ENGINEERING TRANSACTIONS • *Engng. Trans.* • **63**, 3, 273–296, 2015
Polish Academy of Sciences • Institute of Fundamental Technological Research (IPPT PAN)
National Engineering School of Metz (ENIM) • Poznan University of Technology

An Alternative Approach to Initial Stability Analysis of Kirchhoff Plates Resting on Internal Supports by the Boundary Element Method

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An initial stability of Kirchhoff plates supported on boundary and resting on internal supports is analysed in this paper. The internal supports are understood to be part of a plate surface or a line belonging to the plate. The proposed approach avoids Kirchhoff forces at the plate corner and equivalent shear forces at the plate boundary. Two unknown and independent variables are always considered at a boundary element node depending on the type of a plate edge such as the shear force and bending moment for a clamped edge, and the shear force and angle of rotation in normal direction for a simply-supported edge. For a free edge, the deflection and angle of rotation in normal direction are considered as two independent variables with additional angle of rotation in tangent direction which depends on boundary deflections. The two governing integral equations are derived using Betti's theorem. These equations have the form of boundary-domain integral equations. The constant type of boundary element is used. The singular and non-singular formulations of the boundary-domain integral equations with one and two collocation points associated with a single boundary element located slightly outside of a plate edge are presented. To establish a plate curvature by double differentiation of the basic boundary-domain integral equation, the plate domain is divided into rectangular subdomains associated with suitable collocation points. According to the alternative approach, a plate curvature is also established by considering three collocation points located in close proximity to each other along a line parallel to one of the two axes of global coordinate system and establishment of appropriate difference operators.

Key words: boundary element method, Kirchhoff plates, initial stability, fundamental solution.

1. Introduction

The boundary element method (BEM) can be applied to a wide range of engineering analyses of structures. Burczyński [1] described in a comprehensive manner the boundary element method and its application in a variety of fields, the theory of elasticity together with appropriate solutions and basic types of boundary elements. The main advantage of BEM is its relative simplicity of formulating and solving problems of the potential theory and the theory of elasticity. The application of the boundary element method to a plate analysis has particular advantages. Many authors used BEM to solve static, dynamic and initial stability problems of thin plates. There are well-known publications of Altiero and Sikarskie [2], Bèzine and Gamby [3], Stern [4] and Hartmann and Zotemantel [5] who applied BEM to solve thin plate bending problem. Abdel-Akher and Hartley [6] presented evaluation of boundary and boundary-domain integrals of fundamental functions used in plate analysis. A number of contributions devoted to analysis of plates were presented by Debbih [7, 8], Beskos [9], Wen, Aliabadi and Young [10], Katsikadelis [11, 12], Katsikadelis and Yotis [13], Katsikadelis, Sapountzakis and Zorba [14], Katsikadelis and Kandilas [15], Katsikadelis and Sapountzakis [16]. Wrobel and Aliabadi [17] described application of BEM to a thick plate analysis together with procedures for calculating singular and hypersingular integrals in broad aspect. A very interesting approach was presented by Litewka and Sygulski [18, 19] who applied the [20] fundamental solutions by Ganowicz [20] to a static analysis of Reissner's plates. Noteworthy is the publication by Shi [21] who applied a BEM formulation to vibration and initial stability problem of orthotropic thin plates. Ptaszny [22] applied the fast multipole boundary element method to the analysis of plates with many holes. Rashed [23] applied the coupled BEM – flexibility force method to static analysis of thin plates with internal column supports. The major drawback of this approach is the necessary condition of boundary supports, which satisfies kinematic constraints. In order to simplify the calculation procedures Guminiak and Sygulski [24] proposed a modified formulation of the boundary integral equation for a thin plate. This approach was applied to static, dynamic and stability analysis of thin plates and it is presented together with a number of numerical examples in several papers, e.g., [25–30]. Guminiak [31] applied the difference equation model of establishment of curvatures connected to the aforementioned modified BEM approach to solve initial stability problem of thin plates providing also the in-depth review of literature devoted to the BEM application in plate analysis. Myślecki [32, 33] proposed BEM to static analysis of plane girders and BEM combined with approximate fundamental solutions for a problem of plate bending resting on elastic foundation. The author used a non-singular approach of boundary integral equations wherein the derivation of the second boundary integral equation was executed for additional collocation points located outside of a plate domain. The same approach of derivation of the boundary integral equation was applied by Myślecki and Oleńkiewicz [34, 35] to solve the free vibration problem of thin plates. Works by Katsikadelis [36, 37] in which BEM was applied to a wide range of engineering analyses of plates are particularly noteworthy. In this work, the concept of the analog equation method (AEM) is presented as a tool that allows to fully overcome the main drawback of direct BEM, namely its limitation only to linear problems. The AEM is based on the principle of the analog equation of Katsikadelis for differential equations [38]. This concept was established to analyse plate buckling by Nerantzaki and Katsikadelis [39] and Chinnaboon, Chucheepsakul and Katsikadelis [40]. Similarly, Babouskos and Katsikadelis [41, 42] solved the problem of flutter instability of damped plate subjected to a conservative and non-conservative loading. In the present paper, the analysis of initial stability of internally supported thin plates by the direct version of BEM will be presented. The analysis will focus on the modified, simplified [31] formulation of thin plate bending. The Bèzine [3] technique will be applied to introduce internal supports and to establish the vector of curvatures at the internal collocation points.

2. Modelling of internal supports

Internal constraints can have the character of supports at selected points, column or continuous linear supports. When using direct Bèzine technique it is necessary to expand two boundary-domain integral equations [3] to include additional elements where the unknown values are suitable reactions such as concentrated forces (Fig. 1), forces distributed over the column cross-sections and distributed along the continuous linear constraints. The internal column support can be modelled as a surface with one collocation point and constant distribution of reaction (Figs. 2a, 2b and 2c). If the column support has large dimensions in comparison to plate dimensions, several subsurfaces can be introduced to the column surface (Fig. 2d). To calculate elements of the characteristic matrix, it is necessary to integrate suitable fundamental functions on the column surface or subsurfaces. In the case of the column of arbitrary shape (Figs. 2b

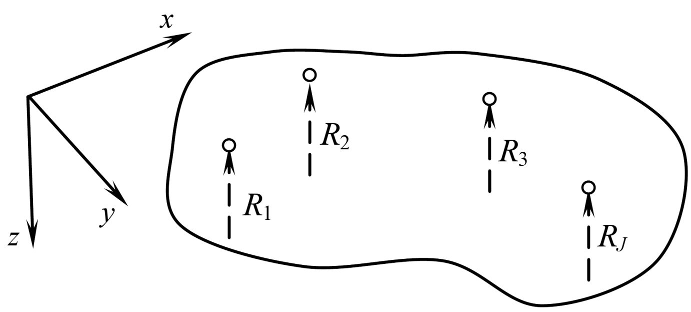

Fig. 1. A plate internally supported at selected points.

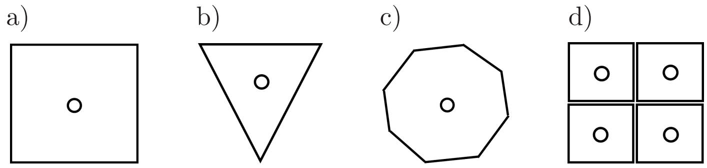

Fig. 2. Examples of the column (plane) supports.

and 2c) the formulae derived by Abdel-Akher and Hartley [6] can be used. The internal linear continuous supports can be modelled as a set of sections (elements) of constant type (Fig. 3). Because the fundamental solution for a thin plate has a singularity of the second order, the collocation point of internal single element can be located at its centre. Using another approach, the internal continuous supports can be treated as a column rectangular support with one edge dimension much smaller than the other one (Fig. 4).

