^*}{1+\nu}\,\dfrac{E}{E^*}\,j_\alpha j_\beta & 0
\end{bmatrix},
$$

where (calculated by means of Cayley-Hamilton's theorem)

$$\overline{\mathbf{W}}_k(\alpha,\beta,z_k;D)=\left[\begin{array}{c:c} \operatorname{ch}(z_k^*\sqrt{\ }) & -\dfrac{1}{\kappa_k^*}\dfrac{\operatorname{sh}(z_k^*\sqrt{\ })}{\sqrt{\ }} \\ \hdashline -\kappa_k^*\sqrt{\ }\operatorname{sh}(z_k^*\sqrt{\ }) & \operatorname{ch}(z_k^*\sqrt{\ }) \end{array}\right] \tag{4.3}$$

and

$$\overline{\mathbf{w}}_k(\alpha,\beta,z_k;t,D)=\left[\begin{array}{c} -\dfrac{h^{*2}}{\kappa_k}\displaystyle\int_0^{z_k^*}\frac{\operatorname{sh}[(z_k^*-\zeta_k^*)\sqrt{\ }]}{\sqrt{\ }}\bar{Q}\,d\zeta_k^* \\ \hdashline \dfrac{h^{*2}}{\kappa^*}\displaystyle\int_0^{z_k^*}\operatorname{ch}[(z_k^*-\zeta_k^*)\sqrt{\ }]\,\bar{Q}\,d\zeta_k^* \end{array}\right] \tag{4.4}$$

with $\bar{Q}=\bar{Q}(\alpha,\beta,\zeta_k;t)$ as Fourier transform of the heat source and with the abbreviations

$$z_k^*=\frac{z_k}{h^*},\quad \zeta_k^*=\frac{\zeta_k}{h^*},\quad \sqrt{\ }=\sqrt{\alpha^{*2}+\beta^{*2}+\frac{h^{*2}}{a_k}D},$$
$$\kappa_k^*=\frac{\kappa_k}{\kappa^*},\quad a_k=\frac{\kappa_k}{\rho_\kappa c_k}. \tag{4.5}$$

Taking $z_k=h_k$ in (4.2) we receive the "transfer equation"

$$\bar{\mathbf{r}}_k(\alpha,\beta,h_k;t)=\overline{\mathbf{W}}_k(\alpha,\beta,h_k;D)\,\bar{\mathbf{r}}_k(\alpha,\beta,0;t)+\bar{\mathbf{w}}_k(\alpha,\beta,h_k;t,D), \tag{4.6}$$

with $\overline{\mathbf{W}}_k=\overline{\mathbf{W}}_k(\alpha,\beta,h_k;D)$ being the thermal transfer matrix of layer k(4). This matrix has the known properties of transfer matrices: cross symmetry, if the elements of the state vector are chosen in a suitable sequence, determinant 1, identical elements with alternating signs for the reciprocal matrix.

Integrating (4.2) with respect to the time under zero initial conditions gives

$$\bar{\mathbf{r}}_k(\alpha,\beta,z_k;t)=\mathscr{L}^{-1}\{\tilde{\overline{\mathbf{W}}}_k(\alpha,\beta,z_k;p)\,\tilde{\bar{\mathbf{r}}}_k(\alpha,\beta,0;p)+\tilde{\bar{\mathbf{w}}}_k(\alpha,\beta,z_k;p)\}, \tag{4.7}$$

where $\tilde{\bar{\mathbf{r}}}_k$ means the Laplace transform of $\bar{\mathbf{r}}_k$ and $\tilde{\overline{\mathbf{W}}}_k$ results from $\overline{\mathbf{W}}_k$ by the substitution $D\to p$ with p as the complex transform parameter [see Appendix (A8), (A9) and (A10)]. Further

$$\tilde{\bar{\mathbf{w}}}_n(\alpha,\beta,z_k;p)=\left[\begin{array}{c} \dfrac{-\dfrac{h^{*2}}{\kappa_k}\displaystyle\int_0^{z_k^*}\operatorname{sh}\left[(z_k^*-\zeta_k^*)\sqrt{\alpha^{*2}+\beta^{*2}+\frac{h^{*2}}{a_k}p}\right]\tilde{\bar{Q}}\,d\zeta_k^*}{\sqrt{\alpha^{*2}+\beta^{*2}+\frac{h^{*2}}{a_k}p}} \\ \hdashline \dfrac{h^{*2}}{\kappa^*}\displaystyle\int_0^{z_k^*}\operatorname{ch}\left[(z_k^*-\zeta_k^*)\sqrt{\alpha^{*2}+\beta^{*2}+\frac{h^{*2}}{a_k}p}\right]\tilde{\bar{Q}}\,d\zeta_k^* \end{array}\right] \tag{4.8}$$

(4) Concerning the method of transfer matrices see the book [30] by PESTEL and LECKIE.

with $\tilde{\bar{Q}}_k = \tilde{\bar{Q}}_k(\alpha, \beta, \zeta_k; p)$ as Laplace transform of $\bar{Q}_k(\alpha, \beta, \zeta_k; t)$. Then the instationary fields of temperature and heat flux in the layer k follow by performing the universe Fourier transforms of (4.1) and (4.7), respectively:

$$\mathbf{p}_k(x, y, z_k; t) = \mathbf{G}'^{-1}\{\bar{\mathbf{P}}_k(\alpha, \beta)\,\bar{\mathbf{r}}_k(\alpha, \beta, z_k; t)\}, \tag{4.9}$$

$$\mathbf{r}_k(x, y, z_k; t) = \mathbf{G}^{-1}\{\bar{\mathbf{r}}_k(\alpha, \beta, z_k; t)\}. \tag{4.10}$$

Generally the inverse Laplace transform has to be carried out numerically.

For a single layer, the (spatial) thermal initial state vector $\tilde{\bar{\mathbf{r}}}_k(0) = \tilde{\bar{\mathbf{r}}}_k(\alpha, \beta, 0; p)$ which is needed in (4.7) has to be determined by means of the boundary conditions for the temperature ϑ and/or heat flux q_{zk}, which are assumed to be of the product type, for instance

$$\begin{aligned} \vartheta_k(x, y, 0; t) &= \vartheta_k^{(1)}(x, y)\,\vartheta_k^{(2)}(t), \\ q_{zk}(x, y, h_k; t) &= q_{zk}^{(1)}(x, y)\,q_{zk}^{(2)}(t), \end{aligned} \tag{4.11}$$

transformed:

$$\begin{aligned} \tilde{\bar{\vartheta}}_k(0) &= \bar{\vartheta}_k^{(1)}\,\tilde{\vartheta}_k^{(2)}, \\ \tilde{\bar{q}}_{zk}(h_k) &= \bar{q}_{zk}^{(1)}\,\tilde{q}_{zk}^{(2)}. \end{aligned} \tag{4.12}$$

If we apply the Laplace transform to (4.6) we get(5)

$$\tilde{\bar{\mathbf{r}}}_k(h_k) = \tilde{\bar{\mathbf{W}}}_k(h_k)\,\tilde{\bar{\mathbf{r}}}_k(0) + \tilde{\bar{\mathbf{w}}}_k(h_k). \tag{4.13}$$

Taking into account (4.12) we can calculate $\tilde{\bar{q}}_{zk}(0)$ and so the initial state vector $\tilde{\bar{\mathbf{r}}}_k(0)$ is known.

4.2. *The layered system*

In the case of a complete contact between the individual layers, the following continuity conditions (for the transformed quantities) have to be fulfilled:

$$\tilde{\bar{\mathbf{r}}}_{k+1}(0) = \tilde{\bar{\mathbf{r}}}_k(h_k) = \tilde{\bar{\mathbf{r}}}_k, \qquad (k = 1, 2, ..., N-1) \tag{4.14}$$

(continuity of temperature and heat flux).

Then the successive application of (4.13) with (4.14) in the form

$$\begin{bmatrix} \tilde{\bar{\mathbf{r}}} \\ 1 \end{bmatrix}_k = \begin{bmatrix} \tilde{\bar{\mathbf{W}}} & \tilde{\bar{\mathbf{w}}} \\ 0 & 1 \end{bmatrix}_k \begin{bmatrix} \tilde{\bar{\mathbf{r}}} \\ 1 \end{bmatrix}_{k-1}, \qquad (k = 1, 2, ..., N-1) \tag{4.15}$$

or shortly written

$$\mathbf{r}_k^* = \mathbf{W}_k^*\,\mathbf{r}_{k-1}^*, \qquad (k = 1, 2, ..., N-1) \tag{4.15'}$$

allows the elimination of the intermediate quantities and yields for an N-layer system

$$\mathbf{r}_N^* = \mathbf{W}_N^*\,\mathbf{W}_{N-1}^* \dots \mathbf{W}_1^*\,\mathbf{r}_0^* = \mathbf{W}^*\,\mathbf{r}_0^*. \tag{4.16}$$

(5) For the matter of brevity in the parentheses we refer here and later only to the dependence on the coordinate z_k.

One component of $\mathbf{r}_0^*$ and one component of $\mathbf{r}_N^*$ are known because of the boundary conditions, the residual ones may be calculated from (4.16). At last all initial state vectors follow from (4.15′) and so according to (4.7), (4.9) and (4.10) the instationary fields of temperature and heat flux are known. Often it is advantageous because of numerical reasons to avoid (4.16) and to solve the system (4.15′) simultaneously by means of Gauss's algorithm [31]. Other possibilities are discussed in [23, 24, 22].

If there is no complete contact between the layers, a jump of the temperature has to be taken into account at a discontinuity face, namely

$$\tilde{\bar{\mathbf{r}}}_k^+ = \mathbf{D}_k \tilde{\bar{\mathbf{r}}}_k^- , \tag{4.17}$$

where

$$\mathbf{D}_k = \begin{bmatrix} 1 & -K_k \\ 0 & 1 \end{bmatrix} \tag{4.18}$$

with K_k as the temperature resistance.

4.3. The stationary case

By formal setting $D \to 0$ in (4.3) there results

$$\overline{\mathbf{W}}_k(\alpha, \beta, z_k) = \left[\begin{array}{c:c} \mathrm{ch}\,(z_k^* \lambda^*) & -\dfrac{1}{\kappa_k^* \lambda^*} \mathrm{sh}\,(z_k^* \lambda^*) \\ \hdashline -\kappa_k^* \lambda^* \,\mathrm{sh}\,(z_k^* \lambda^*) & \mathrm{ch}\,(z_k^* \lambda^*) \end{array}\right] \tag{4.19}$$

with the abbreviation

$$\lambda^* = \sqrt{\alpha^{*2} + \beta^{*2}} . \tag{4.20}$$

Further, taking the heat source Q_k as time independent, (4.4) degenerates to

$$\bar{\mathbf{w}}_k(\alpha, \beta, z_k) = \left[\begin{array}{c} -\dfrac{h^{*2}}{\kappa_k \lambda^*} \displaystyle\int_0^{z_k^*} \mathrm{sh}\,[(z_k^* - \zeta_k^*) \lambda^*]\, \bar{Q} d\zeta_k^* \\ \hdashline \dfrac{h^{*2}}{\kappa^*} \displaystyle\int_0^{z_k^*} \mathrm{ch}\,[(z_k^* - \zeta_k^*) \lambda^*]\, \bar{Q} d\zeta_k^* \end{array}\right]. \tag{4.21}$$

Besides the boundary conditions have to be time independent. As a consequence, instead of $\tilde{\bar{\mathbf{r}}}_k$, $\tilde{\bar{\mathbf{w}}}_k$, $\tilde{\overline{\mathbf{W}}}_k$ in (4.13), (4.14) and (4.15) we have to use $\bar{\mathbf{r}}_k$, $\bar{\mathbf{w}}_k$, $\overline{\mathbf{W}}_k$.

