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VIBRATIONS OF THE THREE-LAYER SHELL WITH DAMPING

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In this paper the analytical method [17–18] has been used for solving the problem of vibrations of the three-layer shell with damping. External layers are modelled as the MINDLIN shells and the internal layer possesses the characteristics of a viscoelastic one-directional WINKLER foundation. Small transverse vibrations of the three-layer shell with damping is excited by the dynamical non-uniform loading moving at the constant velocity v^*. Numerical results are presented in the diagrams.

Key words: vibrations, three-layer shell, damping, numerical results.

1. Introduction

Compound systems coupled together by viscoelastic constraints play an important role in various engineering and building structures. Since 1923, the TIMOSHENKO model [1] for various compound constructions has been applied.

Cylindrical shells with a wide range of geometrical and physical parameters are component parts of many modern structures subjected to the action of different loads. The development of various vibration models becomes an urgent problem due to the structural features of layer shells operating under different conditions of mechanical loading.

The classical theory of cylindrical shells, which is based on the Kirchhoff–Love hypothesis, is widely used for the evaluation of the stress-strain state or vibration of isotropic thin elastic shells. Among numerous precise models applied to the investigation of shells made of modern materials, due to their practical validity, visualization and completeness, the REISSNER model is used. The dynamic problem of elastic homogeneous bodies was presented by GRINCZENKO [2]. The problem of simulation of the acoustic properties of the larger human blood vessels was considered by BORISUK [3]. In the paper by JEMIELITA [4], the criteria of choice of the shear coefficient in plates of medium thickness have been considered.

Coupled problems of the thermomechanical behavior of viscoelastic bodies under harmonic loading were presented by KARNAUCHOW and KIRICZOK [5]. Vibrations of elastic compound systems subjected to inertial moving load was presented by BOGACZ [6] and SZCZEŚNIAK [7, 8]. The problem of non-axisymmetric deformation of flexible rotational shells was solved by PANKRATOVA, NOKOLAEV and ŚWITOŃSKI [9] with the use of the classical Kirchhoff-Love model and the improved Timoshenko model. The problem of vibrations of the elements of shell constructions were described by GRIGORENKO [10]. The problem of laminated plates and shells was presented by LEWIŃSKI, TELEGA [11]. The dynamic problem of elastic homogeneous bodies was presented by TARANTO and MC GRAW [12] and KURNIK and TYLIKOWSKI [13]. The interlayer is a one- or two-directional viscoelastic Winkler layer, but it can also be a multiparametric viscoelastic layer presented by WOŹNIAK [14].

In the above complex cases, especially where viscosity and discrete elements occur, it is recommended to adopt the method of solving the dynamic problem of a system in the domain of function of complex variable following the papers by TSE, MORSE, HINKLE [15], NIZIOL, SNAMINA [16] and CABAŃSKA-PŁACZKIEWICZ [17, 18]. The property of orthogonality of free vibrations of complex types was first described by CREMER, HECKEL, UNGAR [19] and CABAŃSKI [20] for discrete systems with damping, for discrete – continuous systems with damping by NASHIF, JOHNES and HENDERSON [21], and for continuous systems with damping by NOWACKI [22].

The aim of this paper is to present a method of solving the problem and dynamic analysis of free and forced vibrations of a three-layer system with damping, which consists of two elastic shells connected by a viscoelastic interlayer subject to axially symmetric loading $F_2(x,t)$ non-uniform with respect to the axial coordinate, moving at the constant velocity v^*.

2. FORMULATION OF THE PROBLEM

2.1. *Statement of the problem*

Let us consider the free and forced vibrations problem of a three-layer system with damping. The external layers of the complex system are cylindrical shells made of elastic materials, which are connected by a viscoelastic interlayer shown in Fig. 1. The elastic cylindrical shells are described by the Mindlin model and are supported at their ends on circular edges (2.3). The viscoelastic interlayer possesses the characteristics of a homogenous continuous one-directional Winkler foundation and is described by the Voigt–Kelvin model [19, 21, 22].

Small transverse vibrations of the three-layer shell with damping are excited by the axially symmetric loading $F_2(x,t)$, non-uniform with respect to the axial coordinate, moving at the constant velocity v^*, shown in Fig. 1.

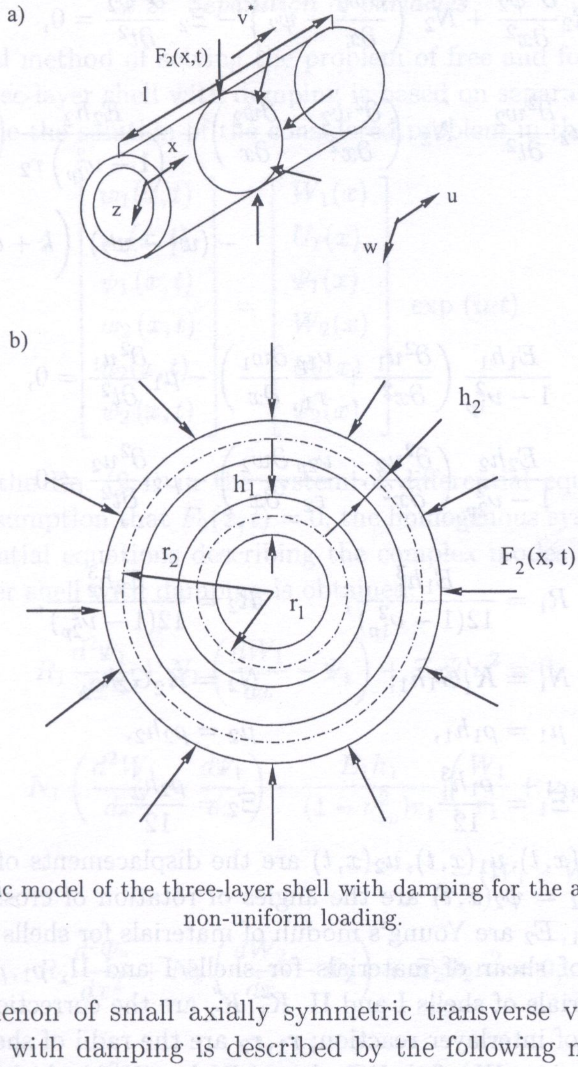


FIG. 1. Dynamic model of the three-layer shell with damping for the axially symmetric non-uniform loading.

The phenomenon of small axially symmetric transverse vibrations for the three-layer shell with damping is described by the following non-homogeneous system of conjugate partial differential equations [17]:

$$R_1 \frac{\partial^2 \psi_1}{\partial x^2} + N_1 \left(\frac{\partial w_1}{\partial x} - \psi_1 \right) - \Xi_1 \frac{\partial^2 \psi_1}{\partial t^2} = 0,$$

$$\mu_1 \frac{\partial^2 w_1}{\partial t^2} - N_1 \left(\frac{\partial^2 w_1}{\partial x^2} - \frac{\partial \psi_1}{\partial x} \right) + \frac{E_1 h_1}{\left(1 - \nu_{1p}^2\right) r_1} \left(\frac{w_1}{r_1} + \nu \frac{\partial u_1}{\partial x} \right) \tag{2.1}$$

$$+ (w_1 - w_2) \left(k + c \frac{\partial}{\partial t} \right) = 0,$$

(2.1) [cont.]
$$R_2 \frac{\partial^2 \psi_2}{\partial x^2} + N_2 \left(\frac{\partial w_2}{\partial x} - \psi_2 \right) - \Xi_2 \frac{\partial^2 \psi_2}{\partial t^2} = 0,$$

$$\mu_2 \frac{\partial^2 w_2}{\partial t^2} - N_2 \left(\frac{\partial^2 w_2}{\partial x^2} - \frac{\partial \psi_2}{\partial x} \right) + \frac{E_2 h_2}{\left(1 - \nu_{2p}^2\right) r_2} \left(\frac{w_2}{r_2} + \nu \frac{\partial u_2}{\partial x} \right) - (w_1 - w_2) \left(k + c \frac{\partial}{\partial t} \right) = F_2(x,t)$$

and

(2.2)
$$\frac{E_1 h_1}{1 - \nu_{1p}^2} \left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\nu_{1p}}{r_1} \frac{\partial w_1}{\partial x} \right) - \mu_1 \frac{\partial^2 u_1}{\partial t^2} = 0,$$
$$\frac{E_2 h_2}{1 - \nu_{2p}^2} \left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\nu_{2p}}{r_2} \frac{\partial w_2}{\partial x} \right) - \mu_2 \frac{\partial^2 u_2}{\partial t^2} = 0$$

where:
$$R_1 = \frac{E_1 h_1^3}{12(1 - \nu_{1p}^2)}, \qquad R_2 = \frac{E_2 h_2^3}{12(1 - \nu_{2p}^2)},$$
$$N_1 = K_1' G_1 h_1, \qquad N_2 = K_2' G_2 h_2,$$
$$\mu_1 = \rho_1 h_1, \qquad \mu_2 = \rho_2 h_2,$$
$$\Xi_1 = \frac{\rho_1 h_1^3}{12}, \qquad \Xi_2 = \frac{\rho_2 h_2^3}{12}.$$

Here $w_1(x,t), w_2(x,t), u_1(x,t), u_2(x,t)$ are the displacements of shells I and II; $\psi_1 = \psi_1(x,t)$, $\psi_2 = \psi_2(x,t)$ are the angles of rotation of cross-sections of the shells I and II; E_1, E_2 are Young's moduli of materials for shells I and II; G_1, G_2 are the moduli of shear of materials for shells I and II; ρ_1, ρ_2 are the mass densities of materials of shells I and II, K_1', K_2' are the correction coefficients; k is the coefficient of interlayer reaction; r_1, r_2 are the radii of shells I and II; c is the coefficient of viscosity of the interlayer; h_1, h_2 are the thicknesses of shells I and II; l is the length of the three-layer shell; ν_{1p}, ν_{2p} are Poisson's coefficients; t is time; x, z are the co-ordinate axes; $F_2(x,t)$ is the axially symmetric loading.

For the assumed support of the shells, the functions appearing in the Eqs. (2.1), (2.2) should satisfy the following boundary conditions:

(2.3)
$$w_1\big|_{x=0} = 0, \quad w_1\big|_{x=l} = 0, \quad w_2\big|_{x=0} = 0, \quad w_2\big|_{x=l} = 0,$$
$$u_1\big|_{x=0} = 0, \quad u_1\big|_{x=l} = 0, \quad u_2\big|_{x=0} = 0, \quad u_2\big|_{x=l} = 0,$$
$$\left.\frac{d\psi_1}{dx}\right|_{x=0} = 0, \quad \left.\frac{d\psi_1}{dx}\right|_{x=l} = 0, \quad \left.\frac{d\psi_2}{dx}\right|_{x=0} = 0, \quad \left.\frac{d\psi_2}{dx}\right|_{x=l} = 0.$$

2.2. *Separation of variables*

An analytical method of solving the problem of free and forced vibration of a cylindrical three-layer shell with damping is based on separation of variables.

Let us assume the solution of the considered problem in the following form:

$$\begin{bmatrix} w_1(x,t) \\ u_1(x,t) \\ \psi_1(x,t) \\ w_2(x,t) \\ u_2(x,t) \\ \psi_2(x,t) \end{bmatrix} = \begin{bmatrix} W_1(x) \\ U_1(x) \\ \Psi_1(x) \\ W_2(x) \\ U_2(x) \\ \Psi_2(x) \end{bmatrix} \exp(i\nu t) \tag{2.4}$$

and substitute the Eq. (2.4) in the system of differential equations (2.1) and (2.2). By the assumption that $F_2(x,t) = 0$, the homogenous system of conjugate ordinary differential equations describing the complex modes of free vibrations of the three-layer shell with damping is obtained:

$$\begin{aligned} & R_1 \frac{d^2\Psi_1}{dx^2} + N_1\left(\frac{dW_1}{dx} - \Psi_1\right) + \Xi_1 \Psi_1 \nu^2 = 0, \\ & N_1\left(\frac{d^2W_1}{dx^2} - \frac{d\Psi_1}{dx}\right) - \frac{E_1 h_1}{(1-\nu_{1p}^2)r_1}\left(\frac{W_1}{r_1} + \nu_{1p}\frac{dU_1}{dx}\right) + \mu_1 W_1 \nu^2 \\ & \qquad - (W_1 - W_2)(k + ic\nu) = 0, \\ & R_2 \frac{d^2\Psi_2}{dx^2} + N_2\left(\frac{dW_2}{dx} - \Psi_2\right) + \Xi_2 \Psi_2 \nu^2 = 0, \\ & N_2\left(\frac{d^2W_2}{dx^2} - \frac{d\Psi_2}{dx}\right) - \frac{E_2 h_2}{(1-\nu_{2p}^2)r_2}\left(\frac{W_2}{r_2} + \nu_{2p}\frac{dU_2}{dx}\right) + \mu_2 W_2 \nu^2 \\ & \qquad + (W_1 - W_2)(k + ic\nu) = 0 \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} & \frac{E_1 h_1}{1-\nu_{1p}^2}\left(\frac{d^2U_1}{dx^2} + \frac{\nu_{1p}}{r_1}\frac{dW_1}{dx}\right) + \mu_1 U_1 \nu^2 = 0, \\ & \frac{E_2 h_2}{1-\nu_{2p}^2}\left(\frac{d^2U_2}{dx^2} + \frac{\nu_{2p}}{r_2}\frac{dW_2}{dx}\right) + \mu_2 U_2 \nu^2 = 0, \end{aligned} \tag{2.6}$$

where $i^2 = -1$. Here $W_1 = W_1(x)$, $W_2 = W_2(x)$, $U_1 = U_1(x)$, $U_2 = U_2(x)$ are the complex vibration modes of shells I and II; $\Psi_1 = \Psi_1(x)$, $\Psi_2 = \Psi_2(x)$ are the complex rotation modes of shells I and II; ν is the complex eigenfrequency of free vibrations of the three-layer shell with damping.

In the case of the simplified model for $u_1 = u_2 = 0$, the system of Eqs. (2.5) has the following form:

(2.7)
$$\begin{aligned} &R_1 \frac{d^2\Psi_1}{dx^2} + N_1\left(\frac{dW_1}{dx} - \Psi_1\right) + \Xi_1 \Psi_1 \nu^2 = 0,\\ &N_1\left(\frac{d^2W_1}{dx^2} - \frac{d\Psi_1}{dx}\right) - s_1\frac{W_1}{r_1} + \mu_1 W_1\nu^2 - (W_1 - W_2)(k + ic\nu) = 0,\\ &R_2 \frac{d^2\Psi_2}{dx^2} + N_2\left(\frac{dW_2}{dx} - \Psi_2\right) + \Xi_2 \Psi_2 \nu^2 = 0,\\ &N_2\left(\frac{d^2W_2}{dx^2} - \frac{d\Psi_2}{dx}\right) - s_2\frac{W_2}{r_2} + \mu_2 W_2\nu^2 + (W_1 - W_2)(k + ic\nu) = 0. \end{aligned}$$

Here:

(2.8)
$$s_1 = \frac{E_1 h_1}{(1 - \nu_{1p}^2) r_1}, \qquad s_2 = \frac{E_2 h_2}{(1 - \nu_{2p}^2) r_2}.$$

2.3. Solution of the boundary value problem

Searching for a particular solution of the system of differential equations (2.7) in the form:

(2.9)
$$\begin{bmatrix} W_1(x) \\ \Psi_1(x) \\ W_2(x) \\ \Psi_2(x) \end{bmatrix} = \begin{bmatrix} A \\ C \\ B \\ D \end{bmatrix} \exp(rx)$$

the homogeneous system of algebraic equations is obtained:

(2.10)
$$\begin{aligned} &A\frac{N_1}{R_1} r + C(r^2 + p_1^*) = 0,\\ &A(r^2 + p_1^{**}) + B\frac{1}{N_1}(k + ic\nu) - Cr = 0,\\ &B\frac{N_2}{R_2} r + D(r^2 + p_2^*) = 0,\\ &A\frac{1}{N_2}(k + ic\nu) + B(r^2 + p_2^{**}) - Dr = 0. \end{aligned}$$

where:

$$p_1^* = \frac{1}{R_1}(\Xi_1\nu^2 - N_1), \qquad p_1^{**} = \frac{1}{N_1}\left[\mu_1\nu^2 - k - ic\nu - \frac{1}{r_1}s_1\right],$$

(2.11)

$$p_2^* = \frac{1}{R_2}(\Xi_2\nu^2 - N_2), \qquad p_2^{**} = \frac{1}{N_2}\left[\mu_2\nu^2 - k - ic\nu - \frac{1}{r_2}s_2\right].$$

The system of equations (2.10) has a nonzero solution provided the determinant of the coefficient matrix of this system is equal to zero

$$\begin{vmatrix} \frac{N_1}{R_1}r & 0 & r^2 + p_1^* & 0 \\ r^2 + p_1^{**} & \frac{1}{N_1}(k + ic\nu) & -r & 0 \\ 0 & \frac{N_2}{R_2}r & 0 & r^2 + p_2^* \\ \frac{1}{N_2}(k + ic\nu) & r^2 + p_2^{**} & 0 & -r \end{vmatrix} = 0. \tag{2.12}$$

The characteristic equation (2.12), after expansion of its determinant, is equivalent to one of the following fourth-order algebraical equation

$$\bar{r}^4 + a_3\bar{r}^3 + a_2\bar{r}^2 + a_1\bar{r} + a_0 = 0 \tag{2.13}$$

where $\bar{r} = r^2$ and a_3, a_2, a_1, a_0 are coefficients of the Eq. (2.13) corresponding to the complex frequency ν.

The roots r_j of the Eqs. (2.13) or (2.12) are described by means of parameters λ_v in the form $r_j = (-1)^{j-1}\, i\lambda_v$, where $j = (2v - 1),\ 2v;\ v = 1, 2, 3, 4$.

Dependence of the parameters λ_v on the complex frequency ν can be described by the Eq. (2.13) or (2.12). The values of the parameters λ_v will be determined by the formula (2.29).

Replacing the parameter r by the roots r_j in the functions (2.9) and applying the Euler formulas, the general solution of the system of differential equations (2.7) is the set of the following functions:

$$W_1(x) = \sum_{v=1}^{4}(A_v^* \sin\lambda_v x + A_v^{**}\cos\lambda_v x),$$

(2.14)

$$\Psi_1(x) = \sum_{v=1}^{4}(C_v^* \cos\lambda_v x + C_v^{**}\sin\lambda_v x),$$

$$W_2(x) = \sum_{v=1}^{4} (B_v^* \sin \lambda_v x + B_v^{**} \cos \lambda_v x),$$

(2.14) [cont.]

$$\Psi_2(x) = \sum_{v=1}^{4} (D_v^* \cos \lambda_v x + D_v^{**} \sin \lambda_v x)$$

Using the Eqs. (2.10), the following relations between the constants appearing in (2.14) have been established:

(2.15)
$$b_v^* = \frac{B_v^*}{A_v^*}, \qquad b_v^{**} = \frac{B_v^{**}}{A_v^{**}},$$
$$c_v^* = \frac{C_v^*}{A_v^*}, \qquad c_v^{**} = \frac{C_v^{**}}{A_v^{**}},$$
$$d_v^* = \frac{D_v^*}{A_v^*}, \qquad d_v^{**} = \frac{D_v^{**}}{A_v^{**}}.$$

where:

(2.16)
$$b_v^* = b_v^{**} = b_v = \frac{(p_1^{**} - \lambda_v^2)(p_1^* - \lambda_v^2) - \frac{N_1}{R_1}\lambda_v^2}{\frac{N_1}{R_1}(p_1^* - \lambda_v^2)},$$
$$c_v^* = c_v = \frac{\frac{N_1}{R_1}\lambda_v}{p_1^* - \lambda_v^2}, \qquad c_v = -c_v^{**},$$
$$d_v^* = d_v = b_v \frac{\frac{N_2}{R_2}\lambda_v}{p_2^* - \lambda_v^2}, \qquad d_v = -d_v^{**}.$$

After substituting (2.15) in (2.14), general solution of the system of differential equations (2.7) takes the following form:

(2.17)
$$W_1(x) = \sum_{v=1}^{4} (A_v^* \sin \lambda_v x + A_v^{**} \cos \lambda_v x),$$
$$\Psi_1(x) = \sum_{v=1}^{4} c_v (A_v^* \cos \lambda_v x - A_v^{**} \sin \lambda_v x),$$
$$W_2(x) = \sum_{v=1}^{4} b_v (A_v^* \sin \lambda_v x + A_v^{**} \cos \lambda_v x),$$
$$\Psi_2(x) = \sum_{v=1}^{4} d_v (A_v^* \cos \lambda_v x - A_v^{**} \sin \lambda_v x).$$

According to the boundary conditions (2.3) concerning $w_1(x)$, $\Psi_1(x)$, $w_2(x)$, $\Psi_2(x)$, the functions $W_1(x)$, $\Psi_1(x)$, $W_2(x)$, $\Psi_2(x)$ in the Eqs. (2.17) should satisfy the following boundary conditions:

$$\begin{aligned} &W_1\,|_{x=0} = 0, \qquad W_1\,|_{x=l} = 0, \qquad W_2\,|_{x=0} = 0, \qquad W_2\,|_{x=l} = 0, \\ &\left.\frac{d\Psi_1}{dx}\right|_{x=0} = 0, \quad \left.\frac{d\Psi_1}{dx}\right|_{x=l} = 0, \quad \left.\frac{d\Psi_2}{dx}\right|_{x=0} = 0, \quad \left.\frac{d\Psi_2}{dx}\right|_{x=l} = 0. \end{aligned} \tag{2.18}$$

Applying the boundary conditions (2.18) to the functions (2.17), one obtains the homogeneous system of algebraic equations, which in the matrix notation takes the following form:

$$\mathbf{Y}\,\mathbf{X} = 0. \tag{2.19}$$

Here $\mathbf{X} = [A_1^*,\ A_2^*,\ A_3^*,\ A_4^*\ A_1^{**},\ A_2^{**},\ A_3^{**},\ A_4^{**}]^T$ are the vectors of unknowns of the system of equations, and

$$\mathbf{Y} = [Y_{i\,*\,j}]_{8\,*8} \tag{2.20}$$

is the characteristic matrix of the system of Eqs. (2.19).