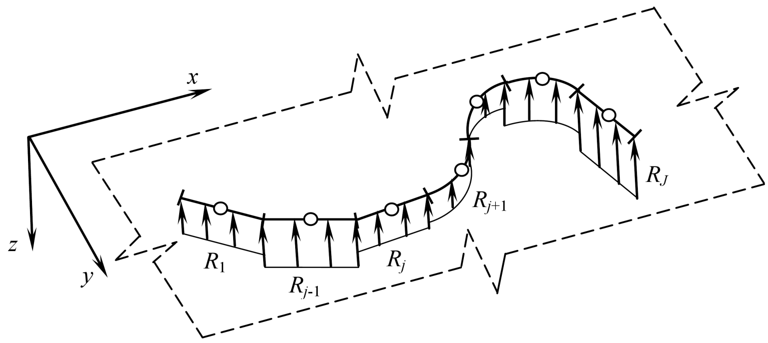

Fig. 3. A plate resting on linear continuous internal supports.

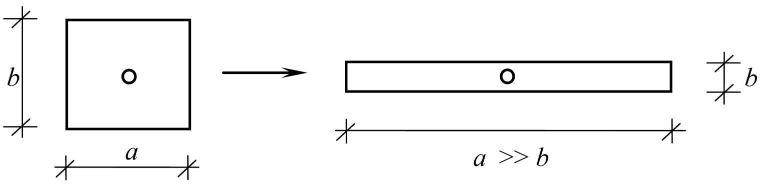

Fig. 4. Internal continuous supports: single element of constant type.

3. Integral formulation of plate bending and initial stability problem considering internal supports

The governing differential equation of plate initial stability has the form [43, 44]:

$$D \cdot \nabla^4 w = -\overline{p}, \tag{3.1}$$

where $w = w(x, y)$ is the unknown function of the plate deflection and $\overline{p}$ is the equivalent load that has the form:

$$\overline{p} = N_x \cdot \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \cdot \frac{\partial^2 w}{\partial x \partial y} + N_y \cdot \frac{\partial^2 w}{\partial y^2}. \tag{3.2}$$

In the majority of contributions devoted to the application of BEM to the thin (Kirchhoff) plate theory, derivation of the boundary integral equation involves the known boundary variables of the classical plate theory, i.e., the shear force and the concentrated corner forces. Thus, on the plate boundary there are considered two physical quantities such as the equivalent shear force V_n, reaction at the plate k-th corner R_k, the bending moment M_n, corner concentrated forces and two geometric variables: the displacement w_b and the angle of rotation in the normal direction φ_n. The boundary integral equation can be derived using Betti's theorem. Two plates are considered, an infinite plate subjected to the unit concentrated force and a real one subjected to the real in-plane loadings N_x, N_{xy} and N_y. The plate bending problem is described in a unique way by two boundary-domain integral equations. The first equation has the form [3, 36, 37]:

$$\begin{aligned} c(\mathbf{x}) \cdot w(\mathbf{x}) &+ \int_\Gamma [V_n^*(\mathbf{y},\mathbf{x}) \cdot w_b(\mathbf{y}) - M_n^*(\mathbf{y},\mathbf{x}) \cdot \varphi_n(\mathbf{y})] \cdot d\Gamma(\mathbf{y}) \\ - \sum_{k=1}^{K} R^*(k,\mathbf{x}) \cdot w(k) &= \int_\Gamma [V_n(\mathbf{y}) \cdot w^*(\mathbf{y},\mathbf{x}) - M_n(\mathbf{y},\mathbf{x}) \cdot \varphi_n^*(\mathbf{y},\mathbf{x})] \cdot d\Gamma(\mathbf{y}) \\ &- \sum_{k=1}^{K} R_k \cdot w^*(k,\mathbf{x}) - \int_{\Omega_r} q_r \cdot w^*(r,\mathbf{x}) \cdot d\Omega_r - \int_{\Gamma_l} q_l \cdot w^*(l,\mathbf{x}) \cdot d\Gamma_l \\ &+ \int_\Omega \left(N_x \cdot \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \cdot \frac{\partial^2 w}{\partial x \partial y} + N_y \cdot \frac{\partial^2 w}{\partial y^2} \right) \cdot w^*(\mathbf{y},\mathbf{x}) \cdot d\Omega(\mathbf{y}), \end{aligned} \tag{3.3}$$

where the fundamental solution of this biharmonic equation

$$\nabla^4 w^*(\mathbf{y},\mathbf{x}) = \frac{1}{D} \cdot \delta(\mathbf{y},\mathbf{x}) \tag{3.4}$$

is the free-space Green's function given as

$$w^*(\mathbf{y},\mathbf{x}) = \frac{1}{8\pi D} \cdot r^2 \cdot \ln(r) \tag{3.5}$$

for a thin isotropic plate, $r = |\mathbf{y} - \mathbf{x}|$, δ is the Dirac delta function, $D = \frac{E\,h^3}{12(1-v^2)}$ is the plate stiffness, $\mathbf{x}$ is the source point and $\mathbf{y}$ is the field point. The coefficient $c(\mathbf{x})$ is taken as

$c(\mathbf{x}) = 1$, when $\mathbf{x}$ is located inside the plate domain,
$c(\mathbf{x}) = 0.5$, when $\mathbf{x}$ is located on the smooth boundary,
$c(\mathbf{x}) = 0$, when $\mathbf{x}$ is located outside the plate domain.

The second boundary-domain integral equation can be obtained by replacing the unit concentrated force $P^* = 1$ by the unit concentrated moment $M_n^* = 1$, which is equivalent to the differentiation of the first boundary integral equation (3.3) with respect to the coordinate n at a point $\mathbf{x}$ belonging to the plate domain, letting this point approach the boundary and taking n to coincide with the normal [3, 36, 37]

$$\begin{aligned}(3.6)\quad c(\mathbf{x})\cdot\varphi_n(\mathbf{x}) + \int_\Gamma \left[\overline{V}_n^*(\mathbf{y},\mathbf{x})\cdot w_b(\mathbf{y}) - \overline{M}_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})\right]\cdot d\Gamma(y)\\
- \sum_{k=1}^{K}\overline{R}^*(k,\mathbf{x})\cdot w(k) = \int_\Gamma \left[V_n(\mathbf{y})\cdot\overline{w}^*(\mathbf{y},\mathbf{x}) - M_n(\mathbf{y})\cdot\overline{\varphi}_n^*(\mathbf{y},\mathbf{x})\right]\cdot d\Gamma(\mathbf{y})\\
- \sum_{k=1}^{K} R_k\cdot\overline{w}^*(k,\mathbf{x}) - \int_{\Omega_r} q_r\cdot\overline{w}^*(r,\mathbf{x})\cdot d\Omega_r - \int_{\Gamma_l} q_l\cdot\overline{w}^*(l,\mathbf{x})\cdot d\Gamma_l\\
+ \int_\Omega \left(N_x\cdot\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y} + N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot\overline{w}^*(\mathbf{y},\mathbf{x})\cdot d\Omega(\mathbf{y}),\end{aligned}$$

where

$$\begin{aligned}&\left\{\overline{V}_n^*(\mathbf{y},\mathbf{x}),\overline{M}_n^*(\mathbf{y},\mathbf{x}),\overline{R}^*(\mathbf{y},\mathbf{x}),\overline{w}^*(\mathbf{y},\mathbf{x}),\overline{w}^*(r,\mathbf{x}),\overline{w}^*(l,\mathbf{x}),\overline{\varphi}_n^*(\mathbf{y},\mathbf{x})\right\}\\
&= \frac{\partial}{\partial n(\mathbf{x})}\left\{V_n^*(\mathbf{y},\mathbf{x}),M_n^*(\mathbf{y},\mathbf{x}),R^*(\mathbf{y},\mathbf{x}),w^*(\mathbf{y},\mathbf{x}),w^*(r,\mathbf{x}),w^*(l,\mathbf{x}),\varphi_n^*(\mathbf{y},\mathbf{x})\right\}.\end{aligned}$$

The second boundary-domain integral equation can be also derived by direct application of the boundary domain integral equation (3.3) to a new set of collocation points located on the same normal line outside the plate edge. This double collocation point approach was presented in the papers [32–35]. The detailed procedure for derivation of the fundamental solution, integral representation of the solution and the two boundary-domain integral equations are presented by KATSIKADELIS in [36, 37].