5. The layered medium: Thermal stresses and displacements

5.1. The infinite single layer

Referring to the viscoelastic layer k the Eq. (3.22) and (3.29) may be written as

$$\bar{\mathbf{b}}_k(\alpha, \beta, z_k; t) = \overline{\mathbf{B}}_k(\alpha, \beta; D)\, \bar{\mathbf{a}}_k(\alpha, \beta, z_k; t) + \bar{\mathbf{e}}_k(\alpha, \beta, z_k; t, D), \tag{5.1}$$

$$\bar{\mathbf{a}}_k(\alpha, \beta, z_k; t) = \overline{\mathbf{T}}_k(\alpha, \beta, z_k; D)\, \bar{\mathbf{a}}_k(\alpha, \beta, 0; t) + \bar{\mathbf{t}}_k(\alpha, \beta, z_k; t, D), \tag{5.2}$$

where according to (3.30), (3.31) and using Cayley-Hamilton's theorem $\overline{\mathbf{T}}_k$ and $\bar{\mathbf{t}}_k$ had been calculated, see (5.3) and (5.4).

$$
(5.3) \qquad \overline{\mathbb{T}}_k(\alpha, \beta, z_k; D) = \frac{c(z_k^*)}{2(1-\nu_k)} \times
$$

$$
\times \left[\begin{array}{c:c:c:c:c:c}
2(1-\nu_k)-\lambda^* z_k^* \, t(z_k^*) & -\frac{\alpha^*}{\lambda^*}[\lambda^* z_k^* + (1-2\nu_k)\, t(z_k^*)] & -\frac{\beta^*}{\lambda^*}[\lambda^* z_k^* + (1-2\nu_k)\, t(z_k^*)] & -\frac{E_k^*}{1+\nu_k} \times \beta^* \lambda^* z_k^* \, t(z_k^*) & -\frac{E_k^*}{1+\nu_k} \times \alpha^* \lambda^* z_k^* \, t(z_k^*) & \frac{E_k^* \lambda^*}{1+\nu_k} \times [\lambda^* z_k^* - t(z_k^*)] \\
\hdashline
\frac{\alpha^*}{\lambda^*}[\lambda^* z_k^* - (1-2\nu_k)\, t(z_k^*)] & 2(1-\nu_k) + \frac{\alpha^{*2}}{\lambda^*} z_k^* \, t(z_k^*) & \frac{\alpha^* \beta^*}{\lambda^*} z_k^* \, t(z_k^*) & \frac{E_k^*}{1+\nu_k} \frac{\alpha^* \beta^*}{\lambda^*} \times [\lambda^* z_k^* + \nu_k \, t(z_k^*)] & \frac{E_k^* \lambda^*}{1+\nu_k}\left[\frac{\alpha^{*2}}{\lambda^*} z_k^* + \left(1-\nu_k \frac{\beta^{*2}}{\lambda^{*2}}\right) t(z_k^*)\right] & \\
\hdashline
\frac{\beta^*}{\lambda^*}[\lambda^* z_k^* - (1-2\nu_k)\, t(z_k^*)] & \frac{\alpha^* \beta^*}{\lambda^*} z_k^* \, t(z_k^*) & 2(1-\nu_k) + \frac{\beta^{*2}}{\lambda^*} z_k^* \, t(z_k^*) & \frac{E_k^* \lambda^*}{1+\nu_k}\left[\frac{\beta^{*2}}{\lambda^*} z_k^* + \left(1-\nu_k \frac{\alpha^{*2}}{\lambda^{*2}}\right) t(z_k^*)\right] & & \\
\hdashline
\frac{1+\nu_k}{E_k^*} \frac{\beta^*}{\lambda^*} z_k^* \, t(z_k^*) & \frac{1+\nu_k}{E_k^*} \frac{\alpha^* \beta^*}{\lambda^{*2}} \times \left[z_k^* - \frac{t(z_k^*)}{\lambda^*}\right] & \frac{1+\nu_k}{E_k^* \lambda^*}\left[4(1-\nu_k)\, t(z_k^*) - \frac{\beta^{*2}}{\lambda^{*2}} \langle t(z_k^*) - \lambda^* z_k^* \rangle\right] & & & \\
\hdashline
\frac{1+\nu_k}{E_k^*} \frac{\alpha^*}{\lambda^*} z_k^* \, t(z_k^*) & \frac{1+\nu_k}{E_k^* \lambda^*}\left[4(1-\nu_k)\, t(z_k^*) - \frac{\alpha^{*2}}{\lambda^{*2}} \langle t(z_k^*) - \lambda^* z_k^* \rangle\right] & & \text{symmetrical} & & \\
\hdashline
\frac{1+\nu_k}{E_k^*} \times \left[z_k^* - \frac{3-4\nu_k}{\lambda^*} t(z_k^*)\right] & & & & &
\end{array}\right]
$$

$$(5.4)\qquad \bar{\mathbf{t}}_k(\alpha, \beta, z_k; t, D) =$$

$$= \begin{bmatrix} \dfrac{E_k^* \gamma_k \lambda^*}{1-\nu_k} \displaystyle\int_0^{z_k^*} s\bar{\vartheta}_k \, d\zeta_k^* - \dfrac{h^*}{2(1-\nu_k)} \displaystyle\int_0^{z_k^*} \left\{ [2(1-\nu_k)\, c - \lambda^* (z_k^* - \zeta_k^*)\, s]\, \bar{f}_{z_k} + \left[(z_k^* - \zeta_k^*)\, c + (1-2\nu_k) \dfrac{s}{\lambda^*} \right] (\alpha^* \overset{\alpha}{\bar{f}}_{x_k} + \beta^* \overset{\beta}{\bar{f}}_{y_k}) \right\} d\zeta_k^* \\ \hdashline -\dfrac{E_k^* \gamma_k \alpha^*}{1-\nu_k} \displaystyle\int_0^{z_k^*} c\bar{\vartheta}_k \, d\zeta_k^* - \dfrac{h^*}{2(1-\nu_k)} \displaystyle\int_0^{z_k^*} \left\{ \left[(z_k^* - \zeta_k^*)\, c - (1-2\nu_k) \dfrac{s}{\lambda^*} \right] \alpha^* \bar{f}_{z_k} + (z_k^* - \zeta_k^*)\, s \dfrac{\alpha^*}{\lambda^*} (\alpha^* \overset{\alpha}{\bar{f}}_{x_k} + \beta^* \overset{\beta}{\bar{f}}_{y_k}) + 2(1-\nu_k)\, c \overset{\alpha}{\bar{f}}_{x_k} \right\} d\zeta_k^* \\ \hdashline -\dfrac{E_k^* \gamma_k \beta^*}{1-\nu_k} \displaystyle\int_0^{z_k^*} c\bar{\vartheta}_k \, d\zeta_k^* - \dfrac{h^*}{2(1-\nu_k)} \displaystyle\int_0^{z_k^*} \left\{ \left[(z_k^* - \zeta_k^*)\, c - (1-2\nu_k) \dfrac{s}{\lambda^*} \right] \beta^* \bar{f}_{z_k} + (z_k^* - \zeta_k^*)\, s \dfrac{\beta^*}{\lambda^*} (\alpha^* \overset{\alpha}{\bar{f}}_{x_k} + \beta^* \overset{\beta}{\bar{f}}_{y_k}) + 2(1-\nu_k)\, c \overset{\beta}{\bar{f}}_{y_k} \right\} d\zeta_k^* \\ \hdashline -\dfrac{1+\nu_k}{1-\nu_k} E^* \gamma_k \dfrac{\beta^*}{\lambda^*} \displaystyle\int_0^{z_k^*} s\bar{\vartheta}_k \, d\zeta_k^* - \dfrac{1+\nu_k}{2(1-\nu_k)} \dfrac{h^*}{E_k^*} \dfrac{\beta^*}{\lambda^*} \displaystyle\int_0^{z_k^*} \left\{ (z_k^* - \zeta_k^*)\, s \bar{f}_{z_k} + \left[(z_k^* - \zeta_k^*)\, c - \dfrac{s}{\lambda^*} \right] \left(\dfrac{\alpha^*}{\lambda^*} \overset{\alpha}{\bar{f}}_{x_k} + \dfrac{\beta^*}{\lambda^*} \overset{\beta}{\bar{f}}_{y_k} \right) + 4(1-\nu_k) \dfrac{s}{\beta^*} \overset{\beta}{\bar{f}}_{y_k} \right\} d\zeta_k^* \\ \hdashline -\dfrac{1+\nu_k}{1-\nu_k} E^* \gamma_k \dfrac{\alpha^*}{\lambda^*} \displaystyle\int_0^{z_k^*} s\bar{\vartheta}_k \, d\zeta_k^* - \dfrac{1+\nu_k}{2(1-\nu_k)} \dfrac{h^*}{E_k^*} \dfrac{\alpha^*}{\lambda^*} \displaystyle\int_0^{z_k^*} \left\{ (z_k^* - \zeta_k^*)\, s \bar{f}_{z_k} + \left[(z_k^* - \zeta_k^*)\, c - \dfrac{s}{\lambda^*} \right] \left(\dfrac{\alpha^*}{\lambda^*} \overset{\alpha}{\bar{f}}_{x_k} + \dfrac{\beta^*}{\lambda^*} \overset{\beta}{\bar{f}}_{y_k} \right) + 4(1-\nu_k) \dfrac{s}{\alpha^*} \overset{\alpha}{\bar{f}}_{x_k} \right\} d\zeta_k^* \\ \hdashline -\dfrac{1+\nu_k}{1-\nu_k} E^* \gamma_k \displaystyle\int_0^{z_k^*} c\bar{\vartheta}_k \, d\zeta_k^* - \dfrac{1+\nu_k}{2(1-\nu_k)} \dfrac{h^*}{E_k^*} \displaystyle\int_0^{z_k^*} \left\{ \left[(z_k^* - \zeta_k^*)\, c - (3-4\nu_k) \dfrac{s}{\lambda^*} \right] \bar{f}_{z_k} + (z_k^* - \zeta_k^*) \dfrac{s}{\lambda^*} (\alpha^* \overset{\alpha}{\bar{f}}_{x_k} + \beta^* \overset{\beta}{\bar{f}}_{y_k}) \right\} d\zeta_k^* \end{bmatrix}$$

In (5.3) and (5.4) we used the abbreviations

$$(5.5)\qquad E_k^* = \frac{E_k}{E^*}, \quad c(z_k^*) = \mathrm{ch}\,(z_k^*\,\lambda^*), \quad t\,(z_k^*) = \mathrm{th}\,(z_k^*\,\lambda^*),$$
$$c = \mathrm{ch}\,[(z_k^* - \zeta_k^*)\,\lambda^*], \quad s = \mathrm{sh}\,[(z_k^* - \zeta_k^*)\,\lambda^*];$$

$\overset{\alpha}{\bar f}_{xk}, \overset{\beta}{\bar f}_{yk}, \bar f_{zk}$ and $\bar\vartheta_k$ mean the Fourier transforms of the corresponding functions. Note the meaning of E_k and ν_k as the operators (2.23).