The first subsystem of the system (2.19) has the form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ b_1 & b_2 & b_3 & b_4 \\ c_1\lambda_1 & c_2\lambda_2 & c_3\lambda_3 & c_4\lambda_4 \\ d_1\lambda_1 & d_2\lambda_2 & d_3\lambda_3 & d_4\lambda_4 \end{bmatrix} \begin{bmatrix} A_1^{**} \\ A_2^{**} \\ A_3^{**} \\ A_4^{**} \end{bmatrix} = 0 \tag{2.21}$$

from which it follows that $A_1^{**} = A_2^{**} = A_3^{**} = A_4^{**} = 0$.

The other subsystem of the system (2.19) leads to the following set of equations:

$$\begin{bmatrix} ss_{11} & ss_{12} & ss_{13} & ss_{14} \\ b_1 ss_{11} & b_2 ss_{12} & b_3 ss_{13} & b_4 ss_{14} \\ c_1 ss_{11} & c_2 ss_{12} & c_3 ss_{13} & c_4 ss_{14} \\ d_1 ss_{11} & d_2 ss_{12} & d_3 ss_{13} & d_4 ss_{14} \end{bmatrix} \begin{bmatrix} A_1^* \\ A_2^* \\ A_3^* \\ A_4^* \end{bmatrix} = 0 \tag{2.22}$$

where: $ss_{11} = \sin\lambda_1 l,\ ss_{12} = \sin\lambda_2 l,\ ss_{13} = \sin\lambda_3 l,\ ss_{14} = \sin\lambda_4 l.$

The condition of solving the system of Eqs. (2.22) is vanishing of the characteristic determinant, that is

$$\text{(2.23)}\qquad \begin{vmatrix} ss_{11} & ss_{12} & ss_{13} & ss_{14} \\ b_1 ss_{11} & b_2 ss_{12} & b_3 ss_{13} & b_4 ss_{14} \\ c_1 ss_{11} & c_2 ss_{12} & c_3 ss_{13} & c_4 ss_{14} \\ d_1 ss_{11} & d_2 ss_{12} & d_3 ss_{13} & d_4 ss_{14} \end{vmatrix} = 0.$$

Expanding the determinant (2.23), the following characteristic equation has been obtained:

$$\text{(2.24)}\qquad \sin\lambda_1 l\ \sin\lambda_2 l \sin\lambda_3 l\ \sin\lambda_4 l = 0.$$

It is obvious that Eq. (2.24) has to be satisfied by each of the following simple equations: $\sin\lambda_1 l = 0$, $\sin\lambda_2 l = 0$, $\sin\lambda_3 l = 0$, $\sin\lambda_4 l = 0$, from which it follows that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$.

The characteristic equation (2.24) may now be rewritten in the form:

$$\text{(2.25)}\qquad \sin\lambda l = 0$$

where:

$$\text{(2.26)}\qquad \lambda = \alpha + i\,\beta$$

is in the general case a complex number.

Owing to the relation (2.26), Eq. (2.25) takes the following form:

$$\text{(2.27)}\qquad \sin\alpha l \cosh\beta l + \cos\alpha l \sinh\beta l = 0.$$

The roots of equation (2.27) are as follows:

$$\text{(2.28)}\qquad \alpha_n = \frac{n\pi}{l}, \qquad \beta_n = 0, \qquad n = 1,\ 2,\ ...,\ \infty.$$

Finally, in accordance with the roots (2.28) and relation (2.26), the roots of characteristic equations (2.25) have the following form:

$$\text{(2.29)}\qquad \lambda_n = \frac{n\pi}{l}\,.$$

The solutions of the system of equations (2.22) corresponding to eigenvalues λ_n are $A_{1n}^* = A_{2n}^* = A_{3n}^* = A_{4n}^* = A_n^*$. In further considerations it is assumed that $A_n^* = 1$.

Substituting $r = i\lambda_n$ into the Eq. (2.13) and carrying out the necessary transformations, one obtains the following equation of frequency:

$$\nu^8 + a_3^{\#}\nu^6 + a_2^{\#}\nu^4 + a_1^{\#}\nu^2 + a_0^{\#} = 0 \tag{2.30}$$

where: $a_3^{\#}$, $a_2^{\#}$, $a_1^{\#}$, $a_0^{\#}$ are coefficients corresponding to eigenvaluesλ_n. The solution of the equation (2.30) is an infinite sequence of complex eigenfrequencies:

$$\nu_n = i\eta_n \pm \omega_n, \qquad n = 1, 2, ..., \infty. \tag{2.31}$$

The relations (2.16) and (2.11) taken for $\lambda_v = \lambda = \lambda_n$ and $\nu = \nu_n$, assume the following form, respectively:

$$\begin{aligned}
b_n^* = b_n^{**} = b_n &= \frac{(p_{1n}^{**} - \lambda_n^2)(p_{1n}^* - \lambda_n^2) - \dfrac{N_1}{R_1}\lambda_n^2}{\dfrac{N_1}{R_1}(p_{1n}^* - \lambda_n^2)}, \\
c_n^* = c_n &= \frac{\dfrac{N_1}{R_1}\lambda_n}{p_{1n}^* - \lambda_n^2}, \qquad c_n = -\,c_n^{**}, \\
d_n^* = d_n &= b_n\frac{\dfrac{N_2}{R_2}\lambda_n}{p_{2n}^* - \lambda_n^2}, \qquad d_n = -\,d_n^{**}
\end{aligned} \tag{2.32}$$

and:

$$\begin{aligned}
p_{1n}^* &= \frac{1}{R_1}\left(\Xi_1\nu_n^2 - N_1\right), \qquad p_{1n}^{**} = \frac{1}{N_1}\left[\mu_1\nu_n^2 - k - ic\nu_n - \frac{1}{r_1}s_1\right], \\
p_{2n}^* &= \frac{1}{R_2}\left(\Xi_2\nu_n^2 - N_2\right), \qquad p_{2n}^{**} = \frac{1}{N_2}\left[\mu_2\nu_n^2 - k - ic\nu_n - \frac{1}{r_2}s_2\right].
\end{aligned} \tag{2.33}$$

Using the previously obtained results, i.e. $A_v^{**} = 0$ for $v = 1, 2, 3, 4$; $A_n^* = 1$; λ_n, ν_n, b_n, c_n, d_n in the Eqs. (2.17), one obtains four following sequences of the complex eigenmodes:

$$\begin{aligned}
W_{1n}(x) &= \sin\lambda_n x, \\
\Psi_{1n}(x) &= c_n\cos\lambda_n x, \\
W_{2n}(x) &= b_n\sin\lambda_n x, \\
\Psi_{2n}(x) &= d_n\cos\lambda_n x.
\end{aligned} \tag{2.34}$$

The system of equations (2.6) can be rewritten in the following form:

$$(2.35)\qquad \begin{aligned} &\frac{d^2U_1}{dx^2}+\frac{1-\nu_{1p}^2}{E_1h_1}\mu_1U_1\nu^2=\frac{\nu_{1p}}{r_1}\frac{dW_1}{dx},\\ &\frac{d^2U_2}{dx^2}+\frac{1-\nu_{2p}^2}{E_2h_2}\mu_2U_2\nu^2=\frac{\nu_{2p}}{r_2}\frac{dW_2}{dx}.\end{aligned}$$

A general solution of the system of differential equations (2.35) for $\lambda=\lambda_n$, $\nu=\nu_n$ and $dW_1/dx=0$, $dW_2/dx=0$ has the form

$$(2.36)\qquad \begin{aligned} U_{1n}^*(x)&=G_n^*\sin\lambda_nx+G_n^{**}\cos\lambda_nx,\\ U_{2n}^*(x)&=H_n^*\sin\lambda_nx+H_n^{**}\cos\lambda_nx.\end{aligned}$$

Here G_n^*, G_n^{**}, H_n^*, H_n^{**} are constants.