The plate bending problem can also be formulated in a modified, simplified way using integral representation of the plate biharmonic equation. Because the concentrated force at the corner is used only to satisfy the differential biharmonic equation of the thin plate, one can assume that it could be distributed along a plate edge segment close to the corner [31]. As a result, the boundary integral equations (3.3) and (3.6) will take the form [29–31]:

$$(3.7)\quad c(\mathbf{x})\cdot w(\mathbf{x})+\int\limits_{\Gamma}\left[T_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-M_{ns}^*(\mathbf{y},\mathbf{x})\cdot\frac{dw(\mathbf{y})}{ds}-M_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})\right]\cdot d\Gamma(\mathbf{y})$$
$$=\int\limits_{\Gamma}\left[\widetilde{T}_n(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\varphi_n^*(\mathbf{y},\mathbf{x})\right]\cdot d\Gamma(\mathbf{y})$$
$$-\int\limits_{\Omega_r} q_r\cdot w^*(r,\mathbf{x})\cdot d\Omega_r-\int\limits_{\Gamma_l} q_l\cdot w^*(l,\mathbf{x})\cdot d\Gamma_l$$
$$+\int\limits_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot w^*(\mathbf{y},\mathbf{x})\cdot d\Omega(\mathbf{y}),$$

$$(3.8)\quad c(\mathbf{x})\cdot\varphi_n(\mathbf{x})+\int\limits_{\Gamma}\left[\overline{T}_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-\overline{M}_{ns}^*(\mathbf{y},\mathbf{x})\cdot\frac{dw(\mathbf{y})}{ds}-\overline{M}_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})\right]\cdot d\Gamma(\mathbf{y})$$
$$=\int\limits_{\Gamma}\left[\widetilde{T}_n(\mathbf{y})\cdot\overline{w}^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\overline{\varphi}_n^*(\mathbf{y},\mathbf{x})\right]\cdot d\Gamma(\mathbf{y})$$
$$-\int\limits_{\Omega_r} q_r\cdot\overline{w}^*(r,\mathbf{x})\cdot d\Omega_r-\int\limits_{\Gamma_l} q_l\cdot\overline{w}^*(l,\mathbf{x})\cdot d\Gamma_l$$
$$+\int\limits_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot\overline{w}^*(\mathbf{y},\mathbf{x})\cdot d\Omega(\mathbf{y}),$$

where

$$\left\{\overline{T}_n^*(\mathbf{y},\mathbf{x}),\overline{M}_n^*(\mathbf{y},\mathbf{x}),\overline{M}_{ns}^*(\mathbf{y},\mathbf{x}),\overline{w}^*(\mathbf{y},\mathbf{x}),\overline{w}^*(r,\mathbf{x}),\overline{w}^*(l,\mathbf{x}),\overline{\varphi}_n^*(\mathbf{y},\mathbf{x})\right\}$$
$$=\frac{\partial}{\partial n(\mathbf{x})}\left\{T_n^*(\mathbf{y},\mathbf{x}),M_n^*(\mathbf{y},\mathbf{x}),M_{ns}^*(\mathbf{y},\mathbf{x}),w^*(\mathbf{y},\mathbf{x}),w^*(r,\mathbf{x}),w^*(l,\mathbf{x}),\varphi_n^*(\mathbf{y},\mathbf{x})\right\}$$

and

$$(3.9)\qquad \widetilde{T}_n(\mathbf{y})=T_n(\mathbf{y})+R_n(\mathbf{y}).$$

The expression (3.9) denotes the shear force for clamped and for simply-supported edges [31], where $\widetilde{T}_n(\mathbf{y})=V_n(\mathbf{y})$ is on the boundary far from the corner and $\widetilde{T}_n(\mathbf{y})=R_n(\mathbf{y})$ is on a small fragment of the boundary close to the corner. In the case of free edge one has to combine the angle of rotation in the tangent direction $\varphi_s(\mathbf{y})$ with the fundamental function $M^*_{ns}(\mathbf{y})$. Because the relation between $\varphi_s(\mathbf{y})$ and the deflection is known: $\varphi_s(\mathbf{y}) = \frac{dw(\mathbf{y})}{ds}$, the angle of rotation $\varphi_s(y)$ can be evaluated using the finite difference scheme of the deflection with two or more adjacent nodal values [31]. In this analysis, the employed finite difference scheme includes deflections at three adjacent nodes.

4. Construction of the set of algebraic equation

The plate boundary is discretized by constant elements. Three approaches of constructing the boundary integral equations, applied also in [31], are considered: the first, singular, where the collocation point is located exactly at a plate edge (Fig. 5),

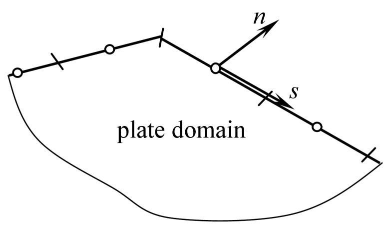

Fig. 5. Collocation point assigned to the boundary element of constant type.

the second, non-singular approach, where the boundary-domain integral equations can be formulated using one set of collocation points (Fig. 6a) and the third one, where two sets of collocation points (Fig. 6b) located outside of the plate boundary on the line normal to the plate edge are considered.

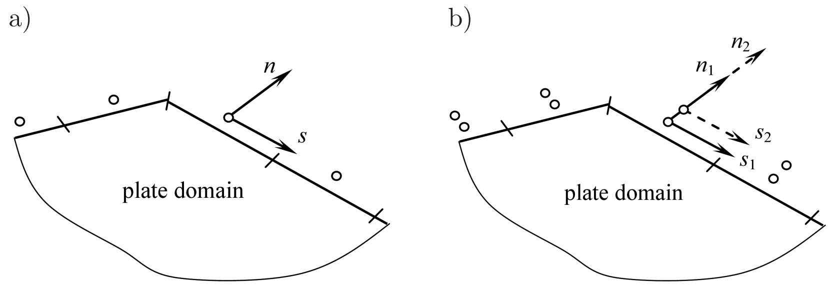

Fig. 6. a) one collocation point, b) two collocation points assigned to the boundary element of constant type.

It is assumed that a rectangular plate is compressed only by N_x forces. Then, in the boundary integral equations (3.8) and (3.9) only the part $N_x \cdot \left(\partial^2 w/\partial x^2\right)$ is present. The unknown variable in internal collocation points is the parameter

$\kappa = \partial^2 w/\partial x^2$, the plate curvature is in x direction [24, 31]. It is also assumed, that the plate has a regular shape without any holes. The distribution of the normal (in-plane) loading along the plate edge perpendicular to the x direction has constant value. The plate domain Ω is divided into the finite number of subdomains just to define the plate curvature at selected internal collocation points associated with these subdomains Ω_m. The normal loading $N_x = N_{cr}$ is constant along the single plate edge (Fig. 7).