Taking $z_k = h_k$ in (5.2) we receive the transfer equation

$$(5.6)\qquad \bar{\mathbf{a}}_k(\alpha, \beta, h_k; t) = \bar{\mathbf{T}}_k(\alpha, \beta, h_k; D)\,\bar{\mathbf{a}}_k(\alpha, \beta, 0; t) + \bar{\mathbf{t}}_k(\alpha, \beta, h_k; t, D)$$

with $\bar{\mathbf{T}}_k$ being the (mechanical) transfer matrix of layer k with the known properties (cross symmetry, determinant 1, identical elements with alternating signs for the reciprocal matrix). In (5.4) the transformed temperature field $\bar\vartheta = \bar\vartheta\,(\alpha, \beta, z_k; t)$ has to be taken from (4.7).

Integrating (5.2) with respect to the time under zero initial conditions by means of Laplace transforms gives

$$(5.7)\qquad \bar{\mathbf{a}}_k(\alpha, \beta, z_k; t) = \mathscr{L}^{-1}\,\{\tilde{\bar{\mathbf{T}}}_k(\alpha, \beta, z_k; p)\,\tilde{\bar{\mathbf{a}}}_k(\alpha, \beta, 0; p) + \tilde{\bar{\mathbf{t}}}_k(\alpha, \beta, z_k; p)\}.$$

In the same manner we receive from (5.1)

$$(5.8)\qquad \bar{\mathbf{b}}_k(\alpha, \beta, z_k; t) = \mathscr{L}^{-1}\,\{\tilde{\bar{\mathbf{B}}}_k(\alpha, \beta; p)\,\tilde{\bar{\mathbf{a}}}_k(\alpha, \beta, z_k; p) + \tilde{\bar{\mathbf{e}}}_k(\alpha, \beta, z_k; p)\}.$$

In (5.7) and (5.8) $\tilde{\bar{\mathbf{a}}}_k$ means the Laplace transform of $\bar{\mathbf{a}}_k$ whereas $\tilde{\bar{\mathbf{T}}}_k$ and $\tilde{\bar{\mathbf{B}}}_k$ result from $\bar{\mathbf{T}}_k$ and $\bar{\mathbf{B}}_k$ by taking $D \to p$ in the operators $E_k(D)$ and $\nu_k(D)$. Further we get (the Eq. (5.9) are on page 114)
and

$$(5.10)\qquad \tilde{\bar{\mathbf{e}}}_k(\alpha, \beta, z_k; p) = \left\{-\frac{2E_k}{1-\nu_k}\,\gamma_k\,\tilde{\bar\vartheta}_k(\alpha, \beta, z_k; p), 0, 0\right\}^T$$

where ${}^{\alpha}\tilde{\bar f}_{xk}, {}^{\beta}\tilde{\bar f}_{yk}, \tilde{\bar f}_{zk}, \tilde{\bar\vartheta}_k$ are the Laplace transforms of $\overset{\alpha}{\bar f}_{xk}, \overset{\beta}{\bar f}_{yk}, \bar f_{zk}, \bar\vartheta_k$.

The instationary fields of the stresses and displacements in the layer k now become

$$(5.11)\qquad \mathbf{a}_k(x, y, z_k; t) = \mathbf{F}^{-1}\,\{\bar{\mathbf{a}}_k(\alpha, \beta, z_k; t)\},$$

$$(5.12)\qquad \mathbf{b}_k(x, y, z_k; t) = \mathbf{F}'^{-1}\,\{\bar{\mathbf{b}}_k(\alpha, \beta, z_k; t)\}.$$

The (spatial) mechanical initial state vector $\tilde{\bar{\mathbf{a}}}_k(0) = \tilde{\bar{\mathbf{a}}}_k(\alpha, \beta, 0; p)$ has to be determined by means of the boundary conditions which are assumed to be of the product type, for instance

$$(5.13)\qquad \begin{aligned} \sigma_{zi_k}(x, y, 0; t) &= \sigma_{zi_k}^{(1)}(x, y)\,\sigma_{zi_k}^{(2)}(t), \\ u_{i_k}(x, y, h_k; t) &= u_{ik}^{(1)}(x, y)\,u_{i_k}^{(2)}(t). \end{aligned} \qquad (i = x, y, z),$$

$$
(5.9)\qquad \tilde{\tilde{\mathbf{t}}}_k(\alpha,\beta,z_k;p)=
$$

$$
=\begin{bmatrix}
\dfrac{E_k^*\gamma_k\lambda^*}{1-\nu_k}\displaystyle\int_0^{z_k^*}s\tilde{\tilde{\vartheta}}_k\,d\zeta_k^*-\dfrac{h^*}{2(1-\nu_k)}\displaystyle\int_0^{z_k^*}\left\{[2(1-\nu_k)\,c-\lambda^*(z_k^*-\zeta_k^*)\,s]\,\tilde{\tilde{f}}_{z_k}+\left[(z_k^*-\zeta_k^*)\,c+(1-2\nu_k)\dfrac{s}{\lambda^*}\right](\alpha^*\,{}^{\alpha}\tilde{\tilde{f}}_{x_k}+\beta^*\,{}^{\beta}\tilde{\tilde{f}}_{y_k})\right\}d\zeta_k^* \\
-\dfrac{E_k^*\gamma_k\alpha^*}{1-\nu_k}\displaystyle\int_0^{z_k^*}c\tilde{\tilde{\vartheta}}_k\,d\zeta_k^*-\dfrac{h^*}{2(1-\nu_k)}\displaystyle\int_0^{z_k^*}\left\{\left[(z_k^*-\zeta_k^*)\,c-(1-2\nu_k)\dfrac{s}{\lambda^*}\right]\alpha^*\tilde{\tilde{f}}_{z_k}+(z_k^*-\zeta_k^*)\,s\,\dfrac{\alpha^*}{\lambda^*}(\alpha^*\,{}^{\alpha}\tilde{\tilde{f}}_{x_k}+\beta^*\,{}^{\beta}\tilde{\tilde{f}}_{y_k})+2(1-\nu_k)\,c\,{}^{\alpha}\tilde{\tilde{f}}_{x_k}\right\}d\zeta_k^* \\
-\dfrac{E_k^*\gamma_k\beta^*}{1-\nu_k}\displaystyle\int_0^{z_k^*}c\tilde{\tilde{\vartheta}}_k\,d\zeta_k^*-\dfrac{h^*}{2(1-\nu_k)}\displaystyle\int_0^{z_k^*}\left\{\left[(z_k^*-\zeta_k^*)\,c-(1-2\nu_k)\dfrac{s}{\lambda^*}\right]\beta^*\tilde{\tilde{f}}_{z_k}+(z_k^*-\zeta_k^*)\,s\,\dfrac{\beta^*}{\lambda^*}(\alpha^*\,{}^{\alpha}\tilde{\tilde{f}}_{x_k}+\beta^*\,{}^{\beta}\tilde{\tilde{f}}_{y_k})+2(1-\nu_k)\,c\,{}^{\beta}\tilde{\tilde{f}}_{y_k}\right\}d\zeta_k^* \\
-\dfrac{1+\nu_k}{1-\nu_k}E^*\gamma_k\dfrac{\beta^*}{\lambda^*}\displaystyle\int_0^{z_k^*}s\tilde{\tilde{\vartheta}}_k\,d\zeta_k^*-\dfrac{1+\nu_k}{2(1-\nu_k)}\,\dfrac{h^*}{E_k^*}\,\dfrac{\beta^*}{\lambda^*}\displaystyle\int_0^{z_k^*}\left\{(z_k^*-\zeta_k^*)\,s\tilde{\tilde{f}}_{z_k}+\left[(z_k^*-\zeta_k^*)\,c-\dfrac{s}{\lambda^*}\right]\left(\dfrac{\alpha^*}{\lambda^*}\,{}^{\alpha}\tilde{\tilde{f}}_{x_k}+\dfrac{\beta^*}{\lambda^*}\,{}^{\beta}\tilde{\tilde{f}}_{y_k}\right)+4(1-\nu_k)\dfrac{s}{\beta^*}\,{}^{\beta}\tilde{\tilde{f}}_{y_k}\right\}d\zeta_k^* \\
-\dfrac{1+\nu_k}{1-\nu_k}E^*\gamma_k\dfrac{\alpha^*}{\lambda^*}\displaystyle\int_0^{z_k^*}s\tilde{\tilde{\vartheta}}_k^*\,d\zeta_k^*-\dfrac{1+\nu_k}{2(1-\nu_k)}\,\dfrac{h^*}{E_k^*}\,\dfrac{\alpha^*}{\lambda^*}\displaystyle\int_0^{z_k^*}\left\{(z_k^*-\zeta_k^*)\,s\tilde{\tilde{f}}_{z_k}+\left[(z_k^*-\zeta_k^*)\,c-\dfrac{s}{\lambda^*}\right]\left(\dfrac{\alpha^*}{\lambda^*}\,{}^{\alpha}\tilde{\tilde{f}}_{x_k}+\dfrac{\beta^*}{\lambda^*}\,{}^{\beta}\tilde{\tilde{f}}_{y_k}\right)+4(1-\nu_k)\dfrac{s}{\alpha^*}\,{}^{\alpha}\tilde{\tilde{f}}_{x_k}\right\}d\zeta_k^* \\
-\dfrac{1+\nu_k}{1-\nu_k}E^*\gamma_k\displaystyle\int_0^{z_k^*}c\tilde{\tilde{\vartheta}}_k\,d\zeta_k^*-\dfrac{1+\nu_k}{2(1-\nu_k)}\,\dfrac{h^*}{E_k^*}\displaystyle\int_0^{z_k^*}\left\{\left[(z_k^*-\zeta_k^*)\,c-(3-4\nu_k)\dfrac{s}{\lambda^*}\right]\tilde{\tilde{f}}_{z_k}+(z_k^*-\zeta_k^*)\dfrac{s}{\lambda^*}(\alpha^*\,{}^{\alpha}\tilde{\tilde{f}}_{x_k}+\beta^*\,{}^{\beta}\tilde{\tilde{f}}_{y_k})\right\}d\zeta_k^*
\end{bmatrix}
$$

Transformed:

$$ {}^{\alpha}\tilde{\bar{\sigma}}_{zx_k}(0) = {}^{\alpha}\bar{\sigma}^{(1)}_{zx_k}\ \tilde{\sigma}^{(2)}_{zx_k}, \dots \tag{5.14} $$

Applying the Laplace transform to (5.6) we get — in a shortened notation —

$$ \tilde{\bar{\mathbf{a}}}_k(h_k) = \tilde{\bar{\mathbf{T}}}(h_k)\,\tilde{\bar{\mathbf{a}}}_k(0) + \tilde{\bar{\mathbf{t}}}(h_k)\,, \tag{5.15} $$

which allows the determination of $\tilde{\bar{\mathbf{a}}}_k(0)$ if one takes into account the (transformed) boundary conditions (5.14).

5.2. The layered system

Besides the boundary conditions at the outer surfaces the following conditions of continuity (for the transformed quantities) have to be fulfilled at the inner surfaces:

$$ \tilde{\bar{\mathbf{a}}}_{k+1}(0) = \tilde{\bar{\mathbf{a}}}_k(h_k) = \tilde{\bar{\mathbf{a}}}_k\,, \qquad (k=1, 2, \dots, N-1) \tag{5.16} $$

(continuity of stress and displacement vector).

Analogously to (4.15) we get

$$ \begin{bmatrix} \tilde{\bar{\mathbf{a}}} \\ 1 \end{bmatrix}_k = \begin{bmatrix} \tilde{\bar{\mathbf{T}}} & \tilde{\bar{\mathbf{t}}} \\ 0 & 1 \end{bmatrix}_k \begin{bmatrix} \tilde{\bar{\mathbf{a}}} \\ 1 \end{bmatrix}_{k-1} \qquad (k=1, 2, \dots, N-1), \tag{5.17'} $$

or shortly written

$$ \mathbf{a}^*_k = \mathbf{T}^*_k\, a^*_{k-1}\,, \qquad (k=1, 2, \dots, N-1)\,. \tag{5.17} $$

Also, for a N-layer system there holds

$$ \mathbf{a}^*_N = \mathbf{T}^*_N\,\mathbf{T}^*_{N-1} \dots \mathbf{T}^*_1\,\mathbf{a}^*_0 = \mathbf{T}^*\,\mathbf{a}^*_0\,. \tag{5.18} $$

By means of the boundary conditions 3 components of $\mathbf{a}^*_0$ and 3 components of $\mathbf{a}^*_N$ are prescribed. Then all mechanical initial state vectors may be calculated from the system (5.17′). At last the instationary fields of stresses and displacements follow from (5.7), (5.8), (5.11), (5.12).

For large values of the transform parameters α and β the matrices $\tilde{\bar{\mathbf{T}}}_k$ become singular and so numerical difficulties arise. In this case it is advantageous to use flexibility matrices instead of transfer matrices. For this purpose the Eq. (5.15) is transformed into

$$ \begin{bmatrix} \tilde{\bar{\mathbf{u}}}_k(0) \\ \tilde{\bar{\mathbf{u}}}_k(h_k) \\ 1 \end{bmatrix} = \begin{bmatrix} \tilde{\bar{\mathbf{N}}}_{11_k} & \tilde{\bar{\mathbf{N}}}_{12_k} & \tilde{\bar{\mathbf{n}}}_{1_k} \\ \tilde{\bar{\mathbf{N}}}_{21_k} & \tilde{\bar{\mathbf{N}}}_{22_k} & \tilde{\bar{\mathbf{n}}}_{2_k} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\tilde{\bar{\boldsymbol{\sigma}}}_k(0) \\ \tilde{\bar{\boldsymbol{\sigma}}}_k(h_k) \\ 1 \end{bmatrix} \tag{5.19} $$

(N being the flexibility matrix given already in [23]) or more detailed

$$
(5.19') \qquad
\begin{bmatrix} {}^{\alpha}\tilde{\tilde{u}}_x^*(0) \\ {}^{\beta}\tilde{\tilde{u}}_y^*(0) \\ \tilde{\tilde{u}}_z^*(0) \\ {}^{\alpha}\tilde{\tilde{u}}_x^*(h_k) \\ {}^{\beta}\tilde{\tilde{u}}_y^*(h_k) \\ \tilde{\tilde{u}}_z^*(h_k) \\ \hdashline 1 \end{bmatrix}
= s
\left[\begin{array}{cccccc:c} a & b & -c & d & e & -f & m \\ & g & -h & e & i & -j & n \\ & & k & f & j & l & o \\ & & & a & b & c & p \\ & \text{symmetrical} & & & g & h & q \\ & & & & & k & r \\ \hdashline & & & 0 & & & 1 \end{array}\right]
\begin{bmatrix} -{}^{\alpha}\tilde{\tilde{\sigma}}_{zx}(0) \\ -{}^{\beta}\tilde{\tilde{\sigma}}_{zy}(0) \\ -\tilde{\tilde{\sigma}}_{zz}(0) \\ {}^{\alpha}\tilde{\tilde{\sigma}}_{zx}(h_k) \\ {}^{\beta}\tilde{\tilde{\sigma}}_{zy}(h_k) \\ \tilde{\tilde{\sigma}}_{zz}(h_k) \\ \hdashline 1 \end{bmatrix}
$$

with (here the body forces are assumed to be zero)

$$
(5.20) \qquad
\begin{aligned}
a &= \beta^{*2}\left[t_k^2 - \left(\frac{\lambda^* h_k^*}{c_k}\right)^2\right] + (1-\nu_k)\,\alpha^{*2}\, t_k\left[t_k - \frac{\lambda^* h_k^*}{c_k^2}\right],\\
b &= \alpha^*\beta^*\left\{-\nu_k t_k^2 + \left(\frac{\lambda^* h_k^*}{c_k}\right)^2 - (1-\nu_k)\, t_k \frac{\lambda^* h_k^*}{c_k^2}\right\},\\
c &= \frac{\alpha^*\lambda^* t_k}{2}\left[(1-2\nu_k)\, t_k^2 + \left(\frac{\lambda^* h_k^*}{c_k}\right)^2\right],\\
d &= (1-\nu_k)\,\alpha^{*2}\frac{t_k}{c_k}(t_k - \lambda^* h_k^*) + \frac{\beta^{*2}}{c_k}\left[t_k^2 - \left(\frac{\lambda^* h_k^*}{c_k}\right)^2\right],\\
e &= -\frac{\alpha^*\beta^*}{c_k}\left\{(1-\nu_k)\,\lambda^* h_k^* t_k + \nu_k t_k^2 - \left(\frac{\lambda^* h_k^*}{c_k}\right)^2\right\},\\
f &= (1-\nu_k)\,\alpha^*\lambda^{*2} h_k^* \frac{t_k^2}{c_k},\\
g &= \alpha^{*2}\left[t_k^2 - \left(\frac{\lambda^* h_k^*}{c_k}\right)^2\right] + (1-\nu_k)\,\beta^{*2}\, t_k\left[t_k - \frac{\lambda^* h_k^*}{c_k^2}\right],\\
h &= \frac{\beta^*\lambda^* t_k}{2}\left[(1-2\nu_k)\, t_k^2 - \left(\frac{\lambda^* h_k^*}{c_k}\right)^2\right],\\
i &= (1-\nu_k)\,\beta^{*2}\frac{t_k}{c_k}(t_k - \lambda^* h_k^*) + \frac{\alpha^{*2}}{c_k}\left[t_k^2 - \left(\frac{\lambda^* h_k^*}{c_k}\right)^2\right],\\
j &= (1-\nu_k)\,\beta^*\lambda^{*2} h_k^* \frac{t_k^2}{c_k},\\
k &= (1-\nu_k)\,\lambda^{*2} t_k\left(t_k + \frac{\lambda^* h_k^*}{c_k^2}\right),\\
l &= (1-\nu_k)\,\lambda^{*2}\frac{t_k}{c_k}(t_k + \lambda^* h_k^*),
\end{aligned}
$$

(5.20) [cont.]
$$m=-E_k\,\gamma_k\,\alpha^*\,\lambda^{*2}\,t_k\int_0^{h_k^*}\left\{\left[\frac{\lambda^*\,h_k^*}{c_k^2}-t_k\right]c(z_k^*)+t_k^2\,s(z_k^*)\right\}\tilde{\bar{\vartheta}}_k\,dz_k^*,$$

$$n=-E_k\,\gamma_k\,\beta^*\,\lambda^{*2}\,t_k\int_0^{h_k^*}\left\{\left[\frac{\lambda^*\,h_k^*}{c_k^2}-t_k\right]c(z_k^*)+t_k^2\,s(z_k^*)\right\}\tilde{\bar{\vartheta}}_k\,dz_k^*,$$

$$o=E_k\,\gamma_k\,\lambda^{*3}\,t_k\int_0^{h_k^*}\left\{\left[\frac{\lambda^*\,h_k^*}{c_k^2}+t_k\right]s(z_k^*)-t_k^2\,c(z_k^*)\right\}\tilde{\bar{\vartheta}}_k\,dz_k^*,$$

$$p=E_k\,\gamma_k\,\alpha^*\,\lambda^{*2}\,\frac{t_k}{c_k}\int_0^{h_k^*}\left\{[t_k-\lambda^*]\,c(z_k^*)+\lambda^*\,t_k\,s(z_k^*)\right\}\tilde{\bar{\vartheta}}_k\,dz_k^*,$$

$$q=E_k\,\gamma_k\,\beta^*\,\lambda^{*2}\,\frac{t_k}{c_k}\int_0^{h_k^*}\{[t_k-\lambda^*]\,c(z_k^*)+\lambda^*\,t_k\,s(z_k^*)\}\,\tilde{\bar{\vartheta}}_k\,dz_k^*,$$

$$r=E_k\,\gamma_k\,\lambda^{*3}\,\frac{t_k}{c_k}\int_0^{h_k^*}\{[t_k+\lambda^*]\,s(z_k^*)-\lambda^*\,t_k\,c(z_k^*)\}\,\tilde{\bar{\vartheta}}_k\,dz_k^*,$$

$$s=\frac{2(1+\nu_k)}{E_k^*\,\lambda^{*3}\,t_k\left[t_k^2-\left(\frac{\lambda^*\,h_k^*}{c_k}\right)^2\right]},$$

where $t_k=t(h_k^*)$, $c_k=c(h_k^*)$.

Now the conditions of continuity read

(5.21)
$$\begin{aligned}\tilde{\bar{\mathbf{u}}}_k(h_k)&=\tilde{\bar{\mathbf{u}}}_{k+1}(0)=\tilde{\bar{\mathbf{u}}}_k,\\ \tilde{\bar{\boldsymbol{\sigma}}}_k(h_k)&=\tilde{\bar{\boldsymbol{\sigma}}}_{k+1}(0)=\tilde{\bar{\boldsymbol{\sigma}}}_k,\end{aligned}\quad(k=1,\ldots,N-1).$$

Using (5.19) there results

(5.22) $$\tilde{\bar{\mathbf{N}}}_{21_k}\,\tilde{\bar{\boldsymbol{\sigma}}}_{k-1}-(\tilde{\bar{\mathbf{N}}}_{22_k}+\tilde{\bar{\mathbf{N}}}_{11_{k+1}})\,\tilde{\bar{\boldsymbol{\sigma}}}_k+\tilde{\bar{\mathbf{N}}}_{12_{k+1}}\,\tilde{\bar{\boldsymbol{\sigma}}}_{k+1}=\tilde{\bar{\mathbf{n}}}_{2_k}-\tilde{\bar{\mathbf{n}}}_{1_{k+1}},\quad(k=1,\ldots,N-1)$$

with the (transformed) stress components as unknowns. Having evaluated these ones the (transformed) displacements follow from (5.19) and so the mechanical initial state vectors $\tilde{\bar{\mathbf{a}}}_k$ are known.

5.3. The instationary elastic case

Here the Fourier transform of the temperature $\bar{\vartheta}_k$ (depending on the time) has to be determined via (4.7). It is needed in the expressions for $\bar{\mathbf{e}}_k$ and $\bar{\mathbf{t}}_k$ where the time operators disappear because of the assumed elastic material. For the same reason the matrices $\bar{\mathbf{B}}_k$ and $\bar{\mathbf{T}}_k$ are free of the time derivatives. Therefore the instationary fields of stresses and displacements follow directly by taking the inverse Fourier transforms of (5.2) and (5.1) in the form

(5.23) $$\bar{\mathbf{a}}_k(\alpha,\beta,z_k;t)=\bar{\mathbf{T}}_k(\alpha,\beta,z_k)\,\bar{\mathbf{a}}_{k-1}(\alpha,\beta;t)+\bar{\mathbf{t}}_k(\alpha,\beta,z_k;t),$$

(5.24) $$\bar{\mathbf{b}}_k(\alpha,\beta,z_k;t)=\bar{\mathbf{B}}_k(\alpha,\beta)\,\bar{\mathbf{a}}_k(\alpha,\beta,z_k;t)+\bar{\mathbf{e}}_k(\alpha,\beta,z_k;t)$$

with

$$\bar{\mathbf{e}}_k(\alpha, \beta, z_k; t) = \left\{ -\frac{2E_k}{1-\nu_k} \gamma_k \bar{\vartheta}_k(\alpha, \beta, z_k; t),\ 0,\ 0 \right\}^T. \tag{5.25}$$

The initial state vectors have to be determined now from

$$\begin{bmatrix} \bar{\mathbf{a}} \\ 1 \end{bmatrix}_k = \begin{bmatrix} \bar{\mathbf{T}} & \bar{\mathbf{t}} \\ 0 & 1 \end{bmatrix}_k \begin{bmatrix} \bar{\mathbf{a}} \\ 1 \end{bmatrix}_{k-1}, \quad (k=1, 2, ..., N) \tag{5.26}$$

[instead of (5.17)] or alternatively from

$$\bar{\mathbf{N}}_{21_k} \bar{\boldsymbol{\sigma}}_{k-1} - (\bar{\mathbf{N}}_{22_k} + \bar{\mathbf{N}}_{11_{k+1}}) \bar{\boldsymbol{\sigma}}_k + \bar{\mathbf{N}}_{12_{k+1}} \bar{\boldsymbol{\sigma}}_{k+1} = \bar{\mathbf{n}}_{2_k} - \bar{\mathbf{n}}_{1_{k+1}}, \quad (k=1, 2, ..., N-1) \tag{5.27}$$

[instead of (5.22)] whereby the boundary conditions may depend on the time.

5.4. The stationary elastic case

Providing that all boundary conditions do not depend on the time, setting the time derivatives in $\bar{\mathbf{W}}_k$ and $\bar{\mathbf{w}}_k$ equal to zero and neglecting — for the matter of simplicity — the heat sources, we get from (4.2), (4.19) and (4.21)

$$\bar{q}^*_{z_k}(\alpha, \beta, z_k) = -\kappa^*_k \lambda^* \operatorname{sh}(z^*_k \lambda^*) \bar{\vartheta}_{k-1} + \operatorname{ch}(z^*_k \lambda^*) \bar{q}^*_{z_{k-1}}, \tag{5.28}$$

$$\bar{\vartheta}_k(\alpha, \beta, z_k) = \operatorname{ch}(z^*_k \lambda^*) \bar{\vartheta}_{k-1} - \frac{1}{\kappa^*_k \lambda^*} \operatorname{sh}(z^*_k \lambda^*) \bar{q}^*_{z_{k-1}}. \tag{5.29}$$

With (5.29) we are able to perform the integrations in (5.4). Neglecting also the body forces there, we receive

$$\bar{\mathbf{t}}_k(\alpha, \beta, z_k) = \frac{\gamma_k z^*_k c(z^*_k)}{2(1-\nu_k)} \times \tag{5.30}$$

$$\times \begin{bmatrix} E_k \lambda^* t(z^*_k) & -\frac{E_k}{\kappa^*_k}\left\{1 - \frac{t(z^*_k)}{z^*_k \lambda^*}\right\} \\ -E_k \alpha^* \left\{1 + \frac{t(z^*_k)}{z^*_k \lambda^*}\right\} & \frac{E_k}{\kappa^*_k} \frac{\alpha^*}{\lambda^*} t(z^*_k) \\ -E_k \beta^* \left\{1 + \frac{t(z^*_k)}{z^*_k \lambda^*}\right\} & \frac{E_k}{\kappa^*_k} \frac{\beta^*}{\lambda^*} t(z^*_k) \\ -(1+\nu_k) \frac{\beta^*}{\lambda^*} t(z^*_k) & \frac{1+\nu_k}{\kappa^*_k} \frac{\beta^*}{\lambda^{*2}} \left\{1 - \frac{t(z^*_k)}{z^*_k \lambda^*}\right\} \\ -(1+\nu_k) \frac{\alpha^*}{\lambda^*} t(z^*_k) & \frac{1+\nu_k}{\kappa^*_k} \frac{\alpha^*}{\lambda^*} \left\{1 - \frac{t(z^*_k)}{z^*_k \lambda^*}\right\} \\ -(1+\nu_k) \left\{1 + \frac{t(z^*_k)}{z^*_k \lambda^*}\right\} & \frac{1+\nu_k}{\kappa^*_k} z^*_k \frac{t(z^*_k)}{z^*_k \lambda^*}. \end{bmatrix} \begin{bmatrix} \bar{\vartheta}_k(\alpha, \beta, 0) \\ \bar{q}^*_{z_k}(\alpha, \beta, 0) \end{bmatrix}$$

with $c(z^*_k) = \operatorname{ch}(\lambda^* z^*_k)$, $\quad t(z^*_k) = \operatorname{th}(\lambda^* z^*_k)$.

Also, from (5.25) with (5.29) we get

$$(5.31)\qquad \bar{\mathbf{e}}_k(\alpha,\beta,z_k)=\left\{-\frac{2E_k\,\gamma_k}{1-\nu_k}\,c(z_k^*)\left[\bar{\vartheta}_{k-1}-\frac{1}{\kappa_k^*\,\lambda^*}\,t(z_k^*)\,\bar{q}^*_{z_{k-1}}\right],0,0\right\}^T;$$

the stresses and displacements now following via (5.23), (5.24), (5.26) or (5.27).

6. The two-dimensional problem

If the heat source Q, the body forces f_i and the boundary conditions do not depend on the coordinate y, we have a plane problem. Also, if these quantities are only dependent on the cylindrical coordinates r and z (i.e. independent of the circumferential coordinate φ), we have an axisymmetrical problem. These two cases allow a common description [32, 22]. It seems worth-while to summarize the results, the left column concerning the plane and the right one the axisymmetrical problem respectively.

6.1. Basic equations

Heat conduction: The quantities needed in (2.3) and (2.4) are

$$(6.1)\qquad \mathbf{r}=\{\vartheta,q_z^*\}^T,$$

$$(6.2)\qquad \mathbf{s}=\left\{0,\frac{h^*}{\kappa^*}\,Q\right\}^T,$$

$$(6.3)\qquad \mathbf{p}=\{q_x^*\},\qquad\Big|\qquad \mathbf{p}=\{q_r^*\},$$

$$(6.4)\qquad \begin{array}{l|l} D_1=\dfrac{\partial}{\partial x}, & D_1=\dfrac{\partial}{\partial r},\\[2ex] D_2=\dfrac{\partial}{\partial x}, & D_2=\dfrac{\partial}{\partial r}+\dfrac{1}{r},\\[2ex] D_3=\dfrac{\partial}{\partial x}, & D_3=\dfrac{\partial}{\partial r}-\dfrac{1}{r},\\[2ex] D_4=\dfrac{\partial^2}{\partial x^2}, & D_4=\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\,\dfrac{\partial}{\partial r}-\dfrac{1}{r^2}, \end{array}$$

$$D=\frac{\partial}{\partial t},$$

$$(6.5)\qquad \mathbf{R}=\left[\begin{array}{c|c} 0 & -\dfrac{\kappa^*}{\kappa h^*}\\ \hline \dfrac{\kappa}{\kappa^*}\,h^*\left(D_2\,D_1-\dfrac{1}{a}\,D\right) & 0\end{array}\right],$$

$$(6.6)\qquad \mathbf{P}=\left[-\frac{\kappa h^*}{\kappa^*}\,D_1,\ 0\right].$$

Thermoelasticity and thermoviscoelasticity: The quantities occurring in (2.14) and (2.15) are

(6.7) $\mathbf{a}=\{\sigma_{zz}, \sigma_{zx}, u_x^*, u_z^*\}^T$ | $\mathbf{a}=\{\sigma_{zz}, \sigma_{zr}, u_r^*, u_z^*\}^T$

(6.8) $\mathbf{b}=\{\sigma_{xx}+\sigma_{yy}, \sigma_{xx}-\sigma_{yy}\}^T$ | $\mathbf{b}=\{\sigma_{rr}+\sigma_{\varphi\varphi}, \sigma_{rr}-\sigma_{\varphi\varphi}\}^T$

(6.9) $\mathbf{d}=\left\{-f_z, -f_x+\frac{E\gamma}{1-\nu}D_1\vartheta, 0, \frac{1+\nu}{1-\nu}\frac{E^*}{h^*}\gamma\vartheta\right\}^T$ | $\mathbf{d}=\left\{-f_z, -f_r+\frac{E\gamma}{1-\nu}D_1\vartheta, 0, \frac{1+\nu}{1-\nu}\frac{E^*}{h^*}\gamma\vartheta\right\}^T,$

$$\mathbf{e}=\left\{-\frac{2E}{1-\nu}\gamma\vartheta, 0\right\}^T, \tag{6.10}$$

$$\mathbf{A}=\begin{bmatrix} 0 & -D_2 & 0 & 0 \\ -\frac{\nu}{1-\nu}D_1 & 0 & -\frac{h^*}{E^*}\frac{E}{1-\nu^2}D_4 & 0 \\ 0 & \frac{E^*}{h^*}\frac{2(1+\nu)}{E} & 0 & -D_1 \\ \frac{E^*}{h^*E}\frac{(1+\nu)(1-2\nu)}{1-\nu} & 0 & -\frac{\nu}{1-\nu}D_2 & 0 \end{bmatrix}, \tag{6.11}$$

$$\mathbf{B}=\begin{bmatrix} \frac{2\nu}{1-\nu} & 0 & \frac{h^*}{1-\nu}\frac{E}{E^*}D_2 & 0 \\ 0 & 0 & \frac{h^*}{1+\nu}\frac{E}{E^*}D_3 & 0 \end{bmatrix}. \tag{6.12}$$

We assumed $u_y=0$ (plane strain) and $u_\varphi=0$ (axisymmetrical problem without torsion). The formulas for the state of plane stress are obtained by the substitutions

$$E\to\frac{1+2\nu}{(1+\nu)^2}E, \quad \nu\to\frac{\nu}{1+\nu}, \quad \gamma\to\frac{1+\nu}{1+2\nu}\gamma. \tag{6.13}$$

6.2. *Integral transforms*

Heat conduction: The operator matrices $\mathbf{G}$ and $\mathbf{G}'$ defining the transforms (3.11) and (3.12) are here

(6.14) $\mathbf{G}=\mathrm{diag}\,\{\mathscr{F}_0, \mathscr{F}_0\},$ | $\mathbf{G}=\mathrm{diag}\,\{\mathscr{H}_0, \mathscr{H}_0\},$

(6.15) $\mathbf{G}'=\mathscr{F}_1,$ | $\mathbf{G}'=\mathscr{H}_1$

(one-dimensional Fourier transforms) | (Hankel transforms)

where the operations $\mathscr{F}_0$, $\mathscr{F}_1$, $\mathscr{H}_0$, $\mathscr{H}_1$ are given in (A5) (Appendix). The matrices $\overline{\mathbf{R}}$ and $\overline{\mathbf{P}}$ in (3.9) and (3.10) — calculated with consideration of the rules (A7) — turn out to be identical for the plane and the axisymmetrical state. They follow from (3.15) and (3.16) by taking formally $\beta^*=0$ (i.e. $\alpha^*=\lambda^*$). Therefore we receive also $\overline{\mathbf{W}}_k$ and $\overline{\mathbf{w}}_k$ from (4.3) and (4.4) respectively by means of the same manipulation. The further procedure now follows that described in chapters 4.1, 4.2 and 4.3.

Thermoelasticity and thermoviscoelasticity: In the two-dimensional case the operator matrices $\mathbf{F}$ and $\mathbf{F}'$ in (3.23) and (3.24) become [see Appendix (A5)]:

(6.16)	$\mathbf{F}=\mathrm{diag}\,\{\mathscr{F}_0, \mathscr{F}_1, \mathscr{F}_1, -\mathscr{F}_0\}$,	$\mathbf{F}=\mathrm{diag}\,\{\mathscr{H}_0, \mathscr{H}_1, \mathscr{H}_1, -\mathscr{H}_0\}$,
(6.17)	$\mathbf{F}'=\mathrm{diag}\,\{\mathscr{F}_0, \mathscr{F}_2\}$,	$\mathbf{F}'=\mathrm{diag}\,\{\mathscr{H}_0, \mathscr{H}_2\}$
	(one-dimensional Fourier transforms)	(Hankel transforms)

The matrices $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ in (3.21) and (3.22) had been calculated using the rules (A7) of the Appendix. They are identical for the state of plane strain and the state of axisymmetrical strain and result from (3.27) and (3.28) by taking $\beta^*=0$ (i.e. $\alpha^*=\lambda^*$) and contracting the matrices owing to the reduced vectors (6.7), (6.8):

$$(6.18)\quad \overline{\mathbf{A}}=\frac{1}{h^*}\begin{bmatrix} 0 & -\lambda^* & 0 & 0 \\ \frac{\nu}{1-\nu}\lambda^* & 0 & \frac{1}{1-\nu^2}\frac{E}{E^*}\lambda^{*2} & \\ 0 & 2(1+\nu)\frac{E^*}{E} & & \\ -\frac{(1+\nu)(1-2\nu)}{1-\nu}\frac{E^*}{E} & & \text{symmetrical} & \end{bmatrix},$$

$$(6.19)\quad \overline{\mathbf{B}}=\begin{bmatrix} \frac{2\nu}{1-\nu} & 0 & \frac{1}{1-\nu}\frac{E}{E^*}\lambda^* & 0 \\ 0 & 0 & -\frac{1}{1+\nu}\frac{E}{E^*}\lambda^* & 0 \end{bmatrix}.$$

Also the matrix $\overline{\mathbf{T}}_k$ given in (5.3) is now reduced to

$$(6.20)\quad \overline{\mathbf{T}}_k(\lambda, z_k; D)=\frac{c(z_k^*)}{2(1-\nu_k)}\times$$

$$\times\begin{bmatrix} 2(1-\nu_k)-\lambda^* z_k^*\,t(z_k^*) & -[\lambda^* z_k^*+(1-2\nu_k)\,t(z_k^*)] & -\frac{E_k^*}{1+\nu_k}\times\lambda^{*2} z_k^*\,t(z_k^*) & \frac{E_k^*\lambda^*}{1+\nu_k}\times[\lambda^* z_k^*-t(z_k^*)] \\ \lambda^* z_k^*-(1-2\nu_k)\,t(z_k^*) & 2(1-\nu_k)+\lambda^* z_k^*\,t(z_k^*) & \frac{E_k^*\lambda^*}{1+\nu_k}\times[\lambda^* z_k^*+t(z_k^*)] & \\ \frac{1+\nu_k}{E_k^*}z_k^*\,t(z_k^*) & \frac{1+\nu_k}{E_k^*\lambda^*}[\lambda^* z_k^*+(3-4\nu_k)\,t(z_k^*)] & & \\ \frac{1+\nu_k}{E_k^*\lambda^*}[z_k^*\lambda^*-(3-4\nu_k)\,t(z_k^*)] & & \text{symmetrical} & \end{bmatrix}$$

and the vector $\overline{\mathbf{t}}_k$ given in (5.4) becomes

$$(6.21)\quad \bar{\mathfrak{t}}_k(\lambda, z_k; t, D) = $$

$$= \begin{bmatrix} \dfrac{E_k^* \gamma_k \lambda^*}{1-\nu_k}\displaystyle\int_0^{z_k^*} s\overset{0}{\bar\vartheta}_k \, d\zeta_k^* - \dfrac{h^*}{2(1-\nu_k)}\displaystyle\int_0^{z_k^*}\{[2(1-\nu_k)c-\lambda^*(z_k^*-\zeta_k^*)s]\overset{0}{\bar f}_{z_k} + [\lambda^*(z_k^*-\zeta_k^*)c+(1-2\nu_k)s]\overset{1}{\bar f}_{x_k}\}\, d\zeta_k^* \\ -\dfrac{E_k^*\gamma_k\lambda^*}{1-\nu_k}\displaystyle\int_0^{z_k^*} c\overset{0}{\bar\vartheta}_k \, d\zeta_k^* - \dfrac{h^*}{2(1-\nu_k)}\displaystyle\int_0^{z_k^*}\{[\lambda^*(z_k^*-\zeta_k^*)c-(1-2\nu_k)s]\overset{0}{\bar f}_{z_k}+[2(1-\nu_k)c+\lambda^*(z_k^*-\zeta_k^*)s]\overset{1}{\bar f}_{x_k}\}\, d\zeta_k^* \\ -\dfrac{1+\nu_k}{1-\nu_k}E^*\gamma_k\displaystyle\int_0^{z_k^*} s\overset{0}{\bar\vartheta}_k \, d\zeta_k^* - \dfrac{1+\nu_k}{2(1-\nu_k)}\dfrac{h^*}{E_k^*}\displaystyle\int_0^{z_k^*}\left\{(z_k^*-\zeta_k^*)s\overset{0}{\bar f}_{z_k} + \left[(z_k^*-\zeta_k^*)c+(3-4\nu_k)\dfrac{s}{\lambda^*}\right]\overset{1}{\bar f}_{x_k}\right\} d\zeta_k^* \\ -\dfrac{1+\nu_k}{1-\nu_k}E^*\gamma_k\displaystyle\int_0^{z_k^*} c\overset{0}{\bar\vartheta}_k \, d\zeta_k^* - \dfrac{1+\nu_k}{2(1-\nu_k)}\dfrac{h^*}{E_k^*}\displaystyle\int_0^{z_k^*}\left\{\left[(z_k^*-\zeta_k^*)c-(3-4\nu_k)\dfrac{s}{\lambda^*}\right]\overset{0}{\bar f}_{z_k}+(z_k^*-\zeta_k^*)s\overset{1}{\bar f}_{x_k}\right\} d\zeta_k^* \end{bmatrix}$$

where $\overset{1}{f_x}$ has to be replaced by $\overset{1}{f_r}$ in the axisymmetrical case. The further procedure takes place in the way described in chapters 5.1, 5.2 and 5.3.

7. Concluding remarks and examples

We arrive at the more technical theories of heat transfer and thermal stresses in sandwich plates by considering the case of thin layers characterized by $(\lambda^* z_k^*)^2 \ll 1$ (i.e. the wave length of "loading" has to be large compared to the thickness of the

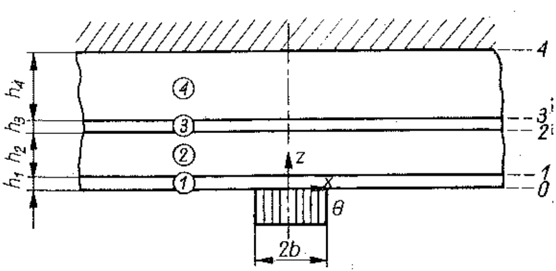


Fig. 2.

layer). Then only the first members of the series (3.19) and (3.30) are relevant leading to considerable simplifications [23]. The same statement holds for the operator matrix (3.2) and the corresponding one in the stress problem.

A computer program has been written for the evaluation of the basic matrices and the thermal and mechanical initial state vectors, for the numerical inversion of the Laplace transforms according to the method of Papoulis and for the numerical inversion of the Fourier and Hankel transforms, respectively using Simpson's integral formula. As an example we considered the plane problem of an elastic four-layer plate shown in Fig. 2 without heat sources and body forces under the boundary conditions

$$\vartheta_0=\vartheta_0(x, z_1=0; t)=\vartheta^{(1)}_{(x)} H(t), \qquad \vartheta^{(1)}_{(x)}=\begin{cases}\theta_0 \\ 0\end{cases} \quad \text{for} \quad |x| \lessgtr b,$$

$$\vartheta_4=\vartheta_4(x; z_4=h_4; t)=0, \qquad \sigma_{zz0}=\sigma_{zz0}(x, z_1=0; t)=0,$$

$$\sigma_{zx0}=\sigma_{zx0}(x, z_1=0; t)=0, \qquad u_{x4}=u_{x4}(x; z_4=h_4; t)=0,$$

$$u_{z4}=u_{z4}(x, z_4=h_4; t)=0,$$

where $H(t)$ means Heaviside's step function. Particularly we chose the following data (Table 1)

Table 1

Layer	E	ν	h	κ	γ	$a=\frac{\kappa}{\rho c}$
	N/cm^2	—	cm	N/sK	1/K	cm^2/s
① Titan alloy	10^7	0.3	0.2	7.5	10^{-5}	0.033
② Honeycomb-core	10^6	0	0.8	2.0	$1.5\cdot10^{-5}$	0.10
③ Titan alloy	10^7	0.3	0.2	7.5	10^{-5}	0.033
④ Isolat. mat.	10^5	0.2	1.2	0.2	$7.5\cdot10^{-5}$	0.002

$2b=h_4$

Temperature, heat flux, stresses and displacements have been computed along all discontinuity surfaces 0, 1, 2, 3, 4 at various timings. Some of the results are shown in Figs. 3 to 10.

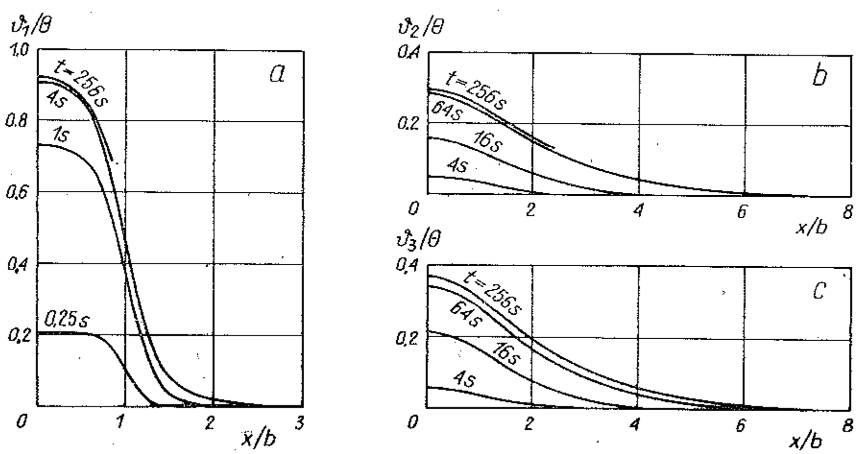


Fig. 3.

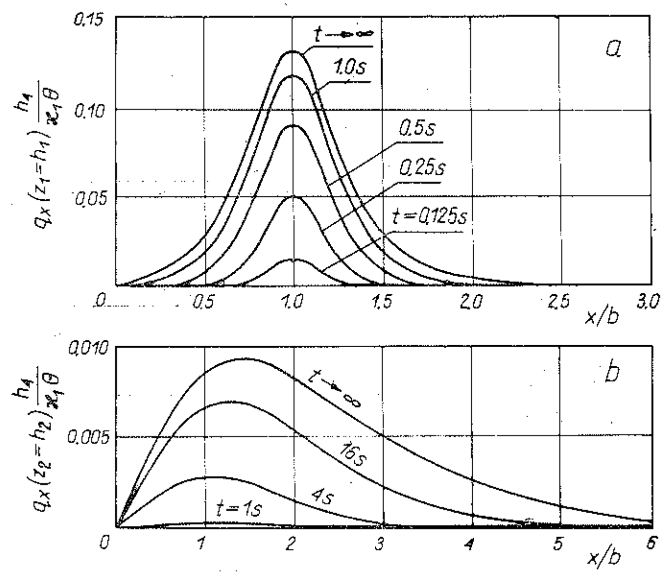

a
b
0.15
0.10
0.05
0
0.5
1.0
1.5
2.0
2.5
3.0
x/b
t → ∞
1.0s
0.5s
0.25s
t=0.125s
0.010
0.005
1
2
3
4
5
6
16s
4s
t=1s

Fig. 4.

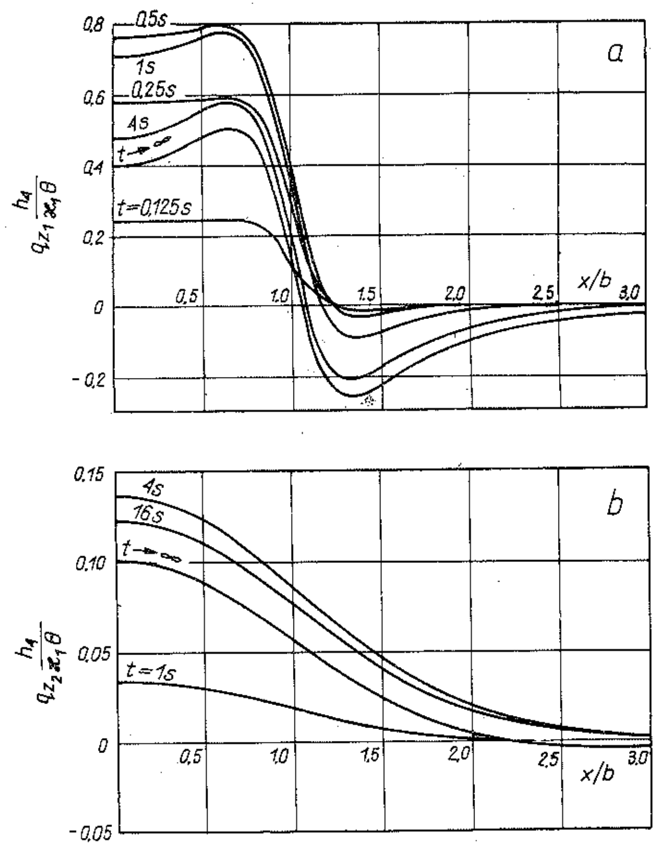
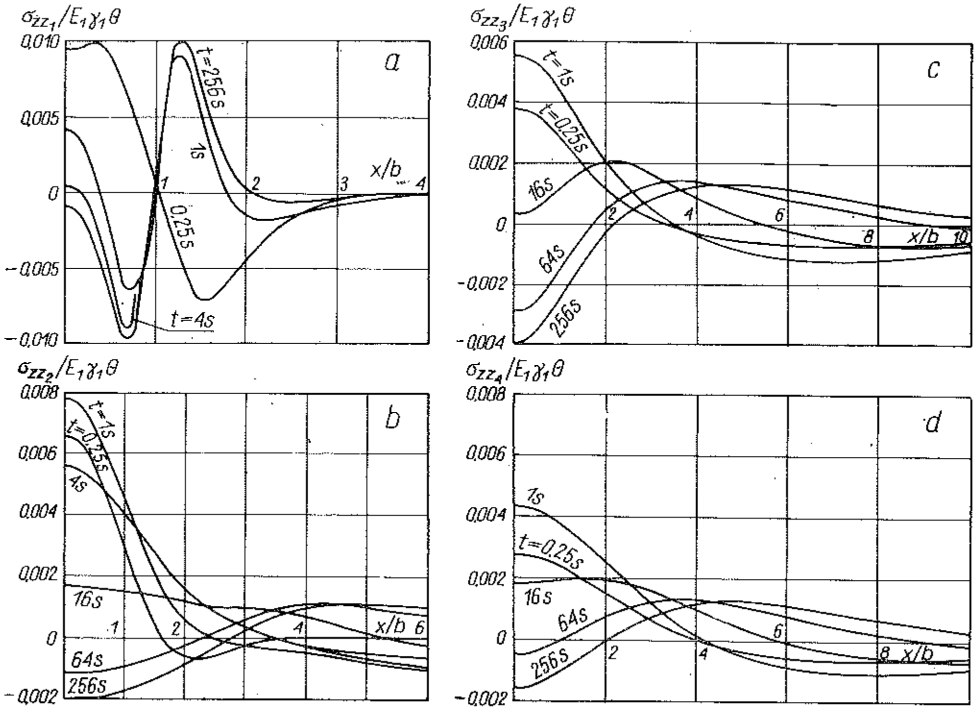
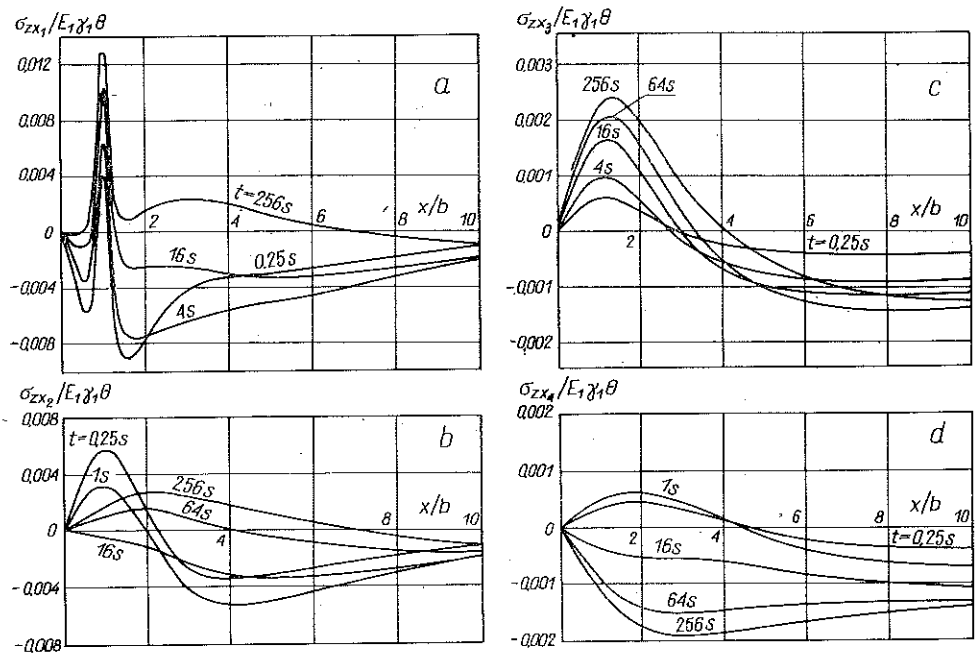

a
b
0.8
0.6
0.4
0.2
0
-0.2
0.5s
1s
0.25s
4s
t → ∞
t=0.125s
0.5
1.0
1.5
2.0
2.5
3.0
x/b
0.15
0.10
0.05
-0.05
4s
16s
t=1s

Fig. 5.



$\sigma_{zz_1}/E_1\gamma_1\theta$
a
t=256s
1s
0,25s
t=4s
x/b
$\sigma_{zz_3}/E_1\gamma_1\theta$
c
t=1s
t=0,25s
16s
64s
256s
x/b
$\sigma_{zz_2}/E_1\gamma_1\theta$
b
t=1s
t=0,25s
4s
16s
64s
256s
x/b
$\sigma_{zz_4}/E_1\gamma_1\theta$
d
1s
t=0,25s
16s
64s
256s
x/b

Fig. 6.



$\sigma_{zx_1}/E_1\gamma_1\theta$
a
t=256s
16s
0,25s
4s
x/b
$\sigma_{zx_3}/E_1\gamma_1\theta$
c
256s
64s
16s
4s
t=0,25s
x/b
$\sigma_{zx_2}/E_1\gamma_1\theta$
b
t=0,25s
1s
256s
64s
16s
x/b
$\sigma_{zx_4}/E_1\gamma_1\theta$
d
1s
16s
t=0,25s
64s
256s
x/b

Fig. 7.

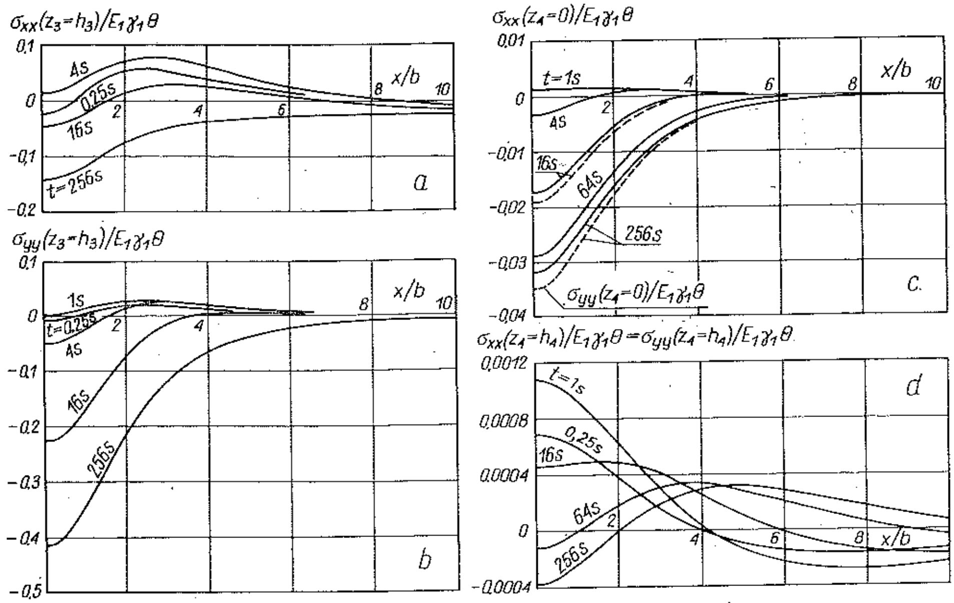

σxx(z3=h3)/E1γ1θ
σyy(z3=h3)/E1γ1θ
σxx(z4=0)/E1γ1θ
σyy(z4=0)/E1γ1θ
σxx(z4=h4)/E1γ1θ=σyy(z4=h4)/E1γ1θ
x/b
t=256s
t=1s
t=0.25s
16s
64s
a
b
c
d

Fig. 8.

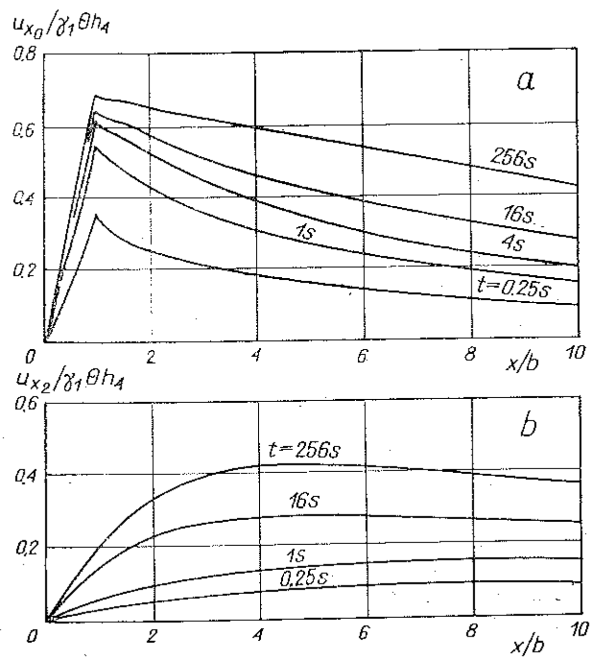

ux0/γ1θh4
ux2/γ1θh4
x/b
256s
16s
4s
1s
t=0.25s
t=256s
0.25s
a
b

Fig. 9.

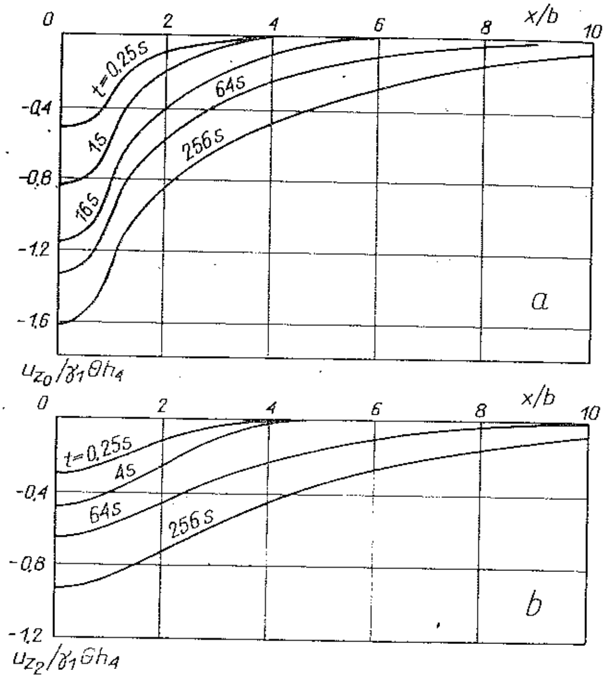


Fig. 10

The heat flux q_x and the stress components σ_{xx} and σ_{yy}, of course, are discontinuous from one layer to the other because of the discontinuity of the material properties.

Appendix

Here we list the definitions of the integral transforms which are needed in this paper.

The *two-dimensional Fourier transforms* of the function $f(x, y)$ are [29, 23]:

$$\text{(A1)}\quad \begin{aligned} \bar{f}(\alpha, \beta) &= \mathscr{F}\{f(x, y)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, e^{i(\alpha x + \beta y)}\, dx\, dy, \\ \overset{\alpha}{\bar{f}}(\alpha, \beta) &= \mathscr{F}_\alpha\{f(x, y)\} = \frac{1}{j_\alpha}\mathscr{F}\{f(x, y)\}, \\ \overset{\beta}{\bar{f}}(\alpha, \beta) &= \mathscr{F}_\beta\{f(x, y)\} = \frac{1}{j_\beta}\mathscr{F}\{f(x, y)\}, \end{aligned}$$

where

$$\text{(A2)}\quad \begin{aligned} &j_\alpha = i\frac{\alpha}{|\alpha|} \quad (\alpha \neq 0), \qquad j_\beta = i\frac{\beta}{|\beta|} \quad (\beta \neq 0), \\ &j_\alpha = i \qquad (\alpha = 0), \qquad j_\beta = i \qquad (\beta = 0), \\ &i = \sqrt{-1} \end{aligned}$$

and α, β denote the real transform parameters.

The inverse transforms are

$$\text{(A3)}\qquad f(x,y)=\begin{bmatrix}\mathscr{F}^{-1}\{\bar{f}(\alpha,\beta)\}\\ \mathscr{F}_\alpha^{-1}\{\overset{\alpha}{\bar{f}}(\alpha,\beta)\}\\ \mathscr{F}_\beta^{-1}\{\overset{\beta}{\bar{f}}(\alpha,\beta)\}\end{bmatrix}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{bmatrix}\bar{f}(\alpha,\beta)\\ j_\alpha\overset{\alpha}{\bar{f}}(\alpha,\beta)\\ j_\beta\overset{\beta}{\bar{f}}(\alpha,\beta)\end{bmatrix}e^{-i(\alpha x+\beta y)}\,d\alpha\,d\beta,$$

if $f(x,y)$ satisfies Dirichlet's conditions for $-\infty<(x,y)<\infty$ and if the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dx\,dy$$

is absolutely convergent.

For the derivatives there hold the relations

$$\overline{\frac{\partial f}{\partial x}}=|\alpha|\overset{\alpha}{\bar{f}}=-j_\alpha j_\beta|\alpha|\overset{\beta}{\bar{f}};\quad \overset{\alpha}{\overline{\frac{\partial f}{\partial x}}}=-|\alpha|\bar{f};\qquad \text{if } f(x,y)\to 0 \quad \text{for}\quad |x|\to\infty,$$

$$\text{(A4)}\qquad \overset{\alpha}{\overline{\frac{\partial^2 f}{\partial x^2}}}=-\alpha^2\overset{\alpha}{\bar{f}};\qquad \overset{\beta}{\overline{\frac{\partial^2 f}{\partial x^2}}}=-\alpha^2\overset{\beta}{\bar{f}};\qquad \text{if}\left.\begin{array}{r}f(x,y)\to 0\\ \dfrac{\partial f}{\partial x}\to 0\end{array}\right\}\quad\text{for}\quad |x|\to\infty,$$

$$\overset{\alpha}{\overline{\frac{\partial^2 f}{\partial x\,\partial y}}}=-|\alpha|\,|\beta|\overset{\beta}{\bar{f}}\qquad \text{if } f(x,y)\to 0\quad\text{for}\quad\begin{cases}|x|\to\infty,\\ |y|\to\infty,\end{cases}$$

and further relations if x and α are replaced by y and β and vice versa.

For the plane problem we use the *one-dimensional Fourier transforms* and for the axisymmetrical problem the *Hankel transforms.* In order to give a unified representation we consider simultaneously the relations for both problems [24].

	Fourier transforms of $f(x)$	Hankel transforms of $f(r)$
(A5)	$\overset{n}{\bar{f}}(\lambda)=\mathscr{F}_n\{f(x)\}=\dfrac{1}{j^n}\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}f(x)\,e^{i\lambda x}dx,$ where $j=i\dfrac{\lambda}{\lvert\lambda\rvert}$ for $\lambda\neq 0$, $j=i$ for $\lambda=0$	$\overset{n}{\bar{f}}(\lambda)=\mathscr{H}_n\{f(r)\}=\displaystyle\int_0^{\infty}rf(r)J_n(\lambda r)dr,$ where J_n denotes Bessel's function of first kind and nth order

and $i=\sqrt{-1}$. Further λ is the real transform parameter.

The inverse transforms are

$$f(x)=\mathscr{F}_n^{-1}\{\overset{n}{\overline{f}}(\lambda)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} j^n \overset{n}{\overline{f}}(\lambda)\, e^{-i\lambda x}d\lambda$$

(A6) if $f(x)$ satisfies Dirichlet's conditions in $-\infty<x<\infty$ and $\int_{-\infty}^{\infty} f(x)\,dx$ is absolutely convergent.

$$f(r)=\mathscr{H}_n^{-1}\{\overset{n}{\overline{f}}(\lambda)\}=\int_0^{\infty}\lambda \overset{n}{\overline{f}}(\lambda)\, J_n(\lambda r)\, d\lambda$$

if $f(r)$ is of bounded variation in the neighbourhood of r and $\int_0^{\infty} f(r)\,dr$ is absolutely convergent.

For the derivatives (6.4) the following relations hold:

$$\overset{1}{\overline{D_1 f}}=-|\lambda|\overset{0}{\overline{f}},\qquad \overset{0}{\overline{D_2 f}}=|\lambda|\overset{1}{\overline{f}},\qquad \overset{2}{\overline{D_3 f}}=-|\lambda|\overset{1}{\overline{f}}$$

if $f(x)\to 0$ for $|x|\to\infty$. | if $rf(r)\to 0$ for $r\to 0$ and $r\to\infty$.

(A7) Further,

$$\overset{1}{\overline{D_4 f}}=-\lambda^2\overset{1}{\overline{f}}$$

provided $f(x)\to 0$ and $\frac{df}{dx}\to 0$ for $|x|\to\infty$.

provided $rf(r)\to 0$ and $r\frac{df(r)}{dr}\to 0$ for $r\to 0$ and $r\to\infty$.

If the conditions listed above in the left column are not satisfied because of physical reasons one has to use generalized Fourier transforms [33] as has been demonstrated by BAKER [34]. However, in this case more complicated expressions arise.

The integration with respect to the time may be performed by means of the *Laplace transform* [35]

(A8) $$\mathscr{L}\{f(t)\}=\tilde{f}(p)=\int_0^{\infty} f(t)\, e^{-pt}dt,$$

the inverse transform being

(A9) $$\mathscr{L}^{-1}\{\tilde{f}(p)\}=f(t)=\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}\tilde{f}(p)\, e^{tp}\, dp$$

with p as the complex transform parameter and ξ lying to the right of all singularities of the integrands. The Laplace transform of the derivative $df/dt=Df$ is given by

(A10) $$\widetilde{Df}=p\tilde{f}$$

provided the initial value $f(+0)$ being zero.

A closed solution of the integral (A9) is only possible in very simple cases. Generally one has to use numerical methods. According to PAPOULIS [25] one gets instead of (A9)

(A11) $$f(t)=\varphi(\tau)=\sum_{\mu=0}^{N\to\infty} c_\nu \sin[(2\mu+1)\tau]$$

with

(A12) $$e^{-\sigma t}=\cos\tau \qquad (\sigma \text{ arbitrary real number}>0),$$

where the coefficients c_ν have to be evaluated from

(A13) $$\frac{2^{2(n+1)}}{\pi}\sigma \bar{f}(p_n)=\sum_{k=0}^{n}\left[2\binom{2n}{k}-\binom{2n+1}{k}\right]c_{n-k} \qquad (n=0,1,\dots N),$$

with

(A14) $$p_n=(2n+1)\sigma \qquad (n=0,1,\dots\infty)$$

being on the real p-axis. Other numerical methods are also available, see [36, 37 and 38].

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STRESZCZENIE

NIESTACJONARNY ROZKŁAD TEMPERATURY I NAPRĘŻEŃ TERMICZNYCH W SPRĘŻYSTYM I LEKPOSRĘŻYSTYM OŚRODKU WARSTWOWYM

W oparciu o metody transformacji całkowych i analizy macierzowej podane zostało formalne rozwiązanie dla niestacjonarnych pól temperatury, strumienia ciepła, naprężeń i przemieszczeń wielowarstwowych ośrodka sprężystego. Efekty nieliniowości, bezwładności i sprzężenia zostały pominięte. Stosowalność teorii pokazano na przykładach policzonych numerycznie, wskazano możliwość rozszerzenia teorii na ośrodki lepkosprężyste.

Резюме

НЕСТАЦИОНАРНОЕ РАСПРЕДЕЛЕНИЕ ТЕМПЕРАТУРЫ И ТЕРМИЧЕСКИХ НАПРЯЖЕНИЙ В УПРУГОЙ И ВЯЗКОУПРУГОЙ СЛОЕВОЙ СРЕДЕ

Опираясь на методы интегральных преобразований и матричного анализа приведено формальное решение для нестационарных полей температуры, потока тепла, напряжений и перемещений многослоевой упругой среды. Эффектами нелинейности, инерции и сопряжения пренебрегается. Применимость теории показана на примерах расчитываемых численно; указана возможность расширения теории на вязкоупругие среды.

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