A particular solution of the system of differential equations (2.35) by means of eigenmodes W_{1n}, W_{2n} of Eqs. (2.34) has the convolution form

$$(2.37)\qquad \begin{aligned} U_{1n}^{**}(x)&=-\frac{\nu_{1p}}{r_1}\int_0^x\sin\lambda_n(x-x^*)\cos\lambda_nx^*dx^*,\\ U_{2n}^{**}(x)&=-\frac{\nu_{2p}}{r_2}b_n\int_0^x\sin\lambda_n(x-x^*)\cos\lambda_nx^*dx^*.\end{aligned}$$

The general solution of the system of equations (2.35) is presented in the following form:

$$(2.38)\qquad \begin{aligned} U_{1n}(x)&=U_{1n}^*(x)+U_{1n}^{**}(x),\\ U_{2n}(x)&=U_{2n}^*(x)+U_{2n}^{**}(x).\end{aligned}$$

According to the Eqs. (2.3) containing u_1 and u_2, the functions U_1 and U_2 have to satisfy the following boundary conditions:

$$(2.39)\qquad \begin{aligned} U_1\,|_{x=0}&=0, &\quad U_1\,|_{x=l}&=0,\\ U_2\,|_{x=0}&=0, &\quad U_2\,|_{x=l}&=0.\end{aligned}$$

Introducing the boundary conditions (2.39) into Eqs. (2.38), two other sequences of complex eigenmodes are obtained:

$$
(2.40)\qquad
\begin{aligned}
U_{1n}(x) &= \sin\lambda_n x - \frac{\nu_{1p}}{r_1}\int_0^x \sin\lambda_n(x-x^*)\cos\lambda_n x^* dx^*,\\
U_{2n}(x) &= \sin\lambda_n x - \frac{\nu_{2p}}{r_2}b_n\int_0^x \sin\lambda_n(x-x^*)\cos\lambda_n x^* dx^*.
\end{aligned}
$$

2.4. Solution of the initial value problem

The complex equation of motion has the form

$$
(2.41)\qquad T^* = \Phi\ \exp\,(i\ \nu\ t).
$$

For $\nu = \nu_n$, the Eqs. (2.41) can be written in the following form:

$$
(2.42)\qquad T_n^* = \Phi_n \exp\,(i\nu_n t)
$$

where Φ_n is the Fourier coefficient.

Free vibrations of the three-layer shell with damping are expanded in the following Fourier series:

$$
(2.43)\qquad
\begin{bmatrix} w_1(x,t)\\ u_1(x,t)\\ \Psi_1(x,t)\\ w_2(x,t)\\ u_2(x,t)\\ \Psi_2(x,t)\end{bmatrix}
=
\begin{bmatrix}
\sum\limits_{n=1}^{\infty} W_{1n}(x)\\
\sum\limits_{n=1}^{\infty} U_{1n}(x)\\
\sum\limits_{n=1}^{\infty} \Psi_{1n}(x)\\
\sum\limits_{n=1}^{\infty} W_{2n}(x)\\
\sum\limits_{n=1}^{\infty} U_{2n}(x)\\
\sum\limits_{n=1}^{\infty} \Psi_{2n}(x)
\end{bmatrix}
\Phi_n \exp(i\nu_n t).
$$

From the system of Eqs. (2.5) and (2.6), performing some algebraical transformations, adding the equations together and then integrating them on both sides in the limits from 0 to 1, the property of orthogonality of eigenmodes for the three-layer viscoelastic shell with Mindlin model is obtained:

$$(2.44)\qquad \int\limits_0^l i\nu_n\Big\{\mu_1\Big[W_{1m}W_{1n}+W_{1n}W_{1m}+U_{1m}U_{1n}+U_{1n}U_{1m} + \frac{h_1^2}{12r_1^2}(\Psi_{1m}\Psi_{1n}+\Psi_{1n}\Psi_{1m})\Big] + \mu_2\Big[W_{2m}W_{2n}+W_{2n}W_{2m}+U_{2m}U_{2n}+U_{2n}U_{2m} + \frac{h_2^2}{12r_2^2}(\Psi_{2m}\Psi_{2n}+\Psi_{2n}\Psi_{2m})\Big] + c(W_{1n}-W_{2n})(W_{1m}-W_{2m})\Big\}dx = N_n\delta_{nm}$$

where:

$$(2.45)\qquad N_n = \int\limits_0^l \Big\{2i\nu_n\Big[\mu_1\Big(W_{1n}^2+U_{1n}^2+\frac{h_1^2}{12r_1^2}\Psi_{1n}^2\Big) + \mu_2\Big(W_{2n}^2+U_{2n}^2+\frac{h_2^2}{12r_2^2}\Psi_{2n}^2\Big)\Big] + c\,(W_{1n}-W_{2n})^2\Big\}\,dx.$$

Here δ_{nm} is the Kronecker delta.

The basis for solving the free vibrations problem are the following initial conditions:

$$(2.46)\qquad \begin{aligned} w_1(x,0)&=w_{01}, & w_2(x,0)&=w_{02},\\ u_1(x,0)&=u_{01}, & u_2(x,0)&=u_{02},\\ \Psi_1(x,0)&=\Psi_{01}, & \Psi_2(x,0)&=\Psi_{02}. \end{aligned}$$

Applying conditions (2.46) in the series (2.43) and taking into account the property of orthogonality (2.44), the formula for the complex Fourier coefficient is obtained:

$$(2.47)\qquad \Phi_n = \frac{1}{N_n}\int\limits_0^l \Big\{i\nu_n\Big[\mu_1\Big(W_{1n}w_{01}+U_{1n}u_{01}+\frac{h_1^2}{12r_1^2}\Psi_{1n}\Psi_{01}\Big) + \mu_2\Big(W_{2n}w_{02}+U_{2n}u_{02}+\frac{h_2^2}{12r_2^2}\Psi_{2n}\Psi_{02}\Big)\Big] + c\,\big[(W_{1n}-W_{2n})\big(w_{01}-w_{02}\big)\big]\Big\}dx.$$

Changing the complex components appearing in the Eqs. (2.43) into the trigonometrical form and having in mind the existence of pairwise conjugated components, the Eqs. (2.43) take the following form:

$$\begin{bmatrix} w_1 \\ u_1 \\ \Psi_1 \\ w_2 \\ u_2 \\ \Psi_2 \end{bmatrix} = \sum_{n=1}^{\infty} e^{-\eta_n t} |\Phi_n| \left\{ \begin{bmatrix} X_{1n} \\ \Theta_{1n} \\ \Lambda_{1n} \\ X_{2n} \\ \Theta_{2n} \\ \Lambda_{2n} \end{bmatrix} \cos(\omega_n t + \varphi_n) - \begin{bmatrix} Y_{1n} \\ \Gamma_{1n} \\ \Omega_{1n} \\ Y_{2n} \\ \Gamma_{2n} \\ \Omega_{2n} \end{bmatrix} \sin(\omega_n t + \varphi_n) \right\} \tag{2.48}$$

where:

$$\begin{aligned}
X_{1n} &= \mathrm{Re} W_{1n}, & Y_{1n} &= \mathrm{Im} W_{1n}, \\
\Theta_{1n} &= \mathrm{Re}\, U_{1n}, & \Gamma_{1n} &= \mathrm{Im} U_{1n}, \\
\Lambda_{1n} &= \mathrm{Re} \Psi_{1n}, & \Omega_{1n} &= \mathrm{Im} \Psi_{1n}, \\
|\Phi_n| &= \sqrt{C_n^2 + D_n^2}, & \Phi_n &= \arg \Phi_n, \\
X_{2n} &= \mathrm{Re}\, W_{2n}, & Y_{2n} &= \mathrm{Im} W_{2n}, \\
\Theta_{2n} &= \mathrm{Re} U_{2n}, & \Gamma_{2n} &= \mathrm{Im} U_{2n}, \\
\Lambda_{2n} &= \mathrm{Re} \Psi_{2n}, & \Omega_{2n} &= \mathrm{Im} \Psi_{2n}, \\
C_n &= \mathrm{Re}\, \Phi_n, & D_n &= \mathrm{Im} \Phi_n.
\end{aligned} \tag{2.49}$$

Finally, the equations (2.48) can also be expressed in the more usual form:

$$\begin{aligned}
w_1 &= \sum_{n=1}^{\infty} e^{-\eta_n t} |\Phi_n| \; |W_{1n}| \cos(\omega_n t + \Phi_n + \chi_{1n}), \\
u_1 &= \sum_{n=1}^{\infty} e^{-\eta_n t} |\Phi_n| \; |U_{1n}| \cos(\omega_n t + \Phi_n + \vartheta_{1n}), \\
\Psi_1 &= \sum_{n=1}^{\infty} e^{-\eta_n t} |\Phi_n| \; |\Psi_{1n}| \cos(\omega_n t + \Phi_n + \theta_{1n}),
\end{aligned} \tag{2.50}$$

$$w_2 = \sum_{n=1}^{\infty} e^{-\eta_n t} |\Phi_n| \, |W_{2n}| \cos(\omega_n t + \Phi_n + \chi_{2n}),$$

(2.50) [cont.]
$$u_2 = \sum_{n=1}^{\infty} e^{-\eta_n t} |\Phi_n| \, |U_{2n}| \cos(\omega_n t + \Phi_n + \vartheta_{2n}),$$

$$\Psi_2 = \sum_{n=1}^{\infty} e^{-\eta_n t} |\Phi_n| \, |\Psi_{2n}| \cos(\omega_n t + \Phi_n + \theta_{2n}),$$

where:

(2.51)
$$\begin{aligned}
|W_{1n}| &= \sqrt{X_{1n}^2 + Y_{1n}^2}, & |W_{2n}| &= \sqrt{X_{2n}^2 + Y_{2n}^2},\\
|U_{1n}| &= \sqrt{\Theta_{1n}^2 + \Gamma_{1n}^2}, & |U_{2n}| &= \sqrt{\Theta_{2n}^2 + \Gamma_{2n}^2},\\
|\Psi_{1n}| &= \sqrt{\Lambda_{1n}^2 + \Omega_{1n}^2}, & |\Psi_{2n}| &= \sqrt{\Lambda_{2n}^2 + \Omega_{2n}^2},\\
\chi_{1n} &= \arg W_{1n}, & \chi_{2n} &= \arg W_{2n},\\
\vartheta_{1n} &= \arg U_{1n}, & \vartheta_{2n} &= \arg U_{2n},\\
\theta_{1n} &= \arg \Psi_{1n}, & \theta_{2n} &= \arg \Psi_{2n}.
\end{aligned}$$

2.5. *Solution of differential equations describing the forced vibrations problem*

Small transverse vibrations of a three-layer shell with damping are excited by axially symmetric inertial loading $F_2(x,t)$ in relation to coordinate axis x, moving in the direction of axis x at the constant velocity v^* (see Fig. 1).

(2.52)
$$F_2(x,t) = P_2 \delta(x - v^* t),$$

where P_2 is the force, $\delta(...)$ denotes the Dirac delta function.

In order to solve differential equations (2.1), (2.2), the function of loading (2.52) is expanded in the series

(2.53)
$$F_2(x,t) = \sum_{n=1}^{\infty} \left[\mu_1 \left(W_{1n} + U_{1n} + \frac{h_1^2}{12 r_1^2} \Psi_{1n} \right) + \mu_2 \left(W_{2n} + U_{2n} + \frac{h_2^2}{12 r_2^2} \Psi_{2n} \right) \right] f_n$$

where W_{1n}, W_{2n}, U_{1n}, U_{2n}, Ψ_{1n}, Ψ_{2n} have been described by the Eqs. (2.34), (2.40).

The functions of displacements of the three-layer shell with damping are expanded in the Fourier series

$$\begin{bmatrix} w_1 \\ u_1 \\ \Psi_1 \\ w_2 \\ u_2 \\ \Psi_2 \end{bmatrix} = \sum_{n=1}^{\infty} \begin{bmatrix} W_{1n} \\ U_{1n} \\ \Psi_{1n} \\ W_{2n} \\ U_{2n} \\ \Psi_{2n} \end{bmatrix} T_n. \tag{2.54}$$

Substituting (2.53) and (2.54) into the differential equations (2.1), (2.2), the following equation of forced motion is obtained:

$$\overset{0}{T}_n - i\nu_n T_n = f_n, \tag{2.55}$$

wheref_n are the coefficients of distribution of the function of loading (2.52) in the Fourier series.

Applying the property of orthogonality of the eigenfunction (2.44), the formulas for coefficients of distribution of the loading are derived:

$$f_n = \frac{1}{N_n} i\nu_n \int_0^l \Big[W_{1n}(x) + W_{2n}(x) + U_{1n}(x) + U_{2n}(x) + \Psi_{1n}(x) + \Psi_{2n}(x)\Big] F_2(x,t)\, dx. \tag{2.56}$$

The solution of the differential equation (2.55) has the form

$$T_n = \int_0^t [\exp(i\nu_n)(t-\tau)] f_n(\tau) d\tau. \tag{2.57}$$

To this end, the Fourier series (2.54) can be also rewritten in the following form:

$$\begin{aligned} w_1 &= \sum_{n=1}^{\infty} |W_{1n}|\,|T_n| \cos(\chi_{1n} + \xi_n), \\ u_1 &= \sum_{n=1}^{\infty} |U_{1n}|\,|T_n| \cos(\vartheta_{1n} + \xi_n) \\ \Psi_1 &= \sum_{n=1}^{\infty} |\Psi_{1n}|\,|T_n| \sin(\theta_{1n} + \xi_n), \end{aligned} \tag{2.58}$$

$$w_2 = \sum_{n=1}^{\infty} |W_{2n}||T_n| \cos(\chi_{2n} + \xi_n),$$

(2.58) [cont.]
$$u_2 = \sum_{n=1}^{\infty} |U_{2n}||T_n| \cos(\vartheta_{2n} + \xi_n),$$

$$\Psi_2 = \sum_{n=1}^{\infty} |\Psi_{2n}||T_n| \sin(\theta_{2n} + \xi_n)$$

where:

(2.59)
$$\chi_{1n} = \arg W_{1n}, \quad \chi_{2n} = \arg W_{2n}, \quad \vartheta_{1n} = \arg U_{1n}, \quad \vartheta_{2n} = \arg U_{2n},$$
$$\theta_{1n} = \arg \Psi_{1n}, \quad \theta_{2n} = \arg \Psi_{2n}, \quad \xi_n = \arg T_n.$$

2.6. Practical problems in the dynamical three-layer cylindrical shell with damping

Here let us consider the dynamic load $F_2(x,t)$, axially symmetric with respect to the axis of symmetry, moving in the direction of the x-axis at a constant velocity v^*, in a three-layer system containing a viscoelastic interlayer. The external layers of the three-layer structure are the cylindrical elastic shells connected by a visco-elastic interlayer.

The elastic shell is described by the Mindlin and the Kirchhoff–Love models and is simply supported at the ends. The viscoelastic interlayer possesses the characteristics of a homogenous continuous one-directional Winkler foundation and is described by the Voigt–Kelvin model.

Small transverse vibrations of a three-layer shell with damping are excited by the dynamical, axially-symmetric loading $F_2(x,t)$ non-uniform with respect to axis x, moving in direction of axis x at the constant velocity v^*, where

(2.60)
$$F_2(x,t) = P_2 \delta(x - v^* t).$$

The numerical results are presented for the following data:

$E_1 = E_2 = 2.1 \cdot 10^{11}$ Pa, $E = 10^8$ Pa, $r_1 = 0.02$ m, $r_2 = \{0.05, 0.06, 0.07\}$ m, $x_0 = 0.5 l$, $h = \{0.01, 0.02, 0.03\}$ m, $\nu_0 = 0.3$, $l = 0.5$ m, $c = 1.5\,\mathrm{Nsm}^{-2}$, $P_2 = 2 \cdot 10^4$ N, $v^* = 25\ \mathrm{ms}^{-1}$.

In order to determine the Fourier Φ_n coefficient, the following initial conditions have been assumed:

(2.61)
$$w_{01} = A_{s1} \sin\left(\frac{\pi x}{l}\right), \qquad w_{02} = A_{s2} \sin\left(\frac{\pi x}{l}\right),$$
$$A_{s1} = 0.008 \text{ cm}, \qquad A_{s2} = 0.01 \text{ cm}.$$

Some results concerning the time-dependence of the distributions of free vibrations w_1, w_2 for external layers I and II in the system of two cylindrical shells connected by a viscoelastic interlayer for $x_0 = 0.5\,l$, are presented in Fig. 2. The thicknesses of the internal and external layers the same.

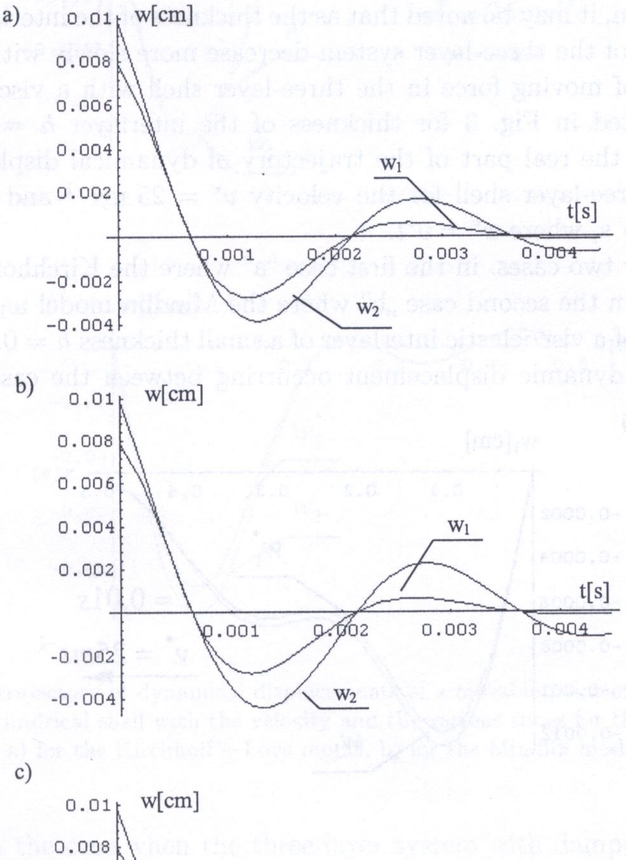


FIG. 2. Free vibrations of the three-layer system of two cylindrical shells coupled by a viscoelastic interlayer for $X_0 = 0.5l$ and various thicknesses of the interlayers: a) $r_1 = 0.02$ m, $r_2 = 0.05$ m, $h = 0.01$ m; b) $r_1 = 0.02$ m, $r_2 = 0.06$ m, $h = 0.02$ m; c) $r_1 = 0.02$ m, $r_2 = 0.07$ m, $h = 0.03$ m.

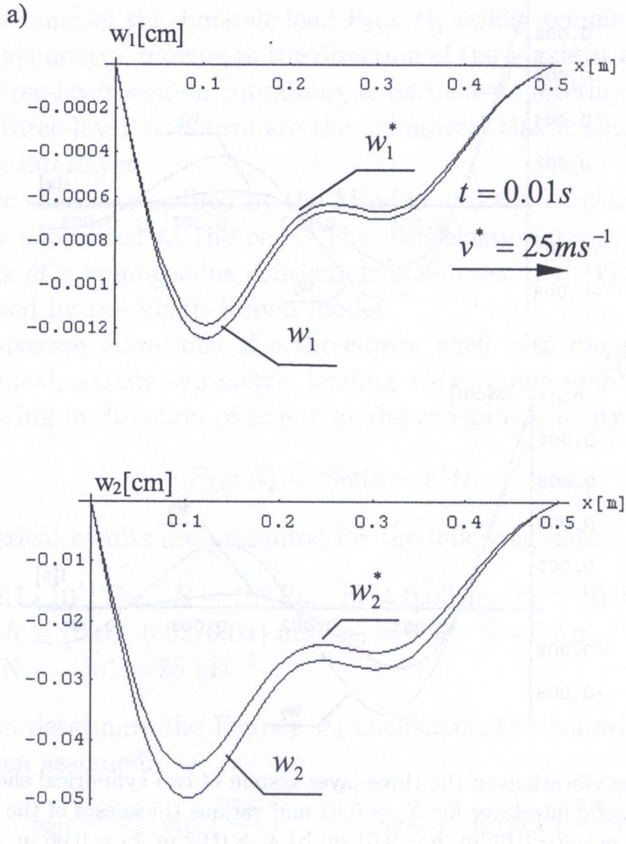
Free vibrations of the three-layer cylindrical shell for the thickness of the interlayer $h = 0.01$ m shown in Fig. 2a reach the values by approximately 5% smaller than those in Fig. 2b for thickness $h = 0.02$ m. The free vibrations of the three-layer cylindrical shell presented in Fig. 2b reach the values by approximately 4% smaller than those in Fig. 2c for thickness $h = 0.03$ m.

In conclusion, it may be noted that as the thickness of the interlayer increases, free vibrations of the three-layer system decrease more slowly with time t.

The effect of moving force in the three-layer shell with a viscoelastic interlayer is presented in Fig. 3 for thickness of the interlayer $h = 0.01$ m. The diagrams show the real part of the trajectory of dynamical displacements of a point of the three-layer shell for the velocity $v^* = 25\ \mathrm{ms}^{-1}$ and various times $t = \{0.01, 0.02\}$ s, where $x^* = v^*t$.

We consider two cases: in the first case "a" where the Kirchhoff–Love model w_1^*, w_2^* occurs, in the second case „b" where the Mindlin model w_1, w_2 occurs.

In the case of a viscoelastic interlayer of a small thickness $h = 0.01$ m (Fig. 3), a difference of dynamic displacement occurring between the cases I and II is



[FIG. 3a]

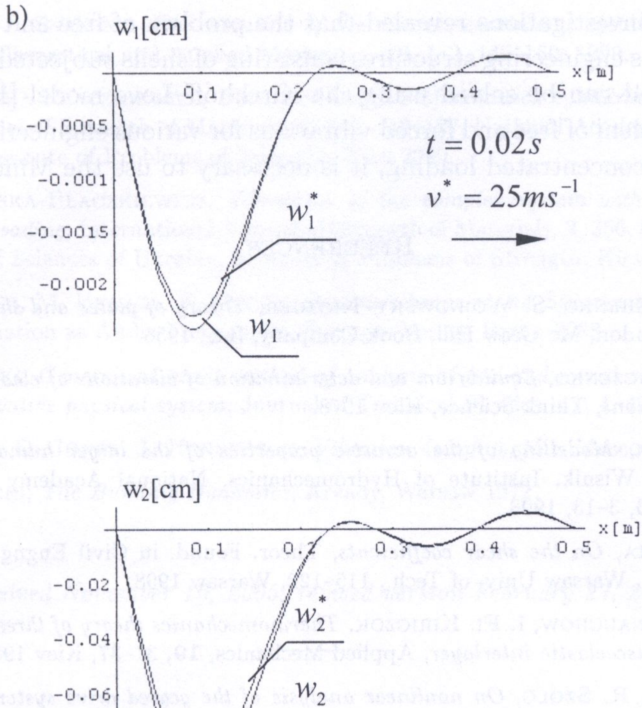


FIG. 3. The trajectory of dynamical displacements of a movable point of the viscoelastic three-layer cylindrical shell with the velocity and the various times for the moving force: a) for the Kirchhoff's–Love model, b) for the Mindlin model.

significant. In the case when the three-layer system with damping is loaded by a moving force (Fig. 3), the amplitudes of forced vibrations for the viscoelastic three-layer cylindrical shell with the Mindlin model w_1, w_2 reach the values approximately by 11% larger than the amplitudes of forced vibrations for the cylindrical shell when the Kirchhoff-Love model w_1^*, w_2^* is used.

3. CONCLUSIONS

The analytical method presented in this paper can be applied to solutions of free and forced vibrations of various engineering structures consisting of shells connected by viscoelastic constraints, subject to dynamical, axially symmetric loading $F_2(x,t)$, non-uniform in relation to axis x and moving at the constant velocity v^*.

Numerical investigations revealed that the problem of free and forced vibrations for various engineering structures consisting of shells subjected to the action of dynamic load can be solved using the Kirchhoff–Love model [18]. When we consider a problem of free and forced vibrations for various engineering structures under moving concentrated loading, it is necessary to use the Mindlin model.

References

1. S. P. Timoshenko, S. Woinowsky–Krygier, *Theory of plates and shells,* New York, Toronto, London, Mc Graw Hill, Book Company, Inc., 1956.

2. W. T. Grinczenko, *Equilibrium and determination of vibrations of elastic bodies of the final dimensions,* Think Science, Kiev 1978.

3. A. Borisuk, *Modelling of the acoustic properties of the larger human blood vessel,* J. Acoustic Wisnik, Institute of Hydromechanics, National Academy of Sciences of Ukraine, **1**, 3, 3–13, 1998.

4. G. Jemielita, *On the shear coefficients,* Theor. Found. in Civil Engng., 6, Faculty of Civil Engng., Warsaw Univ. of Tech., 115–122, Warsaw 1998.

5. W. G. Karnauchow, I. Fi. Kiriczok, *Thermomechanics theory of three-layer isotropic shells with viscoelastic interlayer,* Applied Mechanics, **19,** 31–37, Kiev 1983.

6. R. Bogacz, R. Szolc, *On nonlinear analysis of the geared drive systems by means of the wave method,* Journal of Theoretical and Applied Mechanics, **31**, 2, 393-401, 1993.

7. W. Szcześniak, *The selection of problems of beams and shells subjected to inertial moving load,* Building Engineering, 132, Warsaw Univ. of Tech., 1994.

8. W. Szcześniak, *Vibrations of the plate with mean thikness under the moving loads,* J. Engng. Trans., **35**, 3, 37–53, 1985.

9. N. D. Pankratova, B. Nikolaev, E. Świtoński, *Non-axisymmetrical deformation of flexible rotational shells in classical and improved statement,* Journal of Engineering Mechanics, **3**, 2, 89–96, Brno 1996.

10. Ja. M. Grigorenko, *Solution of problems of theory shells by the methods of numerical analysis,* **20**, 3–22, Applied Mechanics 1984.

11. T. Lewińki, J. J. Telega, *Plates, laminates and shells, Asymptotic analysis and homogenization,* World Scientific, Series on Advances in Mathematics for Applied Sciences, 52, Singapore, New Jersey, London, Hong Kong 2000.

12. R. A. Di Taranto, J. R. McGraw, *Vibratory bending of damped laminated plates,* Journal of Engineering for Industry, **91**, 1081–1090, Transactions of the American Society of Mechanical Engineers 1969.

13. W. Kurnik, A. Tylikowski, *Mechanics of laminated elements,* Warsaw Univ. of Tech., Warsaw 1997.

14. M. Woźniak, *Railway embankment as the building foundation, Mathematical Modelling,* Scientific Treatises and Monograhs, SGGW-AR, Warsaw 1991.

15. F. Tse, I. Morse, R. Hinkle, *Mechanical vibrations, theory and applications,* Allyn and Bacon, Boston 1978.

16. J. Niziol, J. Snamina, *Free vibration of the discrete-continuous system with damping,* Journal of Theoretical and Applied Mechanics, **28**, 1–2, 149–160, 1990.

17. K. Cabańska-Płaczkiewicz, *Dynamics analysis of sandwich cylindrical shell,* International Journal of Strength of Materials, **4**, 346, 119–127, National Academy of Sciences of Ukraine, Institute of Problems of Strength, Kiev 2000.

18. K. Cabańska-Płaczkiewicz, *Vibrations of the complex system with damping under dynamical loading,* International Journal of Strength of Materials, **2**, 356, 82–101, National Academy of Sciences of Ukraine, Institute of Problems of Strength, Kiev 2002.

19. Kl. Cremer, M. Heckel, E. Ungar, *Structure-borne sound,* Structural Vibrations and Sound Radiation at Audio Frequencies, Springer-Verlag, Berlin 1988.

20. J. Cabański, *Generalized exact method of analysis of free and forced oscillations in the non-conservative physical system,* Journal of Technical Physics, **41**, 4, 471–481, 2000.

21. D. Nashif, D. Jones, J. Henderson, *Vibration damping,* Mir, Moscow 1988.

22. W. Nowacki, *The Building Dynamics,* Arkady, Warsaw 1972.

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