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RAYLEIGH WAVES SPEED IN TRANSVERSELY ISOTROPIC MATERIAL

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Rayleigh wave speed in transversely isotropic material is studied. A very simple technique is adopted to solve the secular equation. Speed in some transversely isotropic materials is calculated.

Key words: Rayleigh waves, transversely isotropic material, orthotropic, compressible, strain energy.

1. Introduction

Waves propagated along the plane surface of elastic solid were first studied by Rayleigh [1], an explicit formula was obtained for wave speed. After that Rahman and Barber [2] and Nkemizi [3] derived the secular equation and a formula for Rayleigh waves speed respectively. A computer software MATHEMATICA was also used by some researchers, e.g. Royer [4], to find exact values of the speed. Pham and Ogden [5], Ting [6], Destrade [7], Ogden and Pham [8], Destrade [9] have discussed the explicit secular equation and wave speed. Recently Pham and Ogden [10] presented the formula for Rayleigh wave speed in orthotropic elastic solids.

The aim of this paper is to study the Rayleigh wave speed in a transversely isotropic material. We have found that the secular equation for a transversely isotropic material is exactly the same as that obtained by Pham and Ogden [10] for an orthotropic material if c_{44} is replaced by c_{55}.

2. Boundary value problem and the secular equation

Consider the semi-infinite stress-free surface of a transversely isotropic material. We choose the rectangular co-ordinate system in such a way that the x_3-axis is normal to the boundary and the body occupies the region $x_3 \leq 0$.

Following the paper by PHAM and OGDEN [10] let us consider the plane harmonic waves propagating in the x_1-direction of the x_1x_3-plane, with displacement components (u_1, u_2, u_3) such that

$$u_i = u_i(x_1, x_3, t), \qquad i = 1, 3, \qquad u_2 = 0. \tag{2.1}$$

Generalized Hooke's law for a transversely isotropic body may be written as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{c_{11}-c_{12}}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} \tag{2.2}$$

where ϵ_{ij} is the strain tensor

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad i = 1, 2, 3, \tag{2.3}$$

σ_{ij} is the stress tensor and $c_{ii} > 0$, $i = 1, 3, 4$; $c_{11}c_{33} - c_{13}^2 > 0$, which are the necessary and sufficient conditions for the strain energy of the material to be positive definite.

By using the above equations one can write

$$\begin{aligned} \sigma_{11} &= c_{11}u_{1,1} + c_{13}u_{3,3}, \\ \sigma_{33} &= c_{13}u_{1,1} + c_{33}u_{3,3}, \\ \sigma_{13} &= c_{44}(u_{1,3} + u_{3,1}). \end{aligned} \tag{2.4}$$

Equations of motion for infinitesimal deformation may be written as follows.

$$\sigma_{ij,j} = \rho \ddot{u}_i.$$

In terms of displacements, these equations can be written as

$$\begin{aligned} c_{11}u_{1,11} + c_{44}u_{1,33} + (c_{13} + c_{44})\,u_{3,31} &= \rho\ddot{u}_1, \\ c_{44}u_{3,11} + c_{33}u_{3,33} + (c_{13} + c_{44})\,u_{1,13} &= \rho\ddot{u}_3. \end{aligned} \tag{2.5}$$

The boundary conditions of zero traction, on the plane $x_3 = 0$, are

$$\sigma_{3i} = 0, \qquad i = 1, 3 \tag{2.6}$$

Usual requirements are that the displacement and the stress components decay away from the boundary and vanish far away from the boundary, that is

$$u_i \to 0, \qquad \sigma_{ij} \to 0 \;\; (i, j = 1,3) \quad \text{as} \quad x_3 \to -\infty. \tag{2.7}$$

Considering the harmonic waves propagating in the x_1-direction, by following the paper of PHAM and OGDEN [10] we write;

$$u_j = \varphi_j(kx_3)\exp\left[ik(x_1 - ct)\right], \qquad j = 1\,,\,3, \tag{2.8}$$

where k is the wave number, c is the wave speed and φ_j, $j = 1,3$ are the functions to be determined.

Substituting (2.8) into (2.5) we obtain

$$\begin{aligned} &(c_{11} - \rho c^2)\varphi_1 - c_{44}\varphi_1'' - i(c_{44} + c_{13})\varphi_3' = 0 \\ &(c_{44} - \rho c^2)\varphi_3 - c_{33}\varphi_3'' - i(c_{44} + c_{13})\varphi_1' = 0 \end{aligned} \tag{2.9}$$

and the boundary conditions take the form

$$ic_{13}\varphi_1 + c_{33}\varphi_3' = 0, \qquad \varphi_1' + i\varphi_3 = 0 \qquad \text{on} \quad x_3 = 0 \tag{2.10}$$

and

$$\varphi_j, \varphi_j' \to 0 \qquad \text{as} \quad x_3 = \infty, \tag{2.11}$$

thus the above boundary value problem becomes the same as that of PHAM and OGDEN [10] if we replace c_{44} by c_{55}, and hence the secular equation will also be the same as that of [10], which is as follows.

$$(c_{44} - \rho c^2)\left[c_{13}^2 - c_{33}\left(c_{11} - \rho c^2\right)\right] + \rho c^2\sqrt{c_{33}c_{44}}\sqrt{(c_{11} - \rho c^2)(c_{44} - \rho c^2)} = 0$$

This implies

$$\rho c^2 - \sqrt{\frac{(c_{44} - \rho c^2}{(c_{11} - \rho c^2}}\frac{1}{\sqrt{c_{33}c_{44}}}c_{11}c_{33}\left(1 - \frac{c_{13}^2}{c_{11}c_{33}} - \frac{\rho c^2}{c_{11}}\right) = 0$$

or

$$\frac{\rho c^2}{c_{11}} - \sqrt{\frac{c_{33}}{c_{44}}\frac{\dfrac{c_{44}}{c_{11}} - \dfrac{\rho c^2}{c_{11}}}{1 - \dfrac{\rho c^2}{c_{11}}}}\left(1 - \frac{c_{13}^2}{c_{11}c_{33}} - \frac{\rho c^2}{c_{11}}\right) = 0. \tag{2.12}$$

To simplify, let

$$u = \frac{\rho c^2}{c_{11}}, \qquad a = \frac{c_{44}}{c_{33}}, \qquad b = \frac{c_{44}}{c_{11}}, \qquad p = \frac{c_{13}^2}{c_{11}c_{33}}. \tag{2.13}$$

Therefore, the above mentioned equation (2.12) becomes

$$u - \sqrt{\frac{1}{a}}\sqrt{\frac{b-u}{1-u}}(1-p-u) = 0.$$

This implies

$$au^2(1-u) = (1-p-u)^2(b-u)$$

or

$$(1-a)\,u^3 + \{a - 2(1-p) - b\}\,u^2 + \{(1-p)^2 + 2b(1-p)\}\,u - b(1-p)^2 = 0. \tag{2.14}$$

This is the simplified secular equation and can be solved for u.

3. Rayleigh wave speed for some materials

Consider the following transversely isotropic materials.

1-Cobalt. Elastic constants for cobalt are as follows [11]:

$$c_{11} = 2.59 \times 10^{11}\ \mathrm{N/m^2}, \qquad c_{13} = 1.11 \times 10^{11}\ \mathrm{N/m^2},$$

$$c_{33} = 3.35 \times 10^{11}\ \mathrm{N/m^2}, \qquad c_{44} = 0.71 \times 10^{11}\ \mathrm{N/m^2},$$

$$a = \frac{c_{44}}{c_{33}} = 0.21194, \qquad b = \frac{c_{44}}{c_{11}} = 0.274131, \qquad p = \frac{c_{13}^2}{c_{11}c_{33}} = 0.142004.$$

Thus (2.14) becomes

$$0.788060u^3 - 1.778182u^2 + 1.2065644 - 0.201804 = 0.$$

Put

$$u = z - \frac{(-1.778182)}{3(0.788060)} = z + 0.752135.$$

This implies

$$z^3 - 0.1661065z + 0.044508 = 0$$

$$P = \frac{-0.166065}{3} = -0.055355, \qquad Q = \frac{0.044508}{2} = 0.022254,$$

$$P^3 + Q^2 = 0.000325623 > 0,$$

$$P < 0$$

$$\therefore \quad Z = -[Q + \sqrt{P^3+Q^2}]^{1/3} - [Q - \sqrt{P^3+Q^2}]^{1/3} = -0.504303.$$

This implies

$$u = -0.504303 + 0.752135 = 0.247832.$$

Similarly we can determine the value of u for other transversely isotropic materials what is evident from the following table in which stiffness/elastic constants are taken from [11].

Material	Stiffness 10^{11} (N/m^2)						Density ρ	Raleigh Wave Speed
	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	u	(kg/m^3)	(m/s)
Cobalt	2.59	1.59	1.11	3.35	0.71	0.247832	8900	2685.55
Cadmium	1.16	0.42	0.41	0.509	0.196	0.147016	8642	1404.77
Titanium boride	6.90	4.10	3.20	4.40	2.50	0.233471	4500	5983.28
Zinc	1.628	0.362	0.508	0.627	0.385	0.183415	7140	2045.01
Magnesium	0.5974	0.2624	0.217	0.617	0.1639	0.244048	1740	2894.65

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