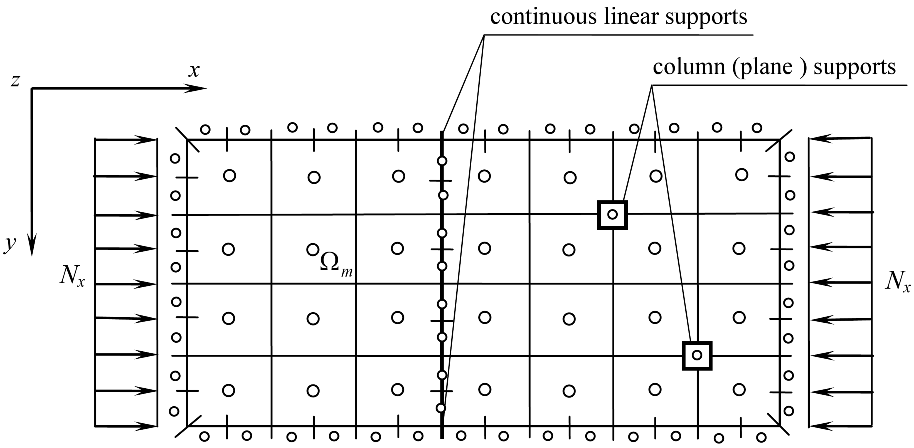

FIG. 7. Distribution of in-plane loading.

The set of algebraic equation can be written in the form (Fig. 8):

$$\begin{bmatrix} \mathbf{G_{BB}} & \mathbf{G_{BS}} & \mathbf{G_{Bq}} & -\lambda\cdot\mathbf{G_{B\kappa}} \\ \mathbf{\Delta} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{G_{qB}} & \mathbf{G_{qS}} & \mathbf{G_{qq}} & -\lambda\cdot\mathbf{G_{q\kappa}} \\ \mathbf{G_{\kappa B}} & \mathbf{G_{\kappa S}} & \mathbf{G_{\kappa q}} & -\lambda\cdot\mathbf{G_{\kappa\kappa}}+\mathbf{I} \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{B} \\ \boldsymbol{\varphi}_\mathbf{S} \\ \mathbf{q} \\ \boldsymbol{\kappa} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{4.1}$$

where $\lambda = N_{cr}$ and $\mathbf{B}$ is the boundary variables vector (column matrix) of the dimension $(2N \times 1)$, where N is the number of boundary nodes (or the number of elements of constant type), $\boldsymbol{\varphi}_\mathbf{S}$ is the vector (column matrix) of boundary angles of rotation in tangent direction depending on boundary deflections, this vector has the dimension $(S \times 1)$, where S is the number of boundary nodes (or the number of elements of constant type) along the free edge, $\mathbf{q}$ is the internal support reaction vector (column matrix) of the dimension $(L \times 1)$, where L is the number of internal supports constraints (subdomains Ω_L or linear sections Γ_L), $\boldsymbol{\kappa}$ is the vector of curvatures established at the selected internal collocation points associated with internal subdomains Ω_m, $\mathbf{G_{BB}}$ and $\mathbf{G_{BS}}$ are the matrices of the dimensions $(2N \times 2N)$ and $(2N \times S)$ respectively, grouping boundary integrals and depending on the type of plate boundary, where N is the number of boundary nodes (or the number of elements of constant type) and S is the number of boundary elements along free edge, $\mathbf{G_{Bq}}$ is the matrix of the dimension $(2N \times L)$ grouping integrals over the internal supports (column or continuous linear) subdomains, integrals over the Ω_r for column supports and over the Γ_l for continuous linear supports, $\mathbf{G_{B\kappa}}$ is the matrix of the dimension $(2N \times M)$ grouping integrals over the internal subdomains Ω_m, $\mathbf{\Delta}$ is the matrix grouping difference operators connecting angles of rotation in tangent direction with deflections of suitable boundary nodes if a plate has free edge. $\mathbf{G_{qB}}$ and $\mathbf{G_{qS}}$ are the matrices of the dimension $(L \times 2N)$ grouping the boundary integrals of the appropriate fundamental functions depending on the type of plate boundary, where L is the number of the internal collocation points associated with internal supports and N is the number of boundary nodes, $\mathbf{G_{qq}}$ is the matrix of the dimension $(L \times L)$ grouping integrals over the internal supports subdomains, $\mathbf{G_{q\kappa}}$ is the matrix of the dimension $(L \times M)$ grouping integrals over the internal subdomains Ω_m.

The fourth matrix Eq. $(4.1)_4$ is obtained by setting up the boundary integral equations for internal collocation points associated with internal subdomains Ω_m. According to the typical approach, in this equation, the plate curvature can be derived by double differentiation of the boundary integral Eq. (3.7) or by constructing a difference operator with respect to the central collocation point '1' (Fig. 9) belonging to each internal subsurface and using Eq. (3.7) in the unchanged form, $\mathbf{G_{\kappa B}}$ is the matrix of the dimension $(M \times 2N)$ grouping the boundary integrals of the second derivatives with respect to the coordinate x of the appropriate fundamental functions depending on the type of plate boundary, where M is the number of internal collocation points and N is the number of boundary nodes, $\mathbf{G_{\kappa S}}$ is the matrix of the dimension $(M \times S)$ grouping the boundary integrals of the second derivatives with respect to the coordinate x of the appropriate fundamental functions depending on the free edge, $\mathbf{G_{\kappa q}}$ is the matrix of the dimension $(M \times L)$ grouping integrals over the internal support (column or linear) subdomains, where $\Omega_L = \Omega_r$ for column supports and $\Omega_L = \Gamma_l$ for continuous linear supports, $\mathbf{G_{\kappa\kappa}}$ is the matrix of the dimension $(M \times M)$ grouping the integrals of the second derivatives with respect to the coordinate x over the internal subsurfaces $\Omega_m \in \Omega$.

The boundary integrals are calculated in the local coordinate system n_i, s_i and then transformed to the coordinate system n_k, s_k connected to the suitable boundary node k (Fig. 8)

All the matrices of boundary and domain integrals present in the matrix equation (4.1) are shown in Fig. 8. Similarly, the domain integrals are calculated in the local coordinate system assigned to the individual element (internal collocation point).

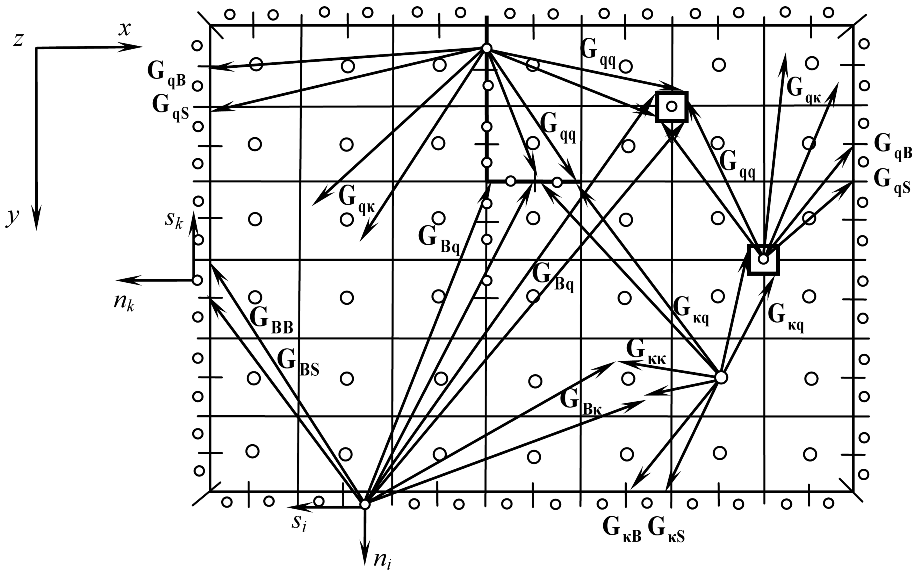

FIG. 8. Calculation of boundary and domain integrals.

In accordance with the simplified approach, the plate curvature can also be established by the addition of two internal collocation points ('2' and '3') [31]. Due to this concept it is necessary to write down three integral equations considering three collocation points ('1', '2' and '3') and to use Eq. (3.7) in unchanged form. These two approaches are illustrated in Fig. 9.

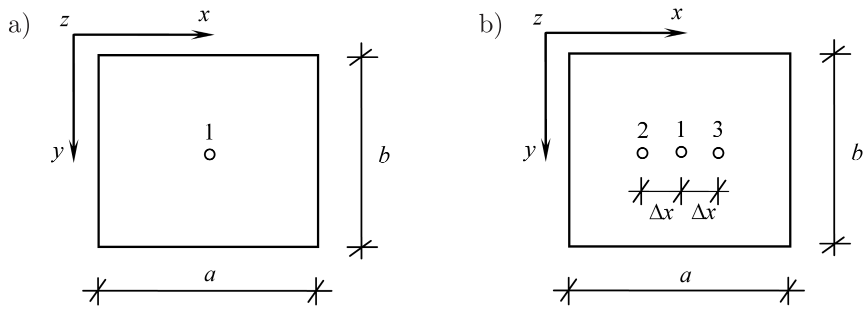

FIG. 9. Definition of curvature at the central collocation point '1' [31].

According to the second approach the plate curvature at the central point '1' is calculated by using the difference quotient

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}_x = \frac{\Delta^2 \mathbf{w}}{\Delta x^2} = \frac{\mathbf{w}_2 - 2 \cdot \mathbf{w}_1 + \mathbf{w}_3}{(\Delta x)^2}. \tag{4.2}$$

Hence, the elements of the matrices $\mathbf{G}_{\kappa\mathbf{B}}$, $\mathbf{G}_{\kappa\mathbf{S}}$, $\mathbf{G}_{\kappa\mathbf{q}}$ and $\mathbf{G}_{\kappa\kappa}$ can be evaluated using three boundary integral equations based only on the boundary integral equation (3.7). Elimination of the boundary variables $\mathbf{B}$, $\boldsymbol{\varphi}_{\mathbf{S}}$ and the internal support reaction vector $\mathbf{q}$ from matrix Eq. (4.1) leads to the standard eigenvalue problem:

$$\left\{\mathbf{A} - \widetilde{\lambda} \cdot \mathbf{I}\right\} \cdot \boldsymbol{\kappa} = \mathbf{0}, \tag{4.3}$$

where $\widetilde{\lambda} = 1/\lambda$,

$$\begin{aligned}\mathbf{A} = \mathbf{G}_{\boldsymbol{\kappa}\boldsymbol{\kappa}} - \widetilde{\mathbf{G}}_{\boldsymbol{\kappa}\mathbf{B}} \cdot \widetilde{\mathbf{G}}_{\mathbf{BB}}^{-1} \cdot \mathbf{G}_{\mathbf{B}\boldsymbol{\kappa}} - \left[\mathbf{G}_{\boldsymbol{\kappa}\mathbf{q}} - \widetilde{\mathbf{G}}_{\boldsymbol{\kappa}\mathbf{B}} \cdot \widetilde{\mathbf{G}}_{\mathbf{BB}}^{-1} \cdot \mathbf{G}_{\mathbf{Bq}}\right] \\ \cdot \left[\mathbf{G}_{\mathbf{qq}} - \widetilde{\mathbf{G}}_{\mathbf{qB}} \cdot \widetilde{\mathbf{G}}_{\mathbf{BB}}^{-1} \cdot \mathbf{G}_{\mathbf{Bq}}\right]^{-1} \cdot \left[\mathbf{G}_{\boldsymbol{\kappa}\mathbf{q}} - \widetilde{\mathbf{G}}_{\mathbf{qB}} \cdot \widetilde{\mathbf{G}}_{\mathbf{BB}}^{-1} \cdot \mathbf{G}_{\mathbf{B}\boldsymbol{\kappa}}\right]\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\widetilde{\mathbf{G}}_{\mathbf{BB}} &= \mathbf{G}_{\mathbf{BB}} + \mathbf{G}_{\mathbf{BS}} \cdot \boldsymbol{\Delta}, \\ \widetilde{\mathbf{G}}_{\mathbf{qB}} &= \mathbf{G}_{\mathbf{qB}} + \mathbf{G}_{\mathbf{qS}} \cdot \boldsymbol{\Delta}, \\ \widetilde{\mathbf{G}}_{\boldsymbol{\kappa}\mathbf{B}} &= \mathbf{G}_{\boldsymbol{\kappa}\mathbf{B}} + \mathbf{G}_{\boldsymbol{\kappa}\mathbf{S}} \cdot \boldsymbol{\Delta}.\end{aligned} \tag{4.5}$$

5. Modes of buckling

The elements of the eigenvector $\boldsymbol{\kappa}$ obtained in the solution of the standard eigenvalue problem (4.3) represent the plate curvatures. The set of the algebraic equations required to calculate the eigenvector $\mathbf{w}$ elements has the form:

$$\begin{bmatrix} \mathbf{G}_{\mathbf{BB}} & \mathbf{G}_{\mathbf{BS}} & \mathbf{G}_{\mathbf{Bq}} & \mathbf{0} \\ \boldsymbol{\Delta} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{G}_{\mathbf{qB}} & \mathbf{G}_{\mathbf{qS}} & \mathbf{G}_{\mathbf{qq}} & \mathbf{0} \\ \mathbf{G}_{\mathbf{wB}} & \mathbf{G}_{\mathbf{wS}} & \mathbf{G}_{\mathbf{wq}} & \mathbf{I} \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{B} \\ \boldsymbol{\varphi}_{\mathbf{S}} \\ \mathbf{q} \\ \mathbf{w} \end{Bmatrix} = \begin{Bmatrix} \lambda \cdot \mathbf{G}_{\mathbf{Bw}} \cdot \boldsymbol{\kappa} \\ \mathbf{0} \\ \lambda \cdot \mathbf{G}_{\mathbf{qw}} \cdot \boldsymbol{\kappa} \\ \lambda \cdot \mathbf{G}_{\mathbf{ww}} \cdot \boldsymbol{\kappa} \end{Bmatrix}. \tag{5.1}$$

In the set of Eqs. (5.1) the first, second and third Eqs. $(5.1)_1$, $(5.1)_2$ and $(5.1)_3$ are obtained from the first, second and third equations of (4.1) and the fourth equation $(5.1)_4$ yields from the boundary integral equations used to calculate plate deflections at internal collocation points. Elimination of the boundary variables $\mathbf{B}$, $\boldsymbol{\varphi}_{\mathbf{S}}$ and the internal support reaction vector $\mathbf{q}$ from Eq. (5.1) gives the elements of the displacement vector:

$$\begin{aligned}\mathbf{w} = \lambda \cdot \Big[\mathbf{G}_{\mathbf{ww}} - \widetilde{\mathbf{G}}_{\mathbf{wB}} \cdot \widetilde{\mathbf{G}}_{\mathbf{BB}}^{-1} \cdot \mathbf{G}_{\mathbf{Bw}} - \left[\mathbf{G}_{\mathbf{wq}} - \widetilde{\mathbf{G}}_{\mathbf{wB}} \cdot \widetilde{\mathbf{G}}_{\mathbf{BB}}^{-1} \cdot \mathbf{G}_{\mathbf{Bw}}\right] \\ \cdot \left[\mathbf{G}_{\mathbf{qq}} - \widetilde{\mathbf{G}}_{\mathbf{qB}} \cdot \widetilde{\mathbf{G}}_{\mathbf{BB}}^{-1} \cdot \mathbf{G}_{\mathbf{Bq}}\right]^{-1} \cdot \left[\mathbf{G}_{\mathbf{wq}} - \widetilde{\mathbf{G}}_{\mathbf{qB}} \cdot \widetilde{\mathbf{G}}_{\mathbf{BB}}^{-1} \cdot \mathbf{G}_{\mathbf{Bw}}\right]\Big] \cdot \boldsymbol{\kappa}\end{aligned} \tag{5.2}$$

and

$$\widetilde{\mathbf{G}}_{\mathbf{wB}} = \mathbf{G}_{\mathbf{wB}} + \mathbf{G}_{\mathbf{wS}} \cdot \boldsymbol{\Delta}. \tag{5.3}$$

6. Numerical examples

The initial stability problem for rectangular plates resting on internal column or linear continuous supports is considered. The considered plates have all the edges supported or are supported along two opposite edges. The in-plane loading N_x is acting along the supported edges. The critical value of the in-plane loading is computed. Each plate edge is divided by the boundary elements of constant type of the same length. Internal continuous linear supports are divided into sections (elements) of constant type of the same length. A column (plane support) has a square cross-section associated with one collocation point and the side length much smaller than the shorter side of the plate. The set of the internal collocation points in which the curvature vector $\boldsymbol{\kappa}$, associated with internal subsurfaces is established, is regular.

Quasi-diagonal terms of the matrix $\mathbf{G_{BB}}$ in Eqs. (4.1) and (5.1) are calculated analytically and remaining ones using the 12-point Gaussian quadrature. All the terms of the matrices $\mathbf{G_{Bq}}$, $\mathbf{G_{B\boldsymbol{\kappa}}}$, $\mathbf{G_{Bw}}$, $\mathbf{G_{qq}}$, $\mathbf{G_{q\boldsymbol{\kappa}}}$, $\mathbf{G_{\boldsymbol{\kappa}q}}$, $\mathbf{G_{\boldsymbol{\kappa}\boldsymbol{\kappa}}}$, $\mathbf{G_{qw}}$, $\mathbf{G_{wq}}$ and $\mathbf{G_{ww}}$ in Eqs. (4.1) and (5.1) are evaluated analytically. The rest of the terms in the matrices $\mathbf{G}_{ik}$ are calculated numerically by the 12-point Gaussian quadrature.

To compare the obtained results with the ones from [29, 30], the following material properties are assumed. For plates resting on internal plane (column) supports the Young's modulus is $E = 1.0$ kPa and the Poisson's ratio $\nu = 0.3$ and for plates resting on internal continuous linear supports the Young's modulus is $E = 30.0$ GPa and the Poisson's ratio $\nu = 0.167$.

The following notation is assumed:

BEM I – singular formulation of governing boundary-domain integral equations (3.7) and (3.8) with the second equation obtained by single differentiation of Eq. (3.7), the vector of curvatures is established by double differentiation of the first governing boundary-domain integral equation (3.7).

BEM II – non-singular formulation of governing boundary-domain integral equations (3.7) and (3.8), with the second equation (3.8) obtained by differentiation of Eq. (3.7), the vector of curvatures is established by double differentiation of the first governing boundary-domain integral equation (3.7). The collocation point of a single boundary element is located outside, near the plate edge. For any collocation point: $\varepsilon_1 = \widetilde{\delta}_1/d$ where $\widetilde{\delta}_1$ is distance of the collocation point from the plate edge and d is the boundary element length.

BEM III – non-singular formulation of governing boundary-domain integral equation (3.7) with the second boundary-domain integral equation obtained for the set of additional collocation points with the same fundamental solution w^*, the vector of curvatures is established by constructing the difference quotient (4.2) and the fundamental solution w^*. Location of two collocation points for any boundary element is determined by ε_1 and $\varepsilon_2 = \widetilde{\delta}_2/d$. For three collocation points belonging to each internal subdomain element: $\varepsilon_\Delta = \Delta x/a$.

FEM – regular finite element mesh, two types of elements: S4R (four nodes with three degree of freedom per node) and S8R (eight nodes with three degree of freedom per node) with reduced integration used in ABAQUS program assumed for comparative analysis.

The critical force N_{cr} is expressed using the non-dimensional term:

$$\widetilde{N}_{cr} = \frac{N_{cr}}{D} \cdot l_x \cdot l_y. \tag{6.1}$$

6.1. Example with the plate simply-supported on two opposite edges with two remaining free edges resting on two internal column supports under constant normal loading

Static and loading scheme is shown in the Fig. 10.

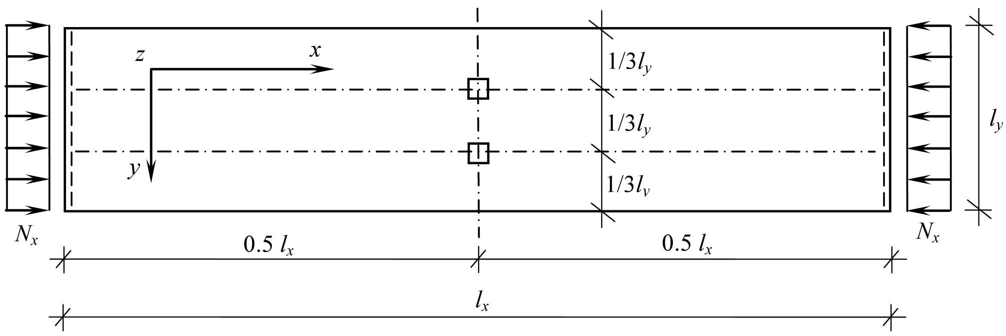

FIG. 10. The plate simply-supported on two opposite edges with two remaining free edges resting on two internal column supports under constant normal loading.

The plate dimensions are $l_y = 0.25 \cdot l_x$ and the internal column (plane) supports dimension (square $b \times b$) is $b = 0.02 \cdot l_y$. Each column surface is associated with one collocation point.

Two boundary and domain discretization patterns are adopted:

a) the number of boundary elements is 96, the number of internal collocation points is 144 and the internal subsurface dimension (square $a \times a$) is $a = 1/6 \cdot l_y$,

b) the number of boundary elements is 120, the number of internal collocation points is 400 and the internal subsurface dimension (square $a \times a$) is $a = 0.05 \cdot l_y$.

The results of calculation are presented in Tables 1–3. The influence of location of the internal collocation points on the critical force values using BEM III approach is presented in Tables 2 and 3. The first buckling mode is shown in Fig. 11.

Table 1. Critical force values. $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$, $\varepsilon_\Delta = 0.01$.

$\widetilde{N}_{cr}$	BEM II(a) [29]	BEM III(a) present	BEM II(b) [29]	BEM III(b) present	FEM S4R [29]	FEM S8R [29]
1	8.8260	9.4300	8.9410	9.7432	8.5580	8.9490
2	19.6310	20.7885	19.6740	21.2720	17.8870	17.5840
3	36.8010	39.1424	37.0280	40.2456	35.7170	34.6450

Table 2. Critical force values, solution BEM III(a) for different values of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\widetilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	9.4304	9.4300	9.4300	9.4289	9.4300
2	20.7870	20.7884	20.7885	20.7869	20.7884
3	39.1460	39.1426	39.1424	39.1438	39.1424

Table 3. Critical force values. solution BEM III(b) for different values of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\widetilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	9.7431	9.7432	9.7432	9.7437	9.7458
2	21.2721	21.2720	21.2720	21.2741	21.2812
3	40.2448	40.2455	40.2456	40.2527	40.2732

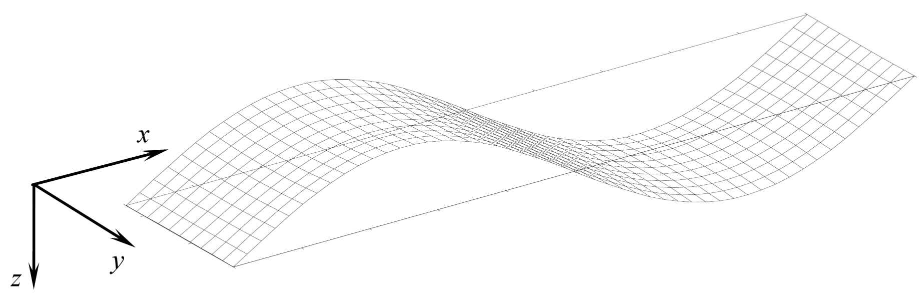

Fig. 11. The first buckling mode.

6.2. Example with the plate clamped on two opposite edges with two remaining free edges resting on two internal column supports under constant normal loading

Static and loading scheme is shown in Fig. 12. The plate properties were assumed identically as in Example 6.1.

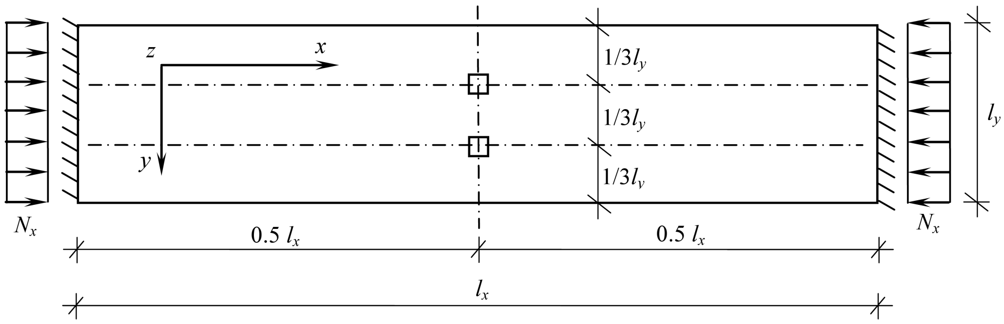

FIG. 12. The plate simply-supported on two opposite edges with two remaining free edges free resting on two internal column supports under constant normal loading.

The results of calculation are presented in Tables 4–6. The influence of location of internal collocation points on the critical force values using BEM III(a) and BEM III(b) approaches is presented in Tables 5 and 6. The first buckling mode is shown in Fig. 13.

Table 4. Critical force values. $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$, $\varepsilon_\Delta = 0.01$.

$\widetilde{N}_{cr}$	BEM II(a) [29]	BEM III(a) present	BEM II(b) [29]	BEM III(b) present	FEM S4R [29]	FEM S8R [29]
1	18.6948	20.3522	18.952	21.0271	18.0400	17.7350
2	37.4158	40.0875	37.669	41.2555	36.0710	34.9630
3	56.7005	60.5224	57.006	62.1079	55.6600	53.0770

Table 5. Critical force values. solution BEM III(a) for different values of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\widetilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	20.3504	20.3522	20.3522	20.3522	20.3522
2	40.0962	40.0875	40.0875	40.0875	40.0875
3	60.5097	60.5221	60.5224	60.5224	60.5228

Table 6. Critical force values. solution BEM III(b) for different values of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\widetilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	21.0270	21.0271	21.0271	21.0286	21.0340
2	41.2552	41.2553	41.2555	41.2616	41.2806
3	62.1090	62.1076	62.1079	62.1224	62.1641

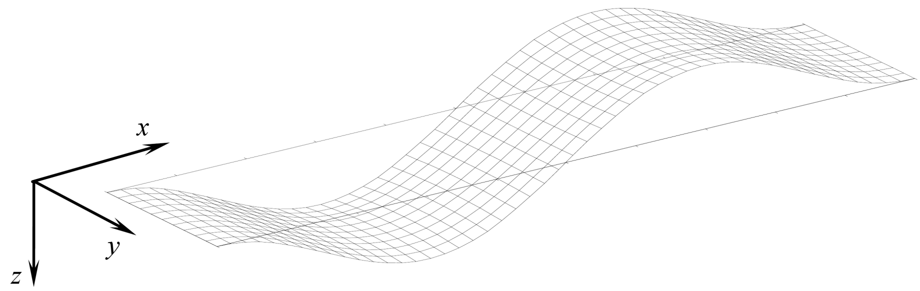

FIG. 13. The first buckling mode.

6.3. Example with the plate simply-supported on two opposite edges with two remaining free edges resting on linear continuous internal support under constant normal loading

Static and loading scheme is shown in the Fig. 14.

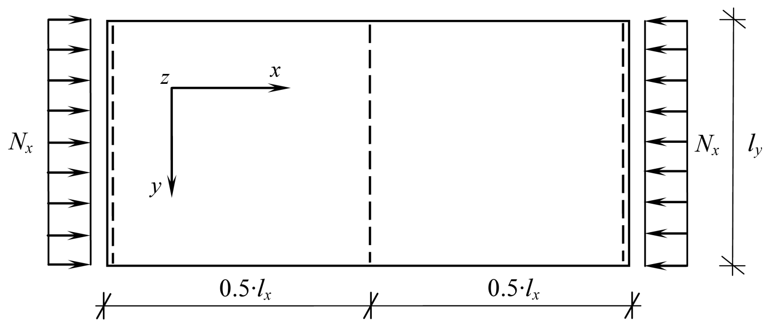

FIG. 14. The plate, simply-supported on two opposite edges with two free edges resting on internal continuous support under constant normal loading.

Each plate edge is divided into number of 40 boundary elements of the same length. The number of internal linear continuous elements of the same length is 40 and the number of internal subsurfaces used to describe the plate curvature is 200. The plate geometry is defined as: $l_x = 2.0 \cdot l_y = 20.0$ m, $h_p = 0.2$ m.

The results of calculation are presented in Tables 7 and 8. The influence of location of internal collocation points on critical force values using BEM III approach is presented in Table 8. The first buckling mode is shown in Fig. 15.

Table 7. Critical force values. $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$, $\varepsilon_\Delta = 0.01$.

$\widetilde{N}_{cr}$	BEM II [30]	BEM III present	FEM S4R [30]
1	19.3976	21.3618	19.4324
2	40.2226	43.4987	40.7566
3	58.6534	59.8541	58.3006

Table 8. Critical force values. Solution BEM III for different values of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\widetilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	21.4618	21.4618	21.4618	21.4596	21.4539
2	43.4985	43.4987	43.4987	43.4983	43.4993
3	59.8548	59.8548	59.8541	59.8353	59.7845

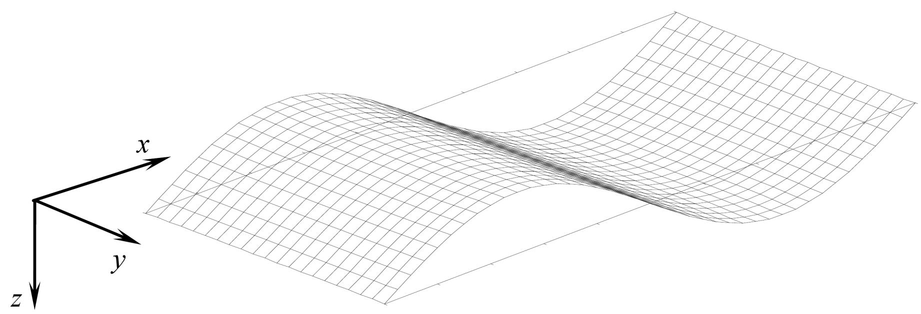

FIG. 15. The first buckling mode.

6.4. Example with the plate simply-supported on two opposite edges resting on two linear continuous internal supports under constant normal loading

Static and loading scheme is shown in Fig. 16. The material properties are assumed identically as in Example 6.3.

Each plate edge is divided into number of 45 boundary elements of the same length. The number of internal linear continuous elements of the same length is 40 and the number of internal subsurfaces used to describe the plate curvature is 300. The plate geometry is defined as: $l_x = 3.0 \cdot l_y = 30.0$ m, $h_p = 0.2$ m.

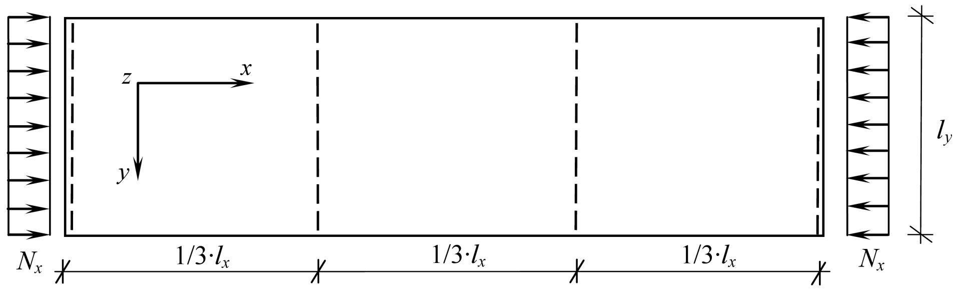

FIG. 16. The plate simply-supported on two opposite and diagonal edges resting on linear continuous internal support under constant normal loading.

The results of calculation are presented in Tables 9 and 10. The influence of location of internal collocation points on the critical force values using BEM III approach is presented in Table 10. The first buckling mode is shown in Fig. 17.

Table 9. Critical force values. $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$, $\varepsilon_\Delta = 0.01$.

$\widetilde{N}_{cr}$	BEM II present	BEM III present	FEM S4R
1	29.0968	32.4116	29.7456
2	43.8981	47.8172	44.0644
3	78.7263	85.0558	78.4790

Table 10. Critical force values. solution BEM III for different value of $\varepsilon\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\widetilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	32.4118	32.4115	32.4116	32.4152	32.4257
2	47.8162	47.8170	47.8172	47.8245	47.8457
3	85.0538	85.0553	85.0558	85.0776	85.1412

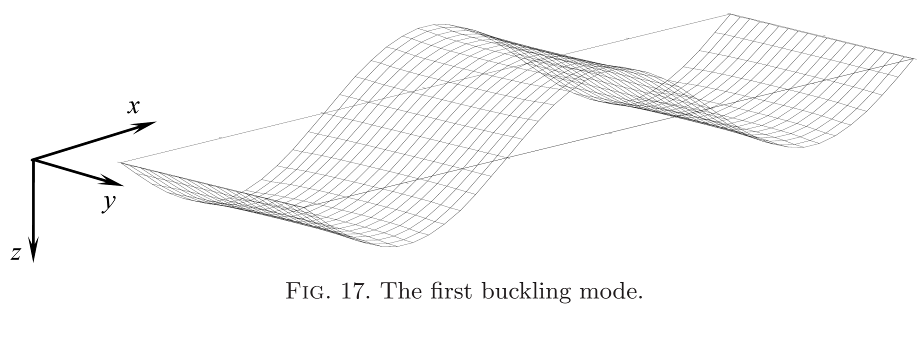

FIG. 17. The first buckling mode.

6.5. Example with the plate simply-supported on all edges resting on two linear continuous internal supports under constant normal loading

Static and loading scheme is shown in Fig. 18. The plate and material properties were assumed identically as in Example 6.4.

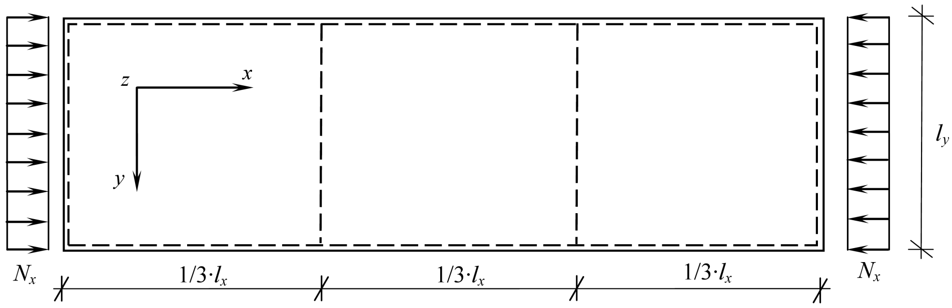

FIG. 18. The plate simply-supported on all edges resting on two linear continuous internal supports under constant normal loading.

The results of calculation are presented in Tables 11 and 12. The influence of location of internal collocation points on the critical force values using BEM III approach is presented in Table 12. The first buckling mode is shown in Fig. 19.

Table 11. Critical force values. $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$, $\varepsilon_\Delta = 0.01$.

$\widetilde{N}_{cr}$	BEM I present	BEM II present	BEM III present	FEM S4R
1	119.7662	119.7672	119.7701	121.2945
2	132.2487	132.2499	132.2535	132.1488
3	161.2000	161.2006	161.2066	163.0539

Table 12. Critical force values, solution BEM III for different values of $\varepsilon_\Delta = \Delta x/a$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.1$.

$\widetilde{N}_{cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	119.7721	119.7701	119.7701	119.7800	119.8105
2	132.2499	132.2535	132.2535	132.2703	132.3207
3	161.1999	161.2061	161.2066	161.2426	161.3524

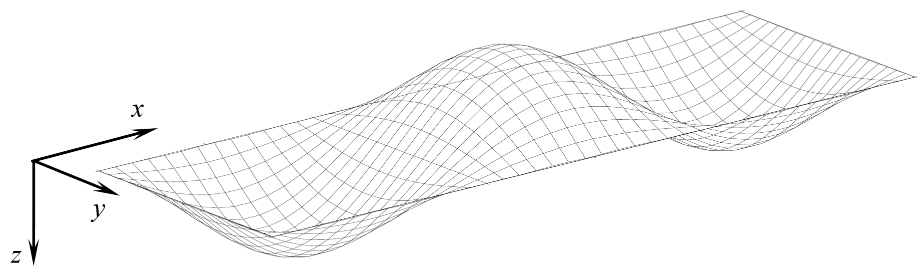

FIG. 19. The first buckling mode.

7. Conclusions

The initial stability of thin plates resting on internal supports was analysed using the boundary element method (BEM). The investigated problem was solved according to the modified and simplified approach, in which the boundary conditions are defined so that there is no need to introduce equivalent boundary quantities dictated by the boundary value problem for the biharmonic differential equation. The collocation versions of the BEM with singular and non-singular calculations of integrals were employed and the constant type of the boundary element was introduced. The Bèzine technique was used to establish the vector of internal support reaction forces and the vector of curvatures inside a plate domain. The plate domain was divided into rectangular subsurfaces which have the character of surface elements of the constant type. In the presented examples, the plates are subjected to in-plane constant loading. The loaded plate edges must be supported which is required in the proposed formulation of buckling analysis [31]. A significant increase in the number of boundary elements and internal subsurfaces does not lead to the significant improvement of the results of calculations, which was shown in the first two examples. In the first example some differences between the results of the critical force values obtained using BEM are observable. This may be due to the shape of the considered plate which may affect the conditioning of the matrix **A**, present in the matrix equation describing the standard eigenvalue problem. Similar differences for the FEM application with two types of finite elements are noticeable, too.

The solution for the critical force is stable for a large range of values $\varepsilon_\Delta = \Delta x/a$. According to the proposed approach, linear continuous supports must be located on the border between internal subdomains Ω_m and surfaces determining the column supports should be located at a necessary distance from the internal subdomains Ω_m. This is a limiting restriction but it does not exclude the application of the considering method. This inconvenience can be fully eliminated by the application of the analog equation method in combination with the classical BEM approach [36, 37].

The presented work relates entirely to the paper [31], in which the buckling problem of rectangular plates was investigated using BEM. The boundary element results obtained for the proposed concept of the thin plate bending problem considering internal linear continuous and plane constraints demonstrate the sufficient effectiveness and efficiency. The proposed method can be applied to the stability analysis of steel construction, e.g., bridge plane girders with orthogonal stiffeners.

Acknowledgment

The author thanks Prof. Ryszard Sygulski and Prof. John T. Katsikadelis for conducting and supervising the scientific and engineering research related to the numerical analysis of structures by BEM.

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Received November 4, 2014; accepted December 30, 2014.