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EXACT REPRESENTATION OF THE DERIVATIVES OF ISOTROPIC TENSOR FUNCTIONS WITH RESPECT TO THE DEFORMATION GRADIENT F

A. Ercolano

Dipartimento di Meccanica, Strutture, Ambiente Territorio

Facoltà di Ingegneria di Cassino
Via G. Di Biasio, n.43
e-mail: ercolano@unicas.it

Expressions for derivatives of isotropic tensor functions with respect to the deformation tensor **F** are derived. Each derivative has the first representation in terms of eigenvectors; then, for computational conveniences, also a basis-free expression, in terms of eigenprojections, is reported. Further, in the same fashion, also the time derivatives are provided. In the paper, a short review of different approaches to the problem existing in literature is presented. In order to make the exposition self-contained, some backgrounds of tensor analysis are also given.

1. Introduction

The measure of the variation of deformation is a fundamental issue of continuum mechanics and is mainly related to finite deformations and to constitutive and evolution problems. The first aspect is concerned with a suitable choice of strain measures. The later two are more involved in the first variation of the deformation in space or in time-like variables. Hence, effective expressions for derivatives and rates of strain tensors with respect to deformation gradient **F** are strongly needed.

In recent times several researchers have provided various expressions of derivatives for different strain tensors. However, most of them are lengthy, especially in the three-dimensional case, and do not preserve any physical meaning.

Basically we find in literature two different approaches the first one is based on the invariants of the tensor argument, while the latter returns to its eigen-projectors or eigen-diads.

In order to introduce the problem, let us consider, for example, the stretch tensor $\mathbf{U}(\mathbf{F}) = \sqrt{\mathbf{F}^T\mathbf{F}} = \sqrt{\mathbf{C}}$.

By the chain rule we have the following derivative with respect to **F**:

$$\frac{\partial \mathbf{U}}{\partial \mathbf{F}} = \frac{\partial \sqrt{\mathbf{F}^T\mathbf{F}}}{\partial \left(\mathbf{F}^T\mathbf{F}\right)} \frac{\partial \left(\mathbf{F}^T\mathbf{F}\right)}{\partial \mathbf{F}}. \tag{1.1}$$

As it is well known, the treatment of the derivative of the square root function is a rather difficult task. In the literature, we basically find two different approaches: the first one is based on invariants, while the other one is based on eigenprojections. GUO [1, 2] and CARLSON and HOGER [3] reduce the problem to the solution of a particular tensor equation $\mathbf{AX} + \mathbf{XA} = \mathbf{H}$, where $\mathbf{A}$ is a symmetric second-order tensor and $\mathbf{X}$ is the unknown second order tensor. This equation can be easily solved recurring to the principal bases of $\mathbf{A}$. This procedure is called implicit because, in order to express the problem in the principal directions of $\mathbf{A}$, one has to find first both the eigenvectors and the eigenvalues of $\mathbf{A}$.

A direct or explicit solution is obtained by finding the $\mathbf{X}$ in terms of $\mathbf{H}$ and $\mathbf{A}$ and returning to the principal invariants. This procedure avoids the evaluation of the eigenvectors and the eigenvalues.

Several papers, completely devoted to this problem (see TING [4, 5] and SCHEIDLER [6]) follow this basis-free procedure. For the latest results and a comprehensive review see ROSATI in [7] and [8].

Other researchers, see WHEELER [9] and CHEN, WHEELER [10], developed a completely different procedure based on the eigenprojectors. Following this approach and exploiting the principal axis method of HILL [11–13], XIAO [15], presented explicit basis-free expressions of some strain derivatives. In [16] XIAO, BRUHNS and MEYERS gave a new accurate evaluation of derivatives of some isotropic functions of symmetric tensor. The basic idea in [16] was the introduction of a class of isotropic functions $\mathbf{G}(\mathbf{A})$ in order to derive a simple expression for $\partial \mathbf{G}_L(\mathbf{C}) / \partial C$. In fact $\partial\sqrt{\mathbf{C}}/\partial \mathbf{C}$ can be then obtained just by choosing $\mathbf{A} = \mathbf{C}$ and using as a Lagrangian isotropic function the square root function $\mathbf{G}_L = \sqrt{\bullet}$.

For the sake of completeness we have to report that a third mixed approach has appeared very recently in literature [17, 18]. As it is well known in the case of non-symmetric tensor arguments the construction of isotropic tensor functions $\mathbf{G}(\mathbf{A})$ is not so straightforward. This depends on the lack of the spectral decomposition for nonsymmetric tensors $\mathbf{A}$. In this case, both Authors make use of the Taylor power series expansion. In order to avoid infinite power series and the related convergence problems, Itskov obtains the closed-form representation combining the Cayley–Hamilton theorem with the so-called Dunford–Taylor integral. Anyway, some convergence problems still remains. On the other hand, LU introduces a generating function $\mathbf{G}(\mathbf{A})$ of the eigenvalues of $\mathbf{A}$ and then considers an invariant representation for $\mathbf{G}(\mathbf{A})$. This approach leads to a finite term representation, even if it is not explicitly shown what happens when dealing with complex eigenvalues and eigenvectors.

In this paper, following XIAO, BRUHNS and MEYERS [16], we get the derivatives of general isotropic functions $\mathbf{G}(\mathbf{A})$ where $\mathbf{A}$ is a symmetric tensor. The first formulation was given by HILL [11–13] but a simple and rigorous proof has been given by SCHEIDLER [14]. In our case, starting from the very definition of the derivative we get this well-known formula by means of a suitable perturbation of the eigenbases of $\mathbf{A}$. This demonstration, is rather different from the one in [16]. Moreover it is simple, accurate and still preserves a physical meaning. Further, an explicit basis-free expression is also given.

The derivative of the product $\partial\left(\mathbf{F}^T\mathbf{F}\right)/\partial\mathbf{F}$ in (1.1) can be obtained using the chain rule and some useful fourth-order tensors of the kind $\mathbf{A}\boxtimes\mathbf{B}$, $\mathbf{A}\widehat{\boxtimes}\mathbf{B}$ where $\boxtimes$ and $\widehat{\boxtimes}$ denote some suitable product operators. This reasoning has been extensively used both by ROSATI [8] and PADOVANI [19]. Thus the difference between the two consists essentially in the treatment of the square derivative. While Rosati returns to the tensor equation $\mathbf{AX}+\mathbf{XA}=\mathbf{H}$, Padovani takes directly the known results of the derivatives of symmetric tensor-valued tensor function [20] and provides explicit and extended formulas for a wide set of derivatives.

Further, in many cases, PADOVANI [19] is able to simplify the terms with the $\boxtimes$ operator so that the correspondent formulas are the simplest in literature.

The main goal of the paper is to extend these formulas to isotropic tensor-valued functions of $\mathbf{F}$. Furthermore, we also provide time derivatives of isotropic tensor functions.

In order to make the exposition self-contained, we first present a certain background of the tensor analysis where a generalization of the spectral decomposition is also included. This decomposition, known as the singular value decomposition, is not very often used in continuum mechanics but, in our opinion, is a very attractive tool in the changes from Lagrangian to Eulerian bases and vice versa.

2. PRELIMINARIES

Let us first introduce some definitions used throughout the paper.

Even though for our purposes we could report us to a 3-dimensional space, we consider a general n-dimensional inner product space V_n. Hence, we denote by *Lin* the set of all linear mappings defined on V_n.

Other particular subsets of *Lin* are:

$$
\begin{aligned}
&Lin^+: && \{A\in Lin : \det A>0\},\\
&Sym: && \{A\in Lin : Ax\cdot y = Ay\cdot x,\ \forall x,y\},\\
&Skw: && \{A\in Lin : Ax\cdot y = -Ay\cdot x,\ \forall x,y\},\\
&PSym: && \{A\in Sym : Ax\cdot x>0,\ \forall x\},\\
&Orth: && \left\{A\in Lin : A^TA=AA^T=I\right\},\\
&Orth^+= && \{A\in Orth \frown Lin^+\}.
\end{aligned}
$$

Obviously we have the following projections of $\mathbf{X}$:

$$\mathbf{X} = \frac{1}{2}(\mathbf{X} + \mathbf{X}^T) + \frac{1}{2}(\mathbf{X} - \mathbf{X}^T) = \hat{\mathbf{X}} + \acute{\mathbf{X}}, \tag{2.1}$$

where $\hat{\mathbf{X}} \in Sym$ and $\acute{\mathbf{X}} \in Skw$, $\forall \mathbf{X} \in Lin$.

Furthermore we define these sets of fourth-order tensors:

$$\mathbb{L}in : \mathbb{X} : Lin \to Lin, \qquad \mathbb{S}ym \; \{\mathbb{X} \;\; Sym \to Sym\}. \tag{2.2}$$

2.1. Some Tensor products

Given $\mathbf{A}$, $\mathbf{B} \in Lin$, the dyadic product $\mathbf{A} \otimes \mathbf{B}$ is the fourth-order tensor defined as follows:

$$\mathbf{A} \otimes \mathbf{B} : \mathbf{X} \Longrightarrow (\mathbf{A} \otimes \mathbf{B})\, \mathbf{X} = (\mathbf{B} \cdot \mathbf{X})\, \mathbf{A} \qquad \forall \mathbf{X} \in Lin, \tag{2.3}$$

where

$$\begin{aligned} (\mathbf{B} \cdot \mathbf{X}) \;&=\; \mathrm{tr}\,(\mathbf{B}^T \mathbf{X}) \;=\; \sum_{i,j} B_{ij} X_{ij}, \\ ((\mathbf{A} \otimes \mathbf{B})\, \mathbf{X})_{hk} \;&=\; \sum_{i,j} B_{ij} X_{ij} A_{hk}. \end{aligned} \tag{2.4}$$

By the above definition we have

$$\begin{aligned} (\mathbf{A} \otimes \mathbf{B})\, \mathbf{C}\mathbf{D} \;&= (\mathbf{A}\mathbf{D} \otimes \mathbf{B})\, \mathbf{C}, \\ \mathbf{D}\, (\mathbf{A} \otimes \mathbf{B})\, \mathbf{C} \;&= (\mathbf{D}\mathbf{A} \otimes \mathbf{B})\, \mathbf{C}. \end{aligned} \tag{2.5}$$

In (2.5) the brackets are used merely to define the fourth-order tensors involved and, consequently, the successions of the operations.

In indicial notation (2.5) becomes

$$\begin{aligned} ((\mathbf{A} \otimes \mathbf{B})\, \mathbf{C}\mathbf{D})_{hl} &= \sum_{i,j,k} (B_{ij} C_{ij})\, (A_{hk} D_{kl}) = \sum_{i,j,k} (A_{hk} D_{kl})\, (B_{ij} C_{ij}) \\ &= ((\mathbf{A}\mathbf{D} \otimes \mathbf{B})\, \mathbf{C})_{hl}, \\ (\mathbf{D}\, (\mathbf{A} \otimes \mathbf{B})\, \mathbf{C})_{ih} &= \sum_{j,k,l} D_{ij}\, (B_{kl} C_{kl})\, A_{jh} = \sum_{j,k,l} D_{ij} A_{jh}\, (B_{kl} C_{kl}) \\ &= ((\mathbf{D}\mathbf{A} \otimes \mathbf{B})\, \mathbf{C})_{ih}, \end{aligned} \tag{2.6}$$

$\forall \mathbf{C}, \mathbf{D} \in Lin$.

Note that the following relations hold:

$$(2.7)\qquad \begin{aligned} (\mathbf{A}\otimes\mathbf{B})\,\widehat{(\mathbf{C}^T\mathbf{D})} &= (\mathbf{A}\otimes\mathbf{CB})\,\mathbf{D} = (\mathbf{A}\otimes\mathbf{DB})\,\mathbf{C},\\ (\mathbf{A}\otimes\mathbf{B})\,\widehat{(\mathbf{C}\mathbf{D}^T)} &= (\mathbf{A}\otimes\mathbf{BC})\,\mathbf{D} = (\mathbf{A}\otimes\mathbf{BD})\,\mathbf{C}. \end{aligned}$$

Let $\mathbf{a}^r$, $\mathbf{a}^s$ be the vectors of an orthonormal base on V_n, then we have

$$(2.8)\qquad \begin{aligned} &[(\mathbf{a}^r\otimes\mathbf{a}^s+\mathbf{a}^s\otimes\mathbf{a}^r)\otimes(\mathbf{a}^r\otimes\mathbf{a}^s+\mathbf{a}^s\otimes\mathbf{a}^r)]\,\mathbf{X}\\ &\qquad = 2\,[(\mathbf{a}^r\otimes\mathbf{a}^r)\,\mathbf{X}\,(\mathbf{a}^s\otimes\mathbf{a}^s)+(\mathbf{a}^s\otimes\mathbf{a}^s)\,\mathbf{X}\,(\mathbf{a}^r\otimes\mathbf{a}^r)] \end{aligned}$$

$\forall\mathbf{X}\in \mathrm{Sym}$.

Moreover, we define these two square products

$$(2.9)\qquad \mathbf{A}\boxtimes\mathbf{B}:\mathbf{X}\Longrightarrow(\mathbf{A}\boxtimes\mathbf{B})\,\mathbf{X}=\mathbf{AXB}^T,\qquad\forall\mathbf{X}$$

and

$$(2.10)\qquad \mathbf{A}\boxtimes^T\mathbf{B}:\mathbf{X}\Longrightarrow(\mathbf{A}\boxtimes^T\mathbf{B})\,\mathbf{X}=\mathbf{AX}^T\mathbf{B}^T,\qquad\forall\mathbf{X}.$$

By means of the above tensors we can immediately derive the following formula:

$$(2.11)\qquad \begin{aligned} \mathbf{A}\hat{\boxtimes}\mathbf{B}:\mathbf{X}&\Longrightarrow\frac{1}{2}\left[(\mathbf{A}\boxtimes\mathbf{B})+(\mathbf{B}\boxtimes^T\mathbf{A})\right]\,\mathbf{X}\\ &=\frac{1}{2}\left[\mathbf{AXB}^T+\mathbf{BX}^T\mathbf{A}^T\right]\,\mathbf{X}. \end{aligned}$$

The notations in (2.9) were introduced by Del Piero [1, 21]. The one in (2.11) has the hat on the top of the square to remember that it is a symmetric tensor-valued function.

The Cartesian components of $\mathbf{A}\boxtimes\mathbf{B}$ and $\mathbf{A}\boxtimes^T\mathbf{B}$ are

$$(2.12)\qquad \begin{aligned} [\mathbf{A}\boxtimes\mathbf{B}]_{ijhk} &= A_{ih}B_{jk},\\ \left[\mathbf{A}\boxtimes^T\mathbf{B}\right]_{ijhk} &= A_{ik}B_{jh}. \end{aligned}$$

Sometimes, for the sake of simplicity, we denote

$$\mathbb{A}\doteq\mathbf{A}\boxtimes\mathbf{A}.$$

The tensor

$$(2.13)\qquad \mathbb{R}_{\mathbf{AB}}\doteq\mathbf{R}_A\boxtimes\mathbf{R}_B,$$

with $\mathbf{R}_A$, $\mathbf{R}_B \in orth^+$, is a rotation on $\mathbb{L}in$

$$\mathbb{R}_{\mathbf{AB}}\mathbb{R}_{\mathbf{AB}}^T = \mathbb{R}_{\mathbf{AB}}^T\mathbb{R}_{\mathbf{AB}} = \mathbb{I}, \tag{2.14}$$

where

$$\left[\mathbb{R}_{\mathbf{AB}}^T\right]_{ijhk} = \left[\mathbb{R}_{\mathbf{AB}}\right]_{hkij}.$$

Let $\mathbf{a}_h$ be the vectors of an orthonormal base x_h on V_n then

$$\mathbf{a}_i \otimes \mathbf{a}_i \tag{2.15}$$

has the meaning of a projection in the x_i direction, while

$$(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i) \qquad i \neq j \tag{2.16}$$

has the meaning of a projection on the $i-j$ plane plus a reflection with respect to the axis of equation $x_i = x_j$.

Thus, if we apply (2.16) twice, we recover the projection on the $i-j$ plane:

$$(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i)^2 = (\mathbf{a}_i \otimes \mathbf{a}_i + \mathbf{a}_j \otimes \mathbf{a}_j). \tag{2.17}$$

The set of the fourth-order tensors

$$\mathbb{B}_{ijhk} \doteq \mathbf{a}_i \otimes \mathbf{a}_j \boxtimes \mathbf{a}_h \otimes \mathbf{a}_k, \qquad i,j,h,k \in \{1,2,3\}$$

defines a base on $\mathbb{L}in$.

It is easy to verify that

$$\mathbb{B}_{ijhk}\mathbb{B}_{ijhk}^T = \mathbb{P}_{ik}, \qquad \mathbb{B}_{ijhk}^T\mathbb{B}_{ijhk} = \mathbb{P}_{jh}, \tag{2.18}$$

where

$$\mathbb{P}_{ij} \doteq \mathbf{a}_i \otimes \mathbf{a}_i \boxtimes \mathbf{a}_j \otimes \mathbf{a}_j, \tag{2.19}$$

are projections:

$$\mathbb{P}_{ij} = \mathbb{P}_{ij}^T, \qquad \mathbb{P}_{ij}^2 = \mathbb{P}_{ij}. \tag{2.20}$$

2.2. Derivatives

We focus our attention on the derivatives of both scalar and tensor-valued tensor functions. A very detailed treatment of the derivatives of tensor functions can be found in ITSKOV [2, 22].

Scalar-valued tensor functions. Let $\alpha(\mathbf{A})$ be a real-valued tensor function, then the derivative of $\alpha(\mathbf{A})$ at a point $\mathbf{A}$ is the linear function $\partial\alpha(\mathbf{A})/\partial\mathbf{A}$: $Lin \rightarrow R$ such that

$$\frac{\partial\alpha(\mathbf{A})}{\partial\mathbf{A}}[\mathbf{X}] = \frac{\partial\alpha(\mathbf{A})}{\partial\mathbf{A}} \cdot \mathbf{X} = \alpha(\mathbf{A}+\mathbf{X}) - \alpha(\mathbf{A}) + o(\mathbf{X}). \tag{2.21}$$

The value of the derivative with respect to the tensor $\mathbf{A}$ in the $\mathbf{X}$ direction can be obtained as

$$\frac{\partial\alpha(\mathbf{A})}{\partial\mathbf{A}}[\mathbf{X}] = \lim_{\epsilon\to 0}\frac{\alpha(\mathbf{A}+\epsilon\mathbf{X}) - \alpha(\mathbf{A})}{\epsilon}. \tag{2.22}$$

Showing explicitly the Cartesian bases, we have in components

$$\frac{\partial\alpha(\mathbf{A})}{\partial\mathbf{A}}[\mathbf{X}] = \sum_{h,k}\frac{\partial\alpha}{\partial(A_{hk}\mathbf{b}_h\otimes\mathbf{b}_k)}X_{hk}\mathbf{b}_h\otimes\mathbf{b}_k, \tag{2.23}$$

$$= \sum_{h,k}\frac{\partial\alpha}{\partial A_{hk}}X_{hk}, = \frac{\partial\alpha(\mathbf{A})}{\partial\mathbf{A}} \cdot \mathbf{X}.$$

Thus we conclude that the derivative is the second order tensor

$$\frac{\partial\alpha(\mathbf{A})}{\partial\mathbf{A}} = \sum_{h,k}\frac{\partial\alpha}{\partial A_{hk}}\mathbf{b}_h\otimes\mathbf{b}_k, \tag{2.24}$$

where $\mathbf{b}_h$ are the Cartesian orthonormal vectors. The scalar value of the derivative in the $\mathbf{X}$ direction is also known as the α differential with respect to the tensor $\mathbf{A}$ for the increment $\mathbf{X}$ and, in this case, it is equal to the scalar product between the second-order tensor representing the derivative $\partial\alpha(\mathbf{A})/\partial\mathbf{A}$ and the tensor increment $\mathbf{X}$

$$d\alpha(\mathbf{A},\mathbf{X}) = \frac{\partial\alpha(\mathbf{A})}{\partial\mathbf{A}}[\mathbf{X}] = \sum_{i,j}\frac{\partial\alpha(\mathbf{A})}{\partial A_{ij}}X_{ij} = \frac{\partial\alpha(\mathbf{A})}{\partial\mathbf{A}} \cdot \mathbf{X}. \tag{2.25}$$

Tensor valued tensor functions. Let $\mathbf{G}(\mathbf{A})$ be a tensor-valued tensor function, then the derivative of $\mathbf{G}$ at a point $\mathbf{A}$ is the linear function $\partial\alpha(\mathbf{A})/\partial\mathbf{A}$: $Lin \rightarrow R$ such that

$$\frac{\partial\mathbf{G}(\mathbf{A})}{\partial\mathbf{A}}[\mathbf{X}] = [\mathbf{G}(\mathbf{A}+\mathbf{X}) - \mathbf{G}(\mathbf{A})] + o(\mathbf{X}). \tag{2.26}$$

$\partial\mathbf{G}(\mathbf{A})/\partial\mathbf{A}[\mathbf{X}]$ is the value of the derivative $\partial\mathbf{G}(\mathbf{A})/\partial\mathbf{A}$ at $\mathbf{A}$ on the increment $\mathbf{X}$ which can be obtained by means of the following limit:

$$\frac{\partial\mathbf{G}(\mathbf{A})}{\partial\mathbf{A}}[\mathbf{X}] = \lim_{\epsilon\to 0}\frac{\mathbf{G}(\mathbf{A}+\epsilon\mathbf{X}) - \mathbf{G}(\mathbf{A})}{\epsilon} = \lim_{\epsilon\to 0}\frac{\Delta\mathbf{G}(\mathbf{A})_{\epsilon\mathbf{X}}}{\epsilon} \tag{2.27}$$

where $\Delta\mathbf{G}(\mathbf{A})_{\epsilon\mathbf{X}} = [\mathbf{G}(\mathbf{A}+\epsilon\mathbf{X}) - \mathbf{G}(\mathbf{A})]$ is the finite difference of $\mathbf{G}$ for the increment $\epsilon\mathbf{X}$.

Also in this case we call the $\mathbf{G}$ differential the value of the derivative for the increment $\mathbf{X}$

$$d\mathbf{G}(\mathbf{A},\mathbf{X}) = \frac{\partial\mathbf{G}(\mathbf{A})}{\partial\mathbf{A}}[\mathbf{X}]. \tag{2.28}$$

Showing explicitly the Cartesian bases, Eq. (2.28) is written as

$$\frac{\partial\mathbf{G}(\mathbf{A})}{\partial\mathbf{A}}[\mathbf{X}] = \sum_{i,j}\sum_{h,k}\frac{\partial(G_{ij}\mathbf{b}_i\otimes\mathbf{b}_j)}{\partial(A_{hk}\mathbf{b}_h\otimes\mathbf{b}_k)}X_{hk}\mathbf{b}_h\otimes\mathbf{b}_k = \sum_{i,j}\sum_{h,k}\frac{\partial G_{ij}}{\partial A_{hk}}X_{hk}\mathbf{b}_i\otimes\mathbf{b}_j. \tag{2.29}$$

Because of the linearity, from the values of the derivatives along all the Cartesian bases $\mathbf{b}_i\otimes\mathbf{b}_j$ we can recover the general expression of $\partial\mathbf{G}(\mathbf{A})/\partial\mathbf{A}$ that is

$$\frac{\partial\mathbf{G}(\mathbf{A})}{\partial\mathbf{A}} = \sum_{i,j}\sum_{h,k}\frac{\partial(G_{ij}\mathbf{b}_i\otimes\mathbf{b}_j)}{\partial(A_{hk}\mathbf{b}_h\otimes\mathbf{b}_k)} = \sum_{i,j}\sum_{h,k}\frac{\partial G_{ij}}{\partial A_{hk}}\mathbf{b}_i\otimes\mathbf{b}. \tag{2.30}$$

In fact, the expression in (2.30) is the only one that satisfies (2.29)

$$\frac{\partial\mathbf{G}(\mathbf{A})}{\partial\mathbf{A}}[\mathbf{X}] = \sum_{i,j,h,k}\left[\frac{\partial G_{ij}}{\partial A_{hk}}\mathbf{b}_i\otimes\mathbf{b}_j\otimes\mathbf{b}_h\otimes\mathbf{b}_k\right][X_{hk}\mathbf{b}_h\otimes\mathbf{b}_k] = \sum_{i,j,h,k}\frac{\partial G_{ij}}{\partial A_{hk}}X_{hk}\mathbf{b}_i\otimes\mathbf{b}_j. \tag{2.31}$$

Using the short-hand notation

$$\mathbf{B}_{ij} = \mathbf{b}_i\otimes\mathbf{b}_j \quad \text{with } \mathbf{B}_i = \mathbf{B}_{ii} \tag{2.32}$$

(2.30) can be rewritten in more compact form as

$$\frac{\partial\mathbf{G}(\mathbf{A})}{\partial\mathbf{A}} = \sum_{i,j,h,k}\frac{\partial G_{ij}}{\partial A_{hk}}\mathbf{B}_{ij}\otimes\mathbf{B}_{hk} \tag{2.33}$$

where the set of the $\mathbf{B}_{ij}\otimes\mathbf{B}_{hk}$ is a base on $\mathbb{L}in$.

In other words, the derivative of $\mathbf{G}(\mathbf{A})$ is represented by a fourth-order tensor and the differential $d\mathbf{G}(\mathbf{A},\mathbf{X})$ is the second-order tensor obtained as the value of the derivative in $\mathbf{X}$.

Obviously, the bases involved are

$$\mathbf{B}_{ij} \ \text{on} \ Lin \qquad \mathbf{B}_{ij}\otimes\mathbf{B}_{hk} \ \text{on} \ \mathbb{L}in. \tag{2.34}$$

Note:

If the tensor $\mathbf{A}$ is symmetric and $\partial\mathbf{G}(\mathbf{A})/\partial\mathbf{A} : Sym \to Sym$, we can use instead of (2.34) the following reduced bases:

$$\hat{\mathbf{B}}_{ij} \quad \text{on} \quad Sym \qquad \hat{\mathbf{B}}_{ij} \otimes \hat{\mathbf{B}}_{hk} \quad \text{on} \quad \mathbb{S}ym \tag{2.35}$$

where the $n(n+1)/2$ elements

$$\hat{\mathbf{B}}_{ij} = \frac{1}{\sqrt{2+2\delta_{ij}}}\left(\mathbf{B}_{ij} + \mathbf{B}_{ji}\right) \qquad j, i \in \{1,..n\}, \qquad j \le i \tag{2.36}$$

are normalized

$$\hat{\mathbf{B}}_{ij} \cdot \hat{\mathbf{B}}_{hk} = \mathrm{tr}\left(\hat{\mathbf{B}}_{ij}\hat{\mathbf{B}}_{hk}\right) = \frac{\left(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}\right)}{1+\delta_{ij}\delta_{hk}}. \tag{2.37}$$

It is interesting to note that, from the kinematical point of view, $\mathbf{B}_i$ and $(\mathbf{B}_{ij} + \mathbf{B}_{ji})$ with $j \neq i$, represent respectively an axial and a shear deformation tensor.

Moreover, the $n^2(n+1)^2/4$ elements of the base in $(2.35)_2$ can represent every fourth-order tensor with first and second minor-diagonal symmetries.

In this case, the first variation given by the derivative has the form

$$\frac{\partial\mathbf{G}(\mathbf{A})}{\partial\mathbf{A}}[\mathbf{X}] = \sum_{i,j,h,k}\left[\frac{\partial G_{ij}}{\partial A_{hk}}\hat{\mathbf{B}}_{ij} \otimes \hat{\mathbf{B}}_{hk}\right]\left[X_{hk}\hat{\mathbf{B}}_{hk}\right] = \sum_{i,j,h,k}\frac{\partial G_{ij}}{\partial A_{hk}} X_{hk}\hat{\mathbf{B}}_{ij}. \tag{2.38}$$

Other important derivatives. The time derivative of a second-order tensor $\mathbf{G}$ is still a second-order tensor $\dot{\mathbf{G}}$ defined as

$$\frac{\partial\mathbf{G}(t)}{\partial t} = \sum_{i,j}\frac{\partial G_{ij}\mathbf{b}_i \otimes \mathbf{b}_j}{\partial t} = \sum_{i,j}\dot{G}_{ij}\mathbf{b}_i \otimes \mathbf{b}_j = \dot{\mathbf{G}}. \tag{2.39}$$

The $\mathbf{G}$ differential with respect to time t or the first time-variation of $\mathbf{G}$ is given by

$$d\mathbf{G}(t) = \frac{\partial\mathbf{G}(t)}{\partial t}dt = \dot{\mathbf{G}}dt.$$

The product $\mathbf{T}(\mathbf{A})$ of two differentiable tensor-valued tensor functions $\mathbf{F}(\mathbf{A})$ and $\mathbf{G}(\mathbf{A})$

$$\mathbf{T}(\mathbf{A}) = \mathbf{F}(\mathbf{A})\,\mathbf{G}(\mathbf{A})$$

is given by

$$\frac{\partial\mathbf{T}(\mathbf{A})}{\partial\mathbf{A}}[\mathbf{B}] = \frac{\partial\mathbf{F}(\mathbf{A})}{\partial\mathbf{A}}[\mathbf{B}]\,\mathbf{G}(\mathbf{A}) + \mathbf{F}(\mathbf{A})\frac{\partial\mathbf{G}(\mathbf{A})}{\partial\mathbf{A}}[\mathbf{B}]. \tag{2.40}$$

Let the tensor-valued tensor function $\mathbf{G}(\mathbf{A})$ be differentiable at $\mathbf{A}$ and let the tensor-valued function $\mathbf{J}$ be differentiable in $\mathbf{G}(\mathbf{A})$.

Then the derivative of the composition

$$\mathbf{M}(\mathbf{A}) = \mathbf{J}(\mathbf{G}(\mathbf{A}))$$

at $\mathbf{A}$ on the increment $\mathbf{B}$, is given by the chain rule

$$\frac{\partial \mathbf{M}(\mathbf{A})}{\partial \mathbf{A}}[\mathbf{B}] = \frac{\partial \mathbf{J}(\mathbf{G}(\mathbf{A}))}{\partial \mathbf{G}} \frac{\partial \mathbf{G}(\mathbf{A})}{\partial \mathbf{A}}[\mathbf{B}]. \tag{2.41}$$

Moreover, if $\mathbf{A} = \mathbf{A}(t)$ we have

$$\dot{\mathbf{M}}(\mathbf{A}(t)) = \frac{\partial \mathbf{M}(\mathbf{A}(t))}{\partial t} = \frac{\partial \mathbf{J}(\mathbf{G}(\mathbf{A}))}{\partial \mathbf{G}} \frac{\partial \mathbf{G}(\mathbf{A})}{\partial \mathbf{A}} \dot{\mathbf{A}}(t). \tag{2.42}$$

3. Kinematics

Let us now consider a body $\mathcal{B} \subset V_3$ and associate each particle to its position vector $\mathbf{x}$ in the reference configuration $\mathcal{B}_o \in V_3$. A motion of $\mathcal{B}$ is represented by a one-parameter mapping $\mathbf{y}_t : \mathcal{B}_o \to \mathcal{B}_t$
where

$$\mathcal{B}_t \doteq \mathbf{y}_t(\mathcal{B}_o) = \{\mathbf{y}(\mathbf{x},t) \in V_3 : \mathbf{x} \in \mathcal{B}_o\}.$$

The actual position $\mathbf{y}(\mathbf{x},t)$ of the material point at time t, can be decomposed into the previous position $\mathbf{x} = \mathbf{y}(\mathbf{x},t_o)$ and a displacement $\mathbf{u}$

$$\mathbf{y}(\mathbf{x},t) = \mathbf{x} + \mathbf{u}(\mathbf{x},t). \tag{3.1}$$

The deformation gradient of $\mathbf{y}$

$$\mathbf{F} = \nabla_{\mathbf{x}} \mathbf{y} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \tag{3.2}$$

represents the tangent mapping between the two tangent spaces of the body B, respectively in $\mathcal{B}_o$ and $\mathcal{B}_t$:

$$\mathbf{F} : T_x(\mathcal{B}_o) \to T_y(\mathcal{B}_t). \tag{3.3}$$

Its determinant can be shown to be always positive; i.e.: $\mathbf{F}$ is an element of Lin^+.

The inverse of $\mathbf{F}$ is obviously

$$\mathbf{F}^{-1} = \nabla_{\mathbf{y}} \mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{y}}. \tag{3.4}$$

In finite strain, the deformation gradient $\mathbf{F}$ is usually decomposed, according to the polar decomposition theorem, [23–27] into

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{3.5}$$

where $\mathbf{U}$ and $\mathbf{V}$ are the right and left stretch tensors and $\mathbf{R}$ is the rotation tensor.

$\mathbf{U}$ and $\mathbf{V}$ are uniquely obtained by the following relations:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2 \tag{3.6}$$

where $\mathbf{C}$ and $\mathbf{B}$ are positive definite symmetric tensors and they are called respectively the right and left Cauchy–Green tensors. The rotation tensor $\mathbf{R}$ is determined by (3.5).

Since both $\mathbf{U}$ and $\mathbf{V}$ are defined in (3.6) as square roots of positive definite symmetric tensors, they are positive definite tensors too and coaxial to $\mathbf{C}$ and $\mathbf{B}$, respectively, i.e. they have the same eigenvectors of $\mathbf{C}$ and $\mathbf{B}$, respectively.

Thus, the following spectral decompositions can be considered:

$$\begin{aligned} \mathbf{C} &= \mathbf{R}_L\,\boldsymbol{\lambda}^2\mathbf{R}_L^T, & \mathbf{B} &= \mathbf{R}_E\,\boldsymbol{\lambda}^2\mathbf{R}_E^T, \\ \mathbf{U} &= \mathbf{R}_L\,\sqrt{\boldsymbol{\lambda}^2}\mathbf{R}_L^T, & \mathbf{V} &= \mathbf{R}_E\,\sqrt{\boldsymbol{\lambda}^2}\mathbf{R}_E^T, \end{aligned} \tag{3.7}$$

where $\mathbf{U}, \mathbf{V}, \boldsymbol{\lambda}^2, \sqrt{\boldsymbol{\lambda}^2} \in PSym$ and $\mathbf{R}_L, \mathbf{R}_E \in orth^+$.

Hence the diagonal tensors $\boldsymbol{\lambda}^2$ and $\sqrt{\boldsymbol{\lambda}^2}$ collect the eigenvalues λ_i^2 and their square roots $\sqrt{\lambda_i^2}$ respectively. The orthonormalized eigenvectors of $\mathbf{F}^T\mathbf{F}$ and $\mathbf{F}\mathbf{F}^T$ are the columns of $\mathbf{R}_L$ and $\mathbf{R}_E$ respectively.

Moreover, the time continuity of relation

$$\det\mathbf{F}\,(t) > 0 \tag{3.8}$$

makes it possible to choose positively orientated eigenvectors so that the tensors $\mathbf{R}_L$ and $\mathbf{R}_E$ can be proper orthogonal i.e.: $\sqrt{\lambda_i^2} = \lambda_i$.

The last two relations in (3.7) can be written

$$\mathbf{U} = \mathbb{R}_{LL}\,\boldsymbol{\lambda}, \qquad \mathbf{V} = \mathbb{R}_{EE}\,\boldsymbol{\lambda}, \tag{3.9}$$

where

$$\mathbb{R}_{LL} = \mathbf{R}_L \boxtimes \mathbf{R}_L, \qquad \mathbb{R}_{EE} = \mathbf{R}_E \boxtimes \mathbf{R}_E. \tag{3.10}$$

Very often if the left and right rotations coincide, we drop an index. For example we write $\mathbb{R}_L$ for $\mathbb{R}_{LL}$. We remind that the rotations $\mathbf{R}_L$ and $\mathbf{R}_E$ are called Lagrangian and Eulerian respectively because they collect the bases of the fundamental tensors $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{B}^{-1} = [\mathbf{F}\mathbf{F}^T]^{-1}$ which describe the deformation in the two approaches.

Generally, the tensors used to measure the deformation are strictly determined by the previous ones. In other words, they exploit the same set of eigenvectors while each eigenvalue is an appropriate invertible function of the corresponding eigenvalue.

Following this reasoning, according to Hill [11–13], Wang and Truesdell [26], Ogden [28] two general classes of deformation tensors known as Hill'strains can be defined

$$\mathbf{G}(\mathbf{U}) = \sum_i f_i(\lambda_i)\,\mathbf{e}_i^L \otimes \mathbf{e}_i^L, \qquad \mathbf{G}(\mathbf{V}) = \sum_i f_i(\lambda_i)\,\mathbf{e}_i^E \otimes \mathbf{e}_i^E. \tag{3.11}$$

The tensors in (3.11) are symmetric and respectively coaxial with $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$ and $\mathbf{V}^{-1} = \sqrt{\mathbf{F}^{-T}\mathbf{F}^{-1}}$ with eigenvalues f_i subject to the normalized conditions

$$f_i(1) = 0, \qquad f_i'(1) = 1, \qquad f_i' > 0. \tag{3.12}$$

Isotropic functions. A tensor-valued tensor function $\mathbf{G}(\mathbf{B})$ is said to be isotropic if

$$\mathbf{Q}[\mathbf{G}(\mathbf{B})]\,\mathbf{Q}^T = [\mathbf{G}(\mathbf{Q}\mathbf{B}\mathbf{Q}^T)], \qquad \forall \mathbf{Q} \in orth^+ \tag{3.13}$$

that is

$$\mathbb{Q}[\mathbf{G}(\mathbf{B})] = [\mathbf{G}(\mathbb{Q}\mathbf{B})]. \tag{3.14}$$

If $\mathbf{B} \in Sym$, $\mathbf{G}(\mathbf{B})$ has the general representation

$$\mathbf{G}(\mathbf{B}) = \sum_i g_i \mathbf{n}_i \otimes \mathbf{n}_i \tag{3.15}$$

where

$$g_i = g_i(\lambda_1, \lambda_2, \lambda_3).$$

The Hill strains in (3.11) can be recognized as a particular subclass of the isotropic functions defined by (3.13).

The following are the particular Hill strains [29, 30]

$$\begin{cases} \mathbf{D}_x^m = \dfrac{1}{m}[\mathbf{U}^m - \mathbf{I}] & m \neq 0 \\ \mathbf{D}_x^0 = \ln \mathbf{U} & m = 0 \end{cases} \qquad \begin{cases} \mathbf{D}_y^m = \dfrac{1}{m}[\mathbf{V}^m - \mathbf{I}] & m \neq 0 \\ \mathbf{D}_y^0 = \ln \mathbf{V} & m = 0 \end{cases} \tag{3.16}$$

These strain tensor classes are broad enough to include almost every Lagrangian and Eulerian strain measures used in the literature.

For $m > 0$, the first class describes the deformation in the Lagrangian approach; for $m < 0$, the second class describes the deformation in the Eulerian approach.

For example, the following formulae

$$\mathbf{D}_x^2 = \frac{1}{2}\Big[\mathbf{F}^T\mathbf{F} - \mathbf{I}\Big], \qquad \mathbf{D}_y^{-2} = \frac{1}{2}\Big[\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}\Big] \tag{3.17}$$

are respectively the Green–Lagrange tensor and the Almansi tensor.

3.1. *Singular value decomposition of* $\mathbf{F}$ *(SVD)*

Because we are concerned with finite strain measures, which can have either a Lagrangan or an Eulerian base, we briefly review the operators that make it possible "to switch" from one kind of strain measure to another. The key role in this change of bases is undoubtedly played by tensor $\mathbf{F}$. In literature we generally find the polar decomposition for $\mathbf{F}$ and the spectral decomposition for the strain measures. In what follows we shall introduce for $\mathbf{F}$ a particular multiplicative decomposition, the singular value decomposition (**SVD**) which generalizes the spectral decomposition. Further it will be shown that, even for the desired switching operators between the Lagrangian frame and the Eulerian frame, a generalization of the singular value decomposition still holds.

Now, going back to (3.9) we observe that $\mathbf{U}$ and $\mathbf{V}$ can be obtained through the same diagonal tensor $\boldsymbol{\lambda}$ but using two different frame rotations.

Unfortunately, for $\mathbf{F}$ the decomposition in (3.9) doesn't hold. This physically means that a real frame rotation leading to principal axis for $\mathbf{F}$ does not exist.

Anyway, another similar decomposition is still available.

Let's consider the spectral decompositions of $\mathbf{F}^T\mathbf{F}$ and $\mathbf{F}\mathbf{F}^T$

$$\mathbf{R}_L^T\mathbf{F}^T\mathbf{F}\mathbf{R}_L = \boldsymbol{\lambda}^2 \qquad \mathbf{R}_E^T\mathbf{F}\mathbf{F}^T\mathbf{R}_E = \boldsymbol{\lambda}^2 \tag{3.18}$$

We can rewrite $(3.18)_1$ in the form:

$$\Big[\mathbf{F}\mathbf{R}_L\ |\boldsymbol{\lambda}|^{-1}\Big]^T\Big[\mathbf{F}\mathbf{R}_L\ |\boldsymbol{\lambda}|^{-1}\Big] = \mathbf{I} \tag{3.19}$$

while from $(3.18)_2$ we obtain

$$\mathbf{F}\mathbf{R}_L\ |\boldsymbol{\lambda}|^{-1} = \mathbf{R}_E. \tag{3.20}$$

From Eq. (3.20) for

$$\Big[\mathbf{R}_L\,|\boldsymbol{\lambda}|^{-1}\Big]^{-1} = |\boldsymbol{\lambda}|\,\mathbf{R}_L^T \tag{3.21}$$

we finally get

$$\mathbf{F} = \mathbf{R}_E \, |\boldsymbol{\lambda}| \, \mathbf{R}_L^T. \tag{3.22}$$

Furthermore, the time continuity of (3.8), together with the initial conditions at $t = t_0$:

$$\mathbf{R}_E (t_0) = \mathbf{R}_L (t_0) = \mathbf{I}, \qquad \boldsymbol{\lambda}_{ij} (t_0) = \delta_{ij}, \tag{3.23}$$

ensure the positiveness of each $\boldsymbol{\lambda}_{ii}$; i.e.: $\boldsymbol{\lambda}_{ii} = |\boldsymbol{\lambda}_{ii}|$.

The decomposition in (3.22) then becomes

$$\mathbf{F} = \mathbf{R}_E \, \boldsymbol{\lambda} \, \mathbf{R}_L^T \tag{3.24}$$

or

$$\mathbf{F} = \mathbb{R}_{EL} \, \boldsymbol{\lambda}, \tag{3.25}$$

where $\mathbb{R}_{EL}$ is defined as

$$\mathbb{R}_{EL} \doteq \mathbf{R}_E \boxtimes \mathbf{R}_L. \tag{3.26}$$

It is easy to verify that the **SVD** in (3.24) is a generalization of the spectral decomposition and implicitly contains the polar decomposition too.

In fact, if we apply the **SVD** to a symmetric tensor, say $\mathbf{U}$, then we have a real frame rotation given by $\mathbb{R}_{LL}$, thus recovering the spectral decomposition in $(3.7)_2$.

Further, the two polar decompositions come directly from the definition in (3.24)

$$\begin{aligned} \mathbf{F} &= \left[\mathbf{R}_E \left[\mathbf{R}_L^T \mathbf{R}_L\right] \boldsymbol{\lambda} \mathbf{R}_L^T\right] = \mathbf{RU}, \\ \mathbf{F} &= \left[\mathbf{R}_E \boldsymbol{\lambda} \left[\mathbf{R}_E^T \mathbf{R}_E\right] \mathbf{R}_L^T\right] = \mathbf{VR}. \end{aligned} \tag{3.27}$$

From (3.27) we also get the multiplicative decomposition of the Cauchy rotation tensor

$$\mathbf{R} = \mathbf{R}_E \mathbf{R}_L^T \tag{3.28}$$

that leads to the corresponding decomposition on $\mathbb{L}in$

$$\mathbb{R} = (\mathbf{R}_E \mathbf{R}_L^T) \boxtimes (\mathbf{R}_E \mathbf{R}_L^T) = \mathbb{R}_E \mathbb{R}_L^T. \tag{3.29}$$

The singular value decomposition (**SVD**) or, in other words, the factorization of a matrix $\mathbf{A}$ into the product

$$\mathbf{A} = \mathbf{Q}_L \boldsymbol{\lambda} \mathbf{Q}_R^T = [\mathbf{Q}_L \boxtimes \mathbf{Q}_R] \, \boldsymbol{\lambda}$$

of unitary matrices $\mathbf{Q}_L$ and $\mathbf{Q}_R$ and a non negative diagonal matrix $\boldsymbol{\lambda}$, has a long and well-established history, see for example the paper of STEWART [31]. The existence theorem for the **SVD** can be found in standard texts on linear algebra such as [32–34]. Although it was introduced in the 1870's by BELTRAMI and JORDAN, **SVD** has become more popular in applied mathematics [32] and mathematical modelling than in continuum mechanics. Singular value analysis are widely performed in least square fitting of data [35] or in data mining applications [38]. Nowadays **SVD** is also currently used by some automated search engines in the Web, e.g., *Alta Vista*.

To the Author's knowledge, in solid mechanics, the explicit application of the **SVD** is very rare [36, 37].

If we look at the three factorizations

$$\mathbf{V} = \mathbf{R_E}\boldsymbol{\lambda}\mathbf{R}_\mathbf{E}^T, \qquad \mathbf{F} = \mathbf{R_E}\boldsymbol{\lambda}\mathbf{R}_\mathbf{L}^T, \qquad \mathbf{U} = \mathbf{R_L}\boldsymbol{\lambda}\mathbf{R}_\mathbf{L}^T \tag{3.30}$$

we recognize that all of them describe the same deformation $\boldsymbol{\lambda}$ seen through the following operators

$$\mathbb{R}_E \qquad \mathbb{R}_{EL} \qquad \mathbb{R}_L \tag{3.31}$$

From the above we recognize $\mathbf{F}$ as the link between the Lagrangian and Eulerian approaches.

Sometimes, for this reason, $\mathbf{U}$ and $\mathbf{V}$ are called one-point tensors while $\mathbf{F}$ is called a two-point tensor in order to underline the role played in the deformation [39]. In our opinion, looking at the relations in (3.30) more appropriate names would be respectively one-base and two-base tensors.

As it is well known, the fourth-order tensor

$$\mathbb{F} = \mathbf{F} \boxtimes \mathbf{F} = \mathbb{R}_E\, \lambda^2\, \mathbb{R}_L^T \tag{3.32}$$

is the operator commonly used to switch from an Eulerian-based deformation tensor to a Lagrangean-based deformation tensor and vice-versa. It's worthwhile to note that (3.32) can be viewed as a singular value decomposition on $\mathbb{L}in$ where the dilatation $\lambda^2 = \boldsymbol{\lambda} \boxtimes \boldsymbol{\lambda}$ is responsible for the change of the measures from λ^{-1} to λ.

For example we can write

$$\left\{\begin{aligned} \mathbb{F}^{-T}\mathbf{D}_x^2, &\;= \mathbf{D}_y^{-2}, \\ \mathbb{F}^{-T}\sqrt{\mathbf{C}}, &\;= \sqrt{\mathbf{B}^{-1}}, \end{aligned}\right. \qquad \left\{\begin{aligned} \mathbb{F}^{T}\mathbf{D}_y^{-2}, &\;= \mathbf{D}_x^2, \\ \mathbb{F}^{T}\sqrt{\mathbf{B}^{-1}}, &\;= \sqrt{\mathbf{C}}, \end{aligned}\right. \tag{3.33}$$

where

$$\left\{\begin{aligned} \mathbb{F}^T &= \mathbf{F}^T \boxtimes \mathbf{F}^T &&= \mathbb{R}_L\lambda^2\mathbb{R}_E^T, \\ \mathbb{F}^{-T} &= \mathbf{F}^{-T} \boxtimes \mathbf{F}^{-T} &&= \mathbb{R}_E\lambda^{-2}\mathbb{R}_L^T, \end{aligned}\right. \tag{3.34}$$

Relations in (3.33) can be easily generalized:

$$\left\{\begin{array}{l}\mathbb{F}_m^{-T}\mathbf{D}_x^m = \mathbf{D}_y^{-m},\\ \mathbb{F}_m^{-T}\mathbf{U}^m = \mathbf{V}^{-m},\end{array}\right. \qquad \left\{\begin{array}{l}\mathbb{F}_m^{T}\mathbf{D}_y^{-m} = \mathbf{D}_x^m,\\ \mathbb{F}_m^{T}\mathbf{V}^{-m} = \mathbf{U}^m,\end{array}\right. \tag{3.35}$$

where

$$\left\{\begin{array}{lll}\mathbb{F}_m^T & = \mathbf{F}_p^T \boxtimes \mathbf{F}_q^T & = \mathbb{R}_L\lambda^m\mathbb{R}_E^T,\\ \mathbb{F}_m^{-T} & = \mathbf{F}_{-p}^{-T} \boxtimes \mathbf{F}_{-q}^{-T} & = \mathbb{R}_E\lambda^{-m}\mathbb{R}_L^T,\end{array}\right. \tag{3.36}$$

and

$$\mathbf{F}_p = \mathbf{R}_E\,\lambda^p\,\mathbf{R}_L^T, \qquad \mathbf{F}_q = \mathbf{R}_E\,\lambda^q\,\mathbf{R}_L^T$$

with $p + q = m$.

Relations (3.35) ensure the change of deformation strains from $\mathbf{D}_x^m$ to $\mathbf{D}_y^{-m}$ and vice-versa. On the other hand, if we want to switch from $\mathbf{U}^m$ to $\mathbf{V}^m$, we need just the rotation $\mathbb{R}$ because in this case, no change in the measures occurs

$$\mathbb{R}\mathbf{U}^m = \left[\mathbb{R}_E\mathbb{R}_L^T\right]\left[\mathbb{R}_L\lambda^m\right] = \mathbb{R}_E\lambda^m = \mathbf{V}^m. \tag{3.37}$$

Note that even for the operators in (3.36) we have a singular value decomposition on $\mathbb{L}in$.

3.1.1. The eigenbases of $\mathbf{F}^T\mathbf{F}$ *and* $\mathbf{F}\mathbf{F}^T$. Let us introduce a slightly different representation of the deformation tensors

$$\left\{\begin{array}{l}\mathbf{C} = \sum_i \lambda_i^2\mathbf{E}_i^L,\\ \mathbf{U} = \sum_i \lambda_i\mathbf{E}_i^L,\end{array}\right. \qquad \left\{\begin{array}{l}\mathbf{B} = \sum_i \lambda_i^2\mathbf{E}_i^E,\\ \mathbf{V} = \sum_i \lambda_i\mathbf{E}_i^E,\end{array}\right. \tag{3.38}$$

where

$$\mathbf{E}_i^L = \mathbf{e}_i^L \otimes \mathbf{e}_i^L, \qquad \mathbf{E}_i^E = \mathbf{e}_i^E \otimes \mathbf{e}_i^E \tag{3.39}$$

are respectively the Lagrangian and the Eulerian eigenbases.

The eigenbases are orthonormal

$$\left\{\begin{array}{ll}\mathbf{I} & = \sum_i \mathbf{E}_i^L,\\ \mathbf{E}_i^L\mathbf{E}_j^L, & = \delta_{ij}\mathbf{E}_j^L,\end{array}\right. \qquad \left\{\begin{array}{ll}\mathbf{I} & = \sum_i \mathbf{E}_i^E,\\ \mathbf{E}_i^E\mathbf{E}_j^E & = \delta_{ij}\mathbf{E}_j^E.\end{array}\right. \tag{3.40}$$

Note that $\mathbf{F}$ can be recovered as

$$\mathbf{F} = \sum_i \lambda_i \mathbf{e}_i^E \otimes \mathbf{e}_i^L = \sum_i \lambda_i \mathbf{E}_i^{EL}, \tag{3.41}$$

where

$$\mathbf{E}_i^{EL} = \mathbf{e}_i^E \otimes \mathbf{e}_i^L \tag{3.42}$$

are the mixed Lagrangan–Eulerian eigenbases.

Hence the following representations

$$\mathbf{R} = \sum_h \mathbf{E}_h^{EL}, \tag{3.43}$$

$$\mathbf{V} = \sum_h \lambda_h \mathbf{E}_h^E, \qquad \mathbf{F} = \sum_h \lambda_h \mathbf{E}_h^{EL}, \qquad \mathbf{U} = \sum_h \lambda_h \mathbf{E}_h^L \tag{3.44}$$

are the projection's counterpart of (3.28) and (3.30) respectively.

3.1.2. The eigenvalues of a symmetric tensor. Let us introduce the modulus and the versor of a symmetric tensor **A**

$$|\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}} = \sqrt{\operatorname{tr}(\mathbf{A}^T\mathbf{A})} \qquad \hat{\mathbf{A}}_u = \frac{1}{|\mathbf{A}|}\mathbf{A}. \tag{3.45}$$

Let us denote by

$$\mathbf{A}^D = \mathbf{A} - \frac{1}{3}\operatorname{tr}(\mathbf{A})\,\mathbf{I} \tag{3.46}$$

the deviatoric component of **A**.

Now the deviatoric component of the versor of **A**, defined as

$$\mathbf{A}_U^D = \frac{1}{|\mathbf{A}^D|}\mathbf{A}^D \tag{3.47}$$

has its characteristic polynomial with the following reduced form:

$$\varepsilon_i^3 + II_{\mathbf{A}_U^D}\,\varepsilon_i - III_{\mathbf{A}_U^D} = 0. \tag{3.48}$$

Hence, from (3.48), see [40], we get the following closed-form solutions for the eigenvalues ε_i

$$\varepsilon_1 = \varepsilon_0 \cos\left(\varphi - \frac{2\pi}{3}\right), \quad \varepsilon_2 = \varepsilon_0 \cos\varphi, \quad \varepsilon_3 = \varepsilon_0 \cos\left(\varphi + \frac{2\pi}{3}\right),$$

where

$$\varepsilon_0 = \sqrt{\frac{2}{3}}, \qquad \cos 3\varphi = 3\sqrt{6}\,III_{\mathbf{A}_U^D}.$$

The eigenvalues a_i of $\mathbf{A}$ are then obtained by the formula

$$a_i = \left|\mathbf{A}^D\right| \varepsilon_i + \frac{1}{3}\mathrm{tr}\,(\mathbf{A})\,.$$

Once the eigenvalues are known, the eigenbases can be obtained either by the principal invariants methods or by projections methods.

Following the second approach we can use Sylvester's formula

$$\mathbf{A}_j = \prod_{i, i \neq j} \frac{(\mathbf{A} - a_i \mathbf{I})}{(a_j - a_i)} = \frac{1}{D_j} \prod_{i, i \neq j} (\mathbf{A} - a_i \mathbf{I})\,, \tag{3.49}$$

where $D_j = \prod_{i, i \neq j} (a_j - a_i)$.

4. Derivatives of some isotropic functions

The aim of the work is to evaluate, as simple as we can, the derivatives of some isotropic functions with respect to the deformation gradient $\mathbf{F}$. First, a simple but effective demonstration of the well-known formula for the derivative of isotropic function is given.

Starting from $\mathbf{C}$ and $\mathbf{B}$ we can consider some Lagrangian and Eulerian isotropic functions of the kind

$$\mathbf{G}_L(\mathbf{C}) = \mathbf{G}_L\left(\mathbf{F}^T\mathbf{F}\right), \qquad \mathbf{G}_E(\mathbf{B}) = \mathbf{G}_E\left(\mathbf{F}\mathbf{F}^T\right), \tag{4.1}$$

whose derivatives with respect to $\mathbf{F}$ can be obtained through the chain rule as

$$\frac{\partial \mathbf{G}_L}{\partial \mathbf{F}} = \frac{\partial \mathbf{G}_L}{\partial\left(\mathbf{F}^T\mathbf{F}\right)} \frac{\partial\left(\mathbf{F}^T\mathbf{F}\right)}{\partial \mathbf{F}}, \qquad \frac{\partial \mathbf{G}_E}{\partial \mathbf{F}} = \frac{\partial \mathbf{G}_E}{\partial\left(\mathbf{F}\mathbf{F}^T\right)} \frac{\partial\left(\mathbf{F}\mathbf{F}^T\right)}{\partial \mathbf{F}}. \tag{4.2}$$

Hence, the first step to undertake is the evaluation of either $\partial \mathbf{G}_L / \partial\left(\mathbf{F}^T\mathbf{F}\right)$ or $\partial \mathbf{G}_E / \partial\left(\mathbf{F}\mathbf{F}^T\right)$.

In what follows, because there is no need to reduce the reasoning to V_3, we shall consider again a general n-dimensional inner product space V_n.

Let us consider the following isotropic function of $\mathbf{A}$:

$$\mathbf{G}(\mathbf{A}) = \sum_i g(a_i)\; \mathbf{a}_i \otimes \mathbf{a}_i, \qquad \mathbf{A} \in \mathrm{Sym}\,(V_n)\,, \tag{4.3}$$

where a_i and $\mathbf{a}_i \otimes \mathbf{a}_i$ are respectively the eigenvalues and the eigenprojections of $\mathbf{A}$ and $g(a_i)$ is a single-scale function satisfying (3.12).

Because the derivative of the function in (4.3) is an element of $\mathbb{S}\mathrm{ym}$, we can use the eigenvectors $\mathbf{a}_i$ to define, as in (2.35), the new reduced bases

$$\hat{\mathbf{A}}_{ij} \quad \text{on} \quad \mathrm{Sym} \qquad \hat{\mathbf{A}}_{ij} \otimes \hat{\mathbf{A}}_{hk} \quad \text{on} \quad \mathbb{S}\mathrm{ym}. \tag{4.4}$$

4.1. Derivative along an eigenbase direction

The value of the derivative along an eigenbase direction $\hat{\mathbf{A}}_{ii} = \mathbf{A}_i = \mathbf{a}_i \otimes \mathbf{a}_i$ of $\mathbf{A}$ is simple to calculate, see [16]:

$$(4.5) \qquad d\mathbf{G}\left(\mathbf{A}\right)_{\mathbf{A}_i} = \lim_{\epsilon\to 0}\frac{\mathbf{G}\left(\mathbf{A}+\epsilon\left(\mathbf{a}_i\otimes\mathbf{a}_i\right)\right)-\mathbf{G}\left(\mathbf{A}\right)}{\epsilon}$$

$$= \lim_{\epsilon\to 0}\frac{\sum_{j\neq i} G\left(a_j\right)\mathbf{a}_j\otimes\mathbf{a}_j + G\left(a_i+\epsilon\right)\mathbf{a}_i\otimes\mathbf{a}_i - G\left(a_i\right)\mathbf{a}_i\otimes\mathbf{a}_i - \sum_{j\neq i} G\left(a_j\right)\mathbf{a}_j\otimes\mathbf{a}_j}{\epsilon}$$

$$= \lim_{\epsilon\to 0}\frac{G\left(a_i+\epsilon\right)\mathbf{a}_i\otimes\mathbf{a}_i - G\left(a_i\right)\mathbf{a}_i\otimes\mathbf{a}_i}{\epsilon} = \frac{\partial G}{\partial a_i}\left(a_i\right)\mathbf{a}_i\otimes\mathbf{a}_i = G'\left(a_i\right)\mathbf{a}_i\otimes\mathbf{a}_i.$$

In other words, the variation $\epsilon\mathbf{A}_i$ along a single eigenbase $\mathbf{A}_i$ produces only a variation $\epsilon = da_i$ to the a_i eigenvalue. In turn da_i causes a variation of G equal to $\frac{\partial G}{\partial a_i}\left(a_i\right)da_i = G'\left(a_i\right)\epsilon$.

We must remark here that the variation $\epsilon\mathbf{A}_i$ does not involve any change in the eigenvectorbase.

4.2. Derivative along a shear base direction

The following procedure is different and, in our opinion, simpler than the one given in [16]. Recurring to a linear pertubation of both eigenvalues and eigenvectors we arrive to the well-known formula within few passages and, maybe not less important, in the way that still preserves a physical meaning.

Let's now consider a derivative along a shear base direction $\hat{\mathbf{A}}_{ij} = \frac{1}{\sqrt{2}}\left(\mathbf{a}_i\otimes\mathbf{a}_j+\mathbf{a}_j\otimes\mathbf{a}_i\right)$.

$$(4.6) \qquad d\mathbf{G}\left(\mathbf{A}\right)_{\left(\sqrt{2}\hat{\mathbf{A}}_{ij}\right)} = \lim_{\epsilon\to 0}\frac{\mathbf{G}\left(\mathbf{A}+\epsilon\left(\mathbf{a}_i\otimes\mathbf{a}_j+\mathbf{a}_j\otimes\mathbf{a}_i\right)\right)-\mathbf{G}\left(\mathbf{A}\right)}{\epsilon}$$

$$= \lim_{\epsilon\to 0}\frac{\Delta\mathbf{G}_{\epsilon\sqrt{2}\hat{\mathbf{A}}_{ij}}}{\epsilon}.$$

Due to the variation of $\mathbf{A}$ equal to $\epsilon\sqrt{2}\hat{\mathbf{A}}_{ij} = \epsilon\left(\mathbf{a}_i\otimes\mathbf{a}_j+\mathbf{a}_j\otimes\mathbf{a}_i\right)$ we have

$$(4.7) \qquad \Delta\mathbf{G}_{\epsilon\sqrt{2}\hat{\mathbf{A}}_{ij}} = \mathbf{G}\left(\sum_{h}^{h\neq i\neq j} a_h\mathbf{a}_h\otimes\mathbf{a}_h + a_i\mathbf{a}_i\otimes\mathbf{a}_i + a_j\mathbf{a}_j\otimes\mathbf{a}_j + \epsilon\left(\mathbf{a}_i\otimes\mathbf{a}_j+\mathbf{a}_j\otimes\mathbf{a}_i\right)\right)$$

$$-\mathbf{G}\left(\sum_h a_h\mathbf{a}_h\otimes\mathbf{a}_h\right),$$

that is

$$(4.8)\qquad \Delta\mathbf{G}_{\epsilon\sqrt{2}\hat{\mathbf{A}}_{ij}} = + \sum_{h}^{h\neq i\neq j} \mathbf{G}\,(a_h)\mathbf{a}_h \otimes \mathbf{a}_h$$

$$+ \,\mathbf{G}\,[a_i\mathbf{a}_i \otimes \mathbf{a}_i + a_j\mathbf{a}_j \otimes \mathbf{a}_j + \epsilon(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i)]$$

$$- \sum_h \mathbf{G}\,(a_h)\,\mathbf{a}_h \otimes \mathbf{a}_h$$

Eq. (4.8) can be simplified to

$$(4.9)\qquad \Delta\mathbf{G}_{\epsilon\sqrt{2}\hat{\mathbf{A}}_{ij}} = + \,\mathbf{G}\,[a_i\mathbf{a}_i \otimes \mathbf{a}_i + a_j\mathbf{a}_j \otimes \mathbf{a}_j + \epsilon(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i)]$$

$$- \,\mathbf{G}\,(a_i\mathbf{a}_i \otimes \mathbf{a}_i + a_j\mathbf{a}_j \otimes \mathbf{a}_j).$$

If we denote by V_{ij} the plane spanned by the couple $(\mathbf{a}_i, \mathbf{a}_j)$, we recognize that

$$(4.10)\qquad a_i\mathbf{a}_i \otimes \mathbf{a}_i + a_j\mathbf{a}_j \otimes \mathbf{a}_j + \epsilon\,(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i)$$

is a vector-valued function on V_{ij} and it can be expressed in its principal base, say $(\bar{\mathbf{a}}_i, \bar{\mathbf{a}}_j)$, through a rotation of the couple $(\mathbf{a}_i, \mathbf{a}_j)$ in V_{ij} and the resulting changes in the eigenvalues a_i and a_j:

$$(4.11)\qquad \bar{a}_i\bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_i + \bar{a}_j\bar{\mathbf{a}}_j \otimes \bar{\mathbf{a}}_j = a_i\mathbf{a}_i \otimes \mathbf{a}_i + a_j\mathbf{a}_j \otimes \mathbf{a}_j + \epsilon\,(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i).$$

Because of (4.11), Eq. (4.9) becomes

$$(4.12)\qquad \Delta\mathbf{G}_{\epsilon\sqrt{2}\hat{\mathbf{A}}_{ij}} = G\,(\bar{a}_i)\,\bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_i + G\,(\bar{a}_j)\,\bar{\mathbf{a}}_j \otimes \bar{\mathbf{a}}_j - G\,(a_i)\,\mathbf{a}_i \otimes \mathbf{a}_i - G\,(a_j)\,\mathbf{a}_j \otimes \mathbf{a}_j.$$

With the result of (4.12) in mind, Eq. (4.6) assumes the more convenient form

$$(4.13)\qquad d\mathbf{G}\,(\mathbf{A})_{\epsilon\sqrt{2}\hat{\mathbf{A}}_{ij}} = \lim_{\epsilon\to 0} \frac{G\,(\bar{a}_i)\,\bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_i + G\,(\bar{a}_j)\,\bar{\mathbf{a}}_j \otimes \bar{\mathbf{a}}_j - G\,(a_i)\,\mathbf{a}_i \otimes \mathbf{a}_i - G\,(a_j)\,\mathbf{a}_j \otimes \mathbf{a}_j}{\epsilon}.$$

Note that $d\mathbf{G}\,(\mathbf{A})_{\epsilon\sqrt{2}\hat{\mathbf{A}}_{ij}} \in V_{ij}$. Now we just have to find the new expressions of both eigenvectors and eigenbases. In order to do that we have to separate the case of two equal eigenvalues.

CASE 1. