ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSACTIONS • 38, 2, 163-176, 1990
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

ANALYSIS OF THE PLASTIC FLOW PROBLEM FOR HARDENING MATERIALS UNDER PLANE STRAIN CONDITIONS

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Plastic yielding problem is considered in the case of material obeying the kinematic and isotropic hardening rules, rotations of the principal stress directions (PSD) being taken into account. The solution is obtained by means of the slip-line theory under plane strain conditions. Plastic nonhomogeneity and anisotropy are described by the Huber-Mises condition and the hardening rules. An example is given, concerning the problem of bending of a strip weakened by two semi-circular notches; the result is compared with the case of rigid-perfectly plastic material.

1. Introduction

The plastic flow problem for hardening materials is described by a system of equations in which the stresses and kinematic fields are coupled by the strain-hardening laws. The attempts to decouple the system can be divided into two groups. In the first group one of the fields, e.g. velocity, is determined by an experiment, and the other one (stress field) can be found numerically [1,2,3,4].

In the second group the solutions are determined by means of numerical methods [5,6,7,8,9,10,11,18] and, in particular, by the iterative method [5,7,8,18]. The general idea of the method consists in replacing the hardening problem by a sequence of nonhomogeneous or anisotropic, perfectly plastic problems which may be solved by the method of characteristics.

In this paper solutions concerning the hardening material are obtained from the plane strain, slip-line theory by means of the numerical method proposed in [7,8,18]. The monotonic and cyclic bending of double-notched bar is presented as an example of the solutions proposed, and compared with that concerning perfectly plastic material.

2. Basic equations

Basic equations describing the plastic flow problem for a hardening material with a kinematic hypothesis under plane strain conditions are

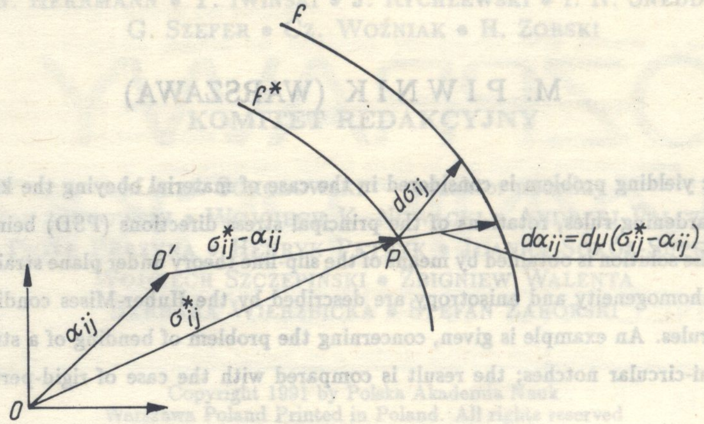

Fig. 1 . The Ziegler law of strain-hardening.

$$\sigma_{x,x} + \tau_{xy,y} = 0,$$
$$\tau_{xy,x} + \sigma_{y,y} = 0,$$
$$(2.1) \qquad f = \tfrac{1}{4}\{(\sigma_x - \alpha_x) - (\sigma_y - \alpha_y)\}^2 + (\tau_{xy} - \alpha_{xy})^2 - K_0^2 = 0,$$
$$V_{x,x} + V_{y,y} = 0,$$
$$\frac{2(\tau_{xy} - \alpha_{xy})}{(\sigma_x - \alpha_x) - (\sigma_y - \alpha_y)} = \frac{V_{x,y} + V_{y,x}}{V_{x,x} - V_{y,y}}$$

representing, respectively, the equations of equilibrium, the Huber-Mises yield condition and the equations of flow (plastic incompressibility and the flow rule associated with the Huber-Mises yield condition). Components of the symmetric tensor α satisfy the set of equations describing evolution of the yield condition.

$$(2.2) \qquad \alpha_{x,t} = \dot{\mu}(\sigma_x - \alpha_x) + 2\alpha_{xy}\omega_{xy} - \alpha_{x,x}V_x - \alpha_{x,y}V_y,$$

(2.2) [cont.]
$$\alpha_{y,t} = \dot{\mu}(\sigma_y - \alpha_y) + 2\alpha_{xy}\omega_{xy} - \alpha_{y,x}V_x - \alpha_{y,y}V_y,$$
$$\alpha_{x,y,t} = \dot{\mu}(\tau_{xy} - \alpha_{xy}) + (\alpha_x - \alpha_y)\omega_{xy} - \alpha_{xy,x}V_x - \alpha_{xy,y}V_y,$$

where

(2.3)
$$\omega_{xy} = \frac{1}{2}(V_{y,x} - V_{x,y})$$

is the component of the rotation tensor of PSD, and

(2.4)
$$\dot{\mu} = \frac{1}{2}\frac{c}{K_0^2}\left\{\left((\sigma_x^* - \alpha_x) - (\sigma_y^* - \alpha_y)\right)V_{x,x}^* + (V_{x,y}^* - V_{y,x}^*)(\tau_{xy}^* - \alpha_{xy})\right\},$$

where K_o, c are material constants $\sigma_x^*, \sigma_y^*, \tau_{xy}^*$ and V_x^*, V_y^* are the stress and velocity components at point P (Fig.1).

The set of equations (2.2) follows from the Jaumann derivative of tensor α [13],

(2.5)
$$\overset{\nabla}{\alpha}_{ij} = \dot{\alpha}_{ij} - \alpha_{ip}\omega_{pj} - \alpha_{jp}\omega_{pi},$$

where

(2.6)
$$\omega_{kl} = \frac{1}{2}(V_{l,k} - V_{k,l}),$$
$$\dot{\alpha}_{ij} = \alpha_{ij,t} + \alpha_{ij,k}V_k,$$

and the Ziegler rule [12] (Fig.1),

(2.7)
$$\overset{\nabla}{\alpha}_{ij} = \dot{\mu}(\sigma_{ij}^* - \alpha_{ij}).$$

Equations (2.1) are most easily handled after reformulation by introducing new stresses p and φ defined by

(2.8)
$$\sigma_x = p + K_0\cos 2\varphi + \frac{1}{2}(\alpha_x - \alpha_y),$$
$$\sigma_y = p - K_0\cos 2\varphi - \frac{1}{2}(\alpha_x - \alpha_y),$$
$$\tau_{xy} = K_0\sin 2\varphi + \alpha_{xy}.$$

Expressions (2.8) satisfy the Huber - Mises yield condition. After substituting (2.8) into (2.1) we have

(2.9)
$$P_{,x} - 2K_0\sin(2\varphi)\cdot\varphi_{,x} + 2K_0\cos(2\varphi)\varphi_{,y} = -(A_{,x} + \alpha_{xy,y}),$$
$$P_{,y} + 2K_0\cos(2\varphi)\cdot\varphi_{,x} + 2K_0\sin(2\varphi)\cdot\varphi_{,y} = A_{,y} - \alpha_{xy,x},$$
$$V_{x,x} + V_{y,y} = 0,$$

$$\frac{V_{x,y} + V_{y,x}}{V_{x,x} - V_{y,y}} = \tan(2\varphi),$$

where

$$A = \frac{1}{2}(\alpha_x - \alpha_y).$$

Assuming that a given instant of time distributions of components $\alpha_x, \alpha_y, \alpha_{xy}$ are known (integration of (2.2) along the trajectory of a particle), Eqs.(2.9) are transformed into characteristic coordinates in a similar way as in the case of a perfectly plastic material [15]. The equations of the characteristic are

$$\begin{aligned} y_{,x} &= \tan(\varphi - 45^o), \quad \text{along } \beta \text{ lines}, \\ y_{,x} &= \tan(\varphi + 45^o), \quad \text{along } \gamma \text{ lines}. \end{aligned} \tag{2.10}$$

The stress equations along the characteristic directions are

$$\begin{aligned} dp - 2K_0 d\varphi &= (\alpha_{xy,x} - A_{,y})dy - (A_{,x} + \alpha_{xy,y})dx, \quad \text{along } \beta, \\ dp + 2K_0 d\varphi &= (A_{,x} + \alpha_{xy,y})dx - (A_{,y} + \alpha_{xy,x})dy, \quad \text{along } \gamma. \end{aligned} \tag{2.11}$$

The velocity equations along the characteristics are

$$\begin{aligned} dV_x + dV_y \tan(\varphi - 45^o) &= 0 \quad \text{along } \beta, \\ dV_x + dV_y \tan(\varphi + 45^o) &= 0 \quad \text{along } \gamma. \end{aligned} \tag{2.12}$$

The velocity and characteristics equations for a perfectly plastic and isotropic hardening material [8,15,18] are identical. Stress equations for a perfectly plastic material along the characteristics are [15]

$$\begin{aligned} dp - 2K_0 d\varphi &= 0 \quad \text{along } \beta, \\ dp + 2K_0 d\varphi &= 0 \quad \text{along } \gamma \end{aligned} \tag{2.13}$$

and for the isotropic hardening material [8,14]

$$\begin{aligned} dp - 2K_0 d\varphi &= K_{,x}dy - K_{,y}dx, \quad \text{along } \beta, \\ dp + 2K_0 d\varphi &= K_{,y}dx - K_{,x}dy, \quad \text{along } \gamma. \end{aligned} \tag{2.14}$$

Here K is a function of the Odqvist parameter ϵ_i

$$\epsilon_i = \sqrt{\frac{2}{3}} \int_0^{\epsilon_{ij}^P} (d\epsilon_{ij} d\epsilon_{ij})^{1/2} \tag{2.15}$$

in the form

$$K = K_0 + c \cdot \epsilon_i. \tag{2.16}$$

3. Numerical method

In the present paper the numerical procedure consists in decomposing the deformation process of a hardening material into a number of stages for each material particle. In each stage we calculate:

1. Stresses from Eqs.(2.10),(2.11) - for a kinematic hypothesis of strain-hardening and (2.10),(2.14)-for the isotropic hypothesis.
2. Flow velocities from Eq.(2.12) for each case.
3. Components of the strain rate tensor by means of numerical differentiation of the flow velocity components.
4. Displacement of an each particle from a plastic region assuming that in sufficiently small time increments the flow velocities are constant.
5. Distribution of the yield point from Eq.(2.16) - for an isotropic hypothesis of strain-hardening.
6. Components of tensor α from Eqs.(2.2) by means of the iterative method [18]. This method is based on the known distribution of components α_{ij} in previous stage of calculations (e.g. in the first stage on distribution α_{ij} for the perfectly-plastic material)

Generally, in this procedure the plastic flow problem for the harden material is formulated as a succession of incremental problems for an anisotropic or nonhomogeneous perfectly plastic material. In the first stage of calculations stress and velocity fields are derived from Eqs.(2.10), (2.11), (2.12); they are the equations of a perfectly plastic material. Further steps of calculation are based on the numerical procedure presented above.

4. Results of numerical calculations

Let us consider the results of calculations for the case of monotonic and cyclic bending of a double-notched bar under plane strain conditions.

Calculations have been performed for $c = 197\text{MPa}$, $K_o = 115{\cdot}5$ MPa and $\omega_o = 5 \cdot 10^{-3} s^{-1}$. The slip-line solution satisfying such conditions was given by GREEN [16]. It is shown in Fig.2c for the case when the radius of the notch is 0.85 times the width of the bar at its minimum cross-section.

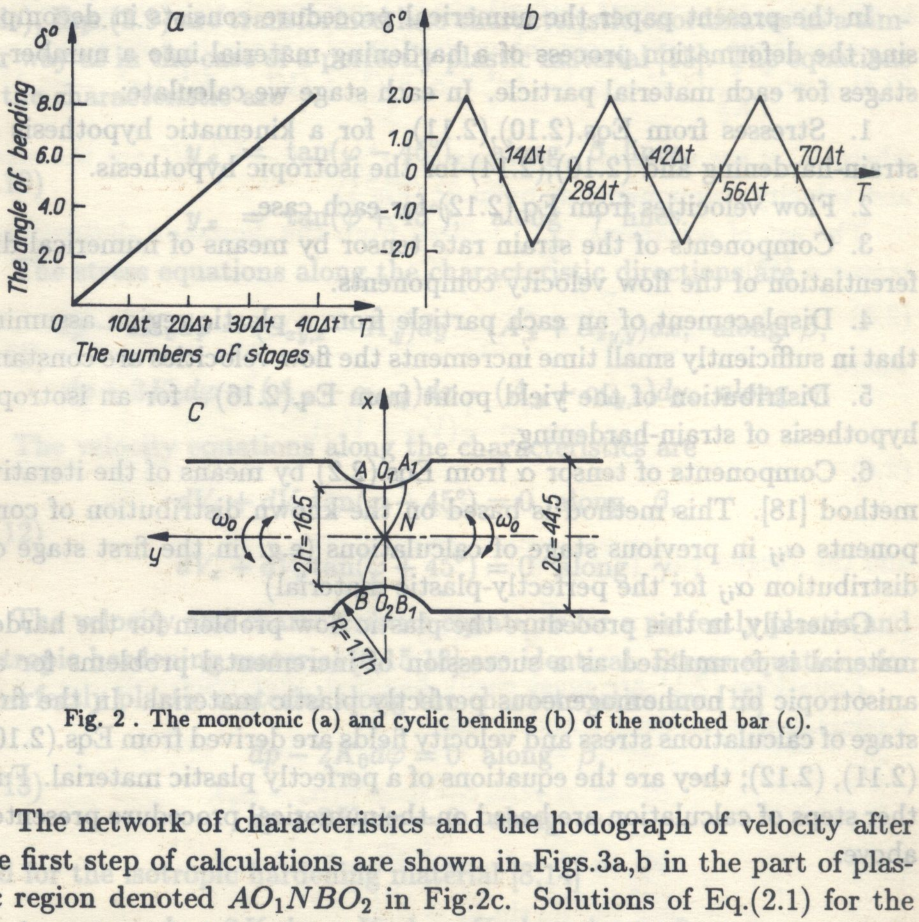

Fig. 2 . The monotonic (a) and cyclic bending (b) of the notched bar (c).

The network of characteristics and the hodograph of velocity after the first step of calculations are shown in Figs.3a,b in the part of plastic region denoted AO_1NBO_2 in Fig.2c. Solutions of Eq.(2.1) for the harden material along the selected characteristic CD and the trajectory of particle Q are presented in Fig.3a. The results are interesting both from the engineering and physical points of view.

Distributions of the Odqvist parameter ϵ_i [15], mean stress p and angle φ after the first and last stage of calculations are shown in Fig.4a-c along the characteristic CD for the case of monotonic bending of a double-notched bar. Values of the Odqvist parameter ϵ_i increase, while

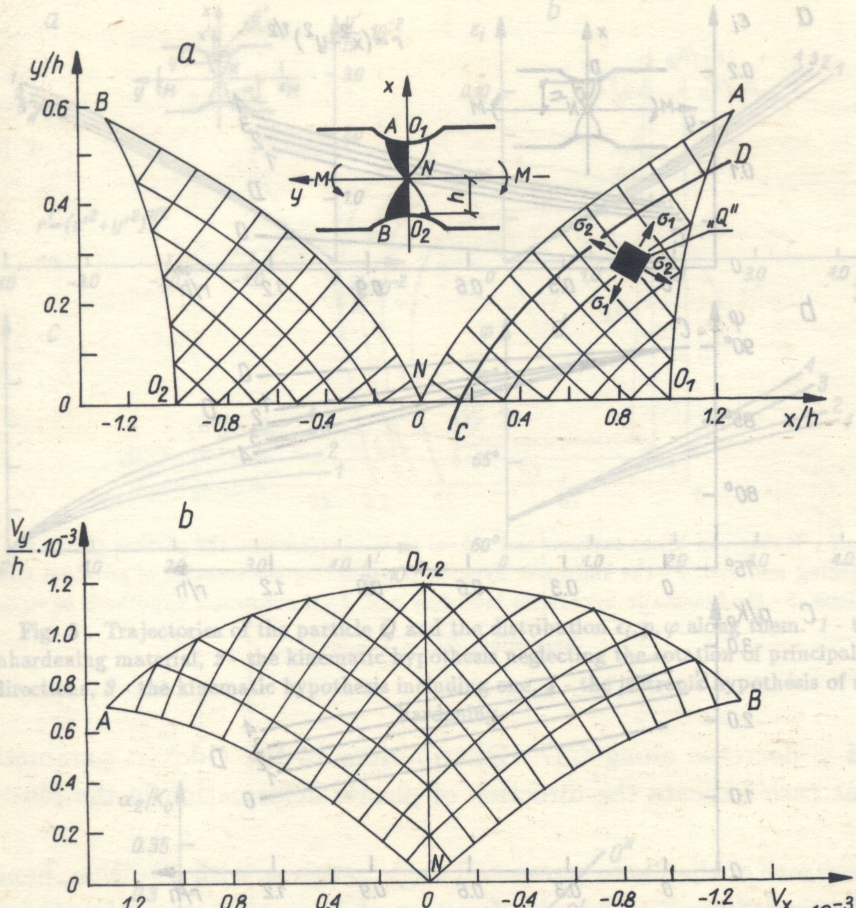

Fig. 3 . The network of characteristics and the hodograph of velocity in the plastic zone O_1ANBO_2.

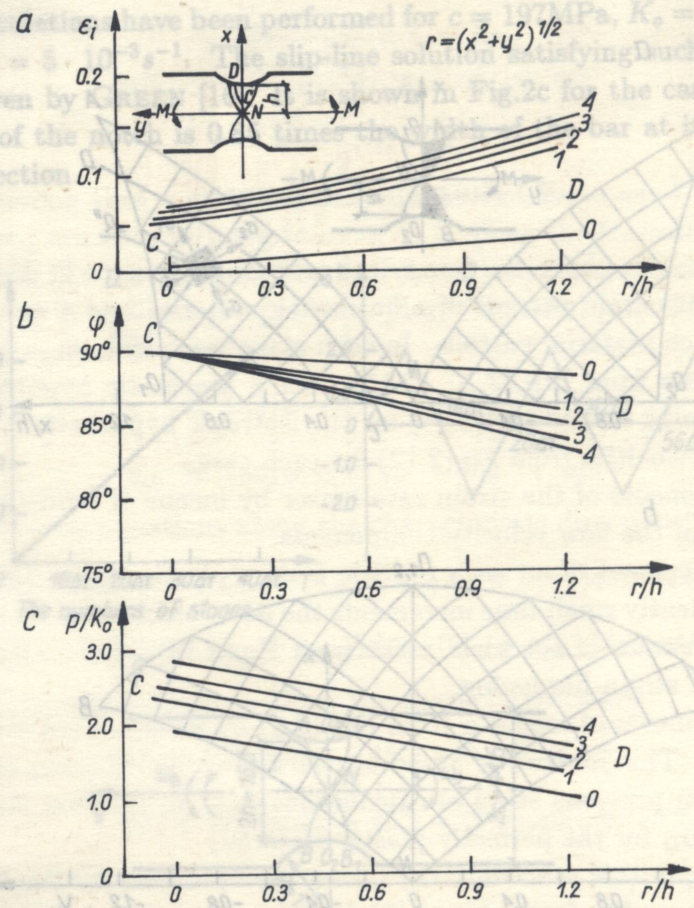

Fig. 4 . Distributions of the Odqvist parameter ϵ_i, the angle φ between σ_1 and x axis and the mean stress p along the intersection CD. (O - the first stage, *1,2,3,4* - the last stage), O - the unhardening material (the first stage), *1* - the unhardening material (the last stage), *2* - the kinematic hypothesis neglecting the rotation of principal stress directions, *3* - the kinematic hypothesis including one, *4* - the isotropic hypothesis of strain hardening.

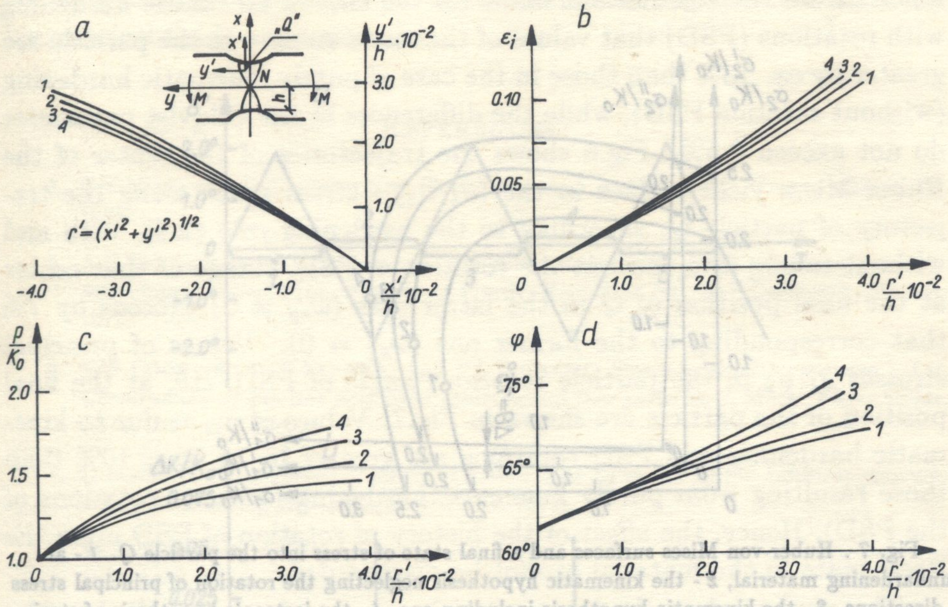

Fig. 5 . Trajectories of the particle Q and the distribution ϵ_i, p, φ along them. *1* - the unhardening material, *2* - the kinematic hypothesis neglecting the rotation of principal stress directions, *3* - the kinematic hypothesis including one, *4* - the isotropic hypothesis of strain hardening.

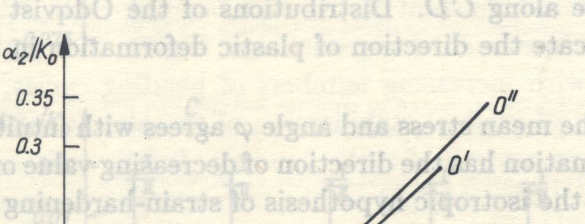

Fig. 6 . Trajectories of the Huber-von Mises surface center. O' - neglecting the rotation of principal stress directions, O'' - including one.

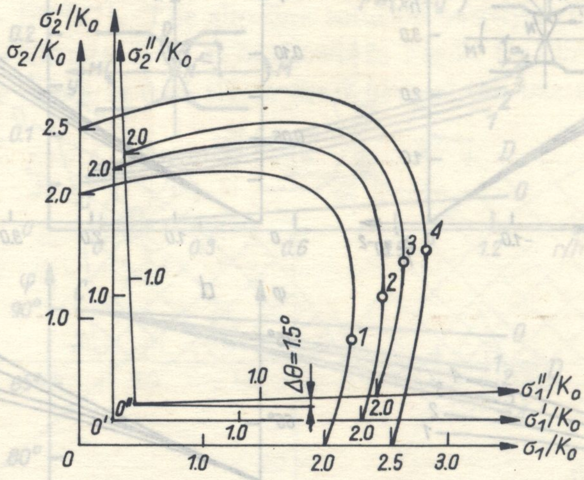

Fig. 7 . Huber-von Mises surfaces and a final state of stress into the particle Q. 1 - an unhardening material, 2 - the kinematic hypothesis neglecting the rotation of principal stress directions, 3 - the kinematic hypothesis including one, 4 - the isotropic hypothesis of strain hardening; $\Delta Q = \int_0^T \omega_{x_y} dt$

p and φ decrease along CD. Distributions of the Odqvist parameter in that case indicate the direction of plastic deformation in the plastic zone.

Decrease of the mean stress and angle φ agrees with intuition, hence the plastic deformation has the direction of decreasing value of p. Calculations made for the isotropic hypothesis of strain-hardening show that values of p are greater by more than 20% than those corresponding to the perfectly plastic material, and less by 5% than those corresponding to the kinematic hypothesis including rotation of the PSD; variations of the Odqvist parameter ϵ_i do not exceed 1% for each of the cases considered.

Detailed analysis of the problem was made along a trajectory of particle Q, under monotonic bendig of a double-notched bar. Trajectory of particle Q and distributions of the Odqvist parameter ϵ_1 [15], mean stress p and angle φ along the trajectory are shown in Figs.5a-d. Particle Q is assumed to lie in the part of the yield zone where the gradient of the Odqvist parameter is the largest (cf. Fig.4a).

Values of ϵ_i, p and φ increase along the trajectory of particle Q; it

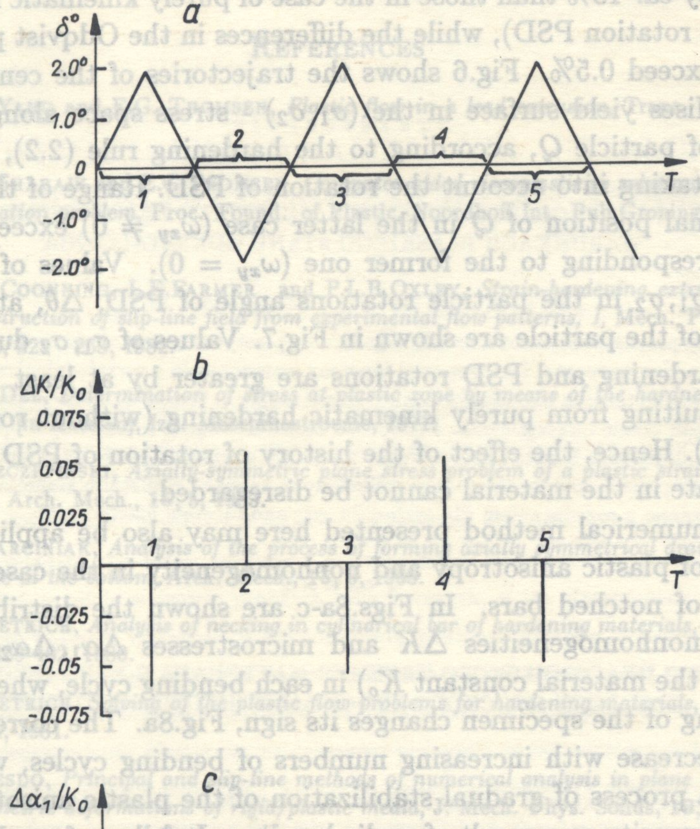

Fig. 8 . The residual nonhomogeneous and microstress in the cyclic bending of the notched bar. I - the kinematic hypothesis including the rotation of principal stress directions (PSD), II - the kinematic hypothesis neglecting the rotation.

follows from the calculations made for the case of kinematic hardening with rotations (PSD) that values of the mean stress p in the particle are greater by ca. 13% than those in the case of purely kinematic hardening (without rotation PSD), while the differences in the Odqvist parameter do not exceed 0.5%. Fig.6 shows the trajectories of the center of the Huber-Mises yield surface in the (σ_1, σ_2) - stress space along the trajectory of particle Q, according to the hardening rule (2.2), with and without taking into account the rotation of PSD. Range of that center at the final position of Q in the latter case ($\omega_{xy} \neq 0$) exceeds by 7% that corresponding to the former one ($\omega_{xy} = 0$). Values of principal stresses σ_1, σ_2 in the particle rotations angle of PSD, $\Delta\theta$, at the final position of the particle are shown in Fig.7. Values of σ_1, σ_2 due to kinematic hardening and PSD rotations are greater by at least 10% than those resulting from purely kinematic hardening (without rotations of the PSD). Hence, the effect of the history of rotation of PSD upon the stress state in the material cannot be disregarded.

The numerical method presented here may also be applied to the analysis of plastic anisotropy and nonhomogeneity in the case of cyclic bending of notched bars. In Figs.8a-c are shown the distributions of residual nonhomogeneities ΔK and microstresses $\Delta\alpha_1, \Delta\alpha_2$ [17] (referred to the material constant K_o) in each bending cycle, when angle σ of bending of the specimen changes its sign, Fig.8a. The corresponding values decrease with increasing numbers of bending cycles, what may suggest a process of gradual stabilization of the plastic anisotropy and nonhomogeneity as a result of cyclic bending. It follows from Fig.8 that the influence of rotation of PSD on the microstresses is small.

5. Conclusions

1. Problem of plastic yielding of materials with hardening may be replaced by a sequence of plastic flows of nonhomogeneous or anisotropic rigid - perfectly plastic materials.

2. Rotation of PSD exerts significant effects on the stress field, kinematics of the plastic yielding process remaining practically unaffected.

3. For the assumed model of material (linear relationship between the effective stresses and strains), the effects of the strain-hardening rule

are significant in the case of the stress fields, and negligible as far as the kinematical fields are concerned.

References

1. C.T.Yang and E.G. Thomsen, *Plastic flow in a lead extrusion*, Trans. ASME, **75**, 1953.

2. A.H.Shabaik and E.G.Thomsen, *Computer-aided viscoplasticity solution of some deformation problem*, Proc. Found. of Plastic, Noordhoff Int. Pub.Groningen, Warsaw 1972.

3. S.W.Coonning, L.E.Farmer and P.L.B.Oxley, *Strain-hardening extrusion - I,II, Construction of slip-line field from experimental flow patterns, I,* Mech. Phys. Solids, **30**, 4, 225 - 263, 1982.

4. G.D.Del, *Determination of stress at plastic zone by means of the hardness measurement* [in Russian], Izd. Maschinostroenie, 1971.

5. W.Szczepiński, *Axially-symmetric plane stress problem of a plastic strain-hardening body*, Arch. Mech., **15**, 5, 1963.

6. Z.Marciniak, *Analysis of the process of forming axially symmetrical drawpieces with a hole at the bottom*, Arch. Mech., **15**, 6, 1963.

7. L.Dietrich, *Analysis of necking in cylindrical bar of hardening materials*, Arch.Mech. **32**, 389-402, 1980.

8. L.Dietrich, *Solving of the plastic flow problems for hardening materials*, J. de Mec., **20**, 4, 1981.

9. D.Besdo, *Principal and slip-line methods of numerical analysis in plane and axially-symmetric deformations of rigid/plastic media*, J. Mech. Phys. Solids, **19**, p.6, 1971.

10. W.Szczepiński, *Introduction to the mechanics of plastic forming of metals*, PWN, NIP, 1979.

11. I.F.Collins, *The application of singular perturbation techniques to the analysis of forming processes for strain-hardening materials*, IUTAM symp., Metal Forming Plasticity, Springer, Berlin 1978.

12. H.Ziegler, *A modification of Prager's hardening rule*, Quart. Appl. Math., **17**, 55-60,1959.

13. K.Waschizu, *Variational methods in elasticity and plasticity*, Pergamon Press, 1982.

14. A.I.Kuznecow, *Plane strain problems of non-homogeneous plastic solids* [in Russian], Vest. Leningrads. Uniw., **13**, 3, 1958.

15. R.Hill, *The mathematical theory of plasticity*, Oxford 1950.

16. A.P.Green, *The plastic yielding of notched bars due to bending*, Quart. I. Mech. Appl. Math., **6**, 223-239, 1953.

17. J.I.Kodashevitsch, W.W.Novozhilov, *Theory of plasticity including the microstress* [in Russian], Prikl. Math. Mech., **22**, 1958.

18. M.Piwnik, Ph.D.Thesis [in Polish], IFTR Rep., 21, 1987.

Streszczenie

ANALIZA PLASTYCZNEGO PŁYNIĘCIA MATERIAŁU ZE WZMOCNIENIEM W WARUNKACH PŁASKIEGO STANU ODKSZTAŁCENIA

W pracy przedstawiono rozwiązanie zagadnienia plastycznego płynięcia materiału ze wzmocnieniem dla kinematycznej i izotropowej hipotezy wzmocnienia z uwzględnieniem obrotu kierunków głównych tensora naprężenia (PSD). Rozwiązanie uzyskano na gruncie teorii linii poślizgu w płaskim stanie odkształcenia. Plastyczna niejednorodność i anizotropia opisane zostały przez warunek Hubera-Misesa i prawa wzmocnienia. Jako przykład przedstawiono rozwiązanie dla zginanego pasma osłabionego dwoma półkolistymi karbami w płaskim stanie odkształcenia, które porównano z rozwiązaniem dla materiału sztywno-idealnie plastycznego.

Резюме

АНАЛИЗ ПЛАСТИЧЕСКОГО ТЕЧЕНИЯ МАТЕРИАЛА С УПРОЧНЕНИЕМ В УСЛОВИЯХ ПЛОСКОГО ДЕФОРМАЦИОННОГО СОСТОЯНИЯ

В работе представлено решение задачи пластического течения материала с упрочнением для кинематической и изотроповой гипотезы упрочнения с учетом вращения главных направлений тензора напряжений. Решение получено на грунте теории линии скольжения в плоском деформационном состоянии. Пластическая неоднорость и анизотропия описаны условием Губера-Мизеса и законами упрочнения. Как пример показано решение для изгибаемой полосы, ослабленной двумя полукруговыми надрезами, в плоском деформационном состоянии, которое сравнено с решением для жесткого-идеально пластического материала.

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Received August 24, 1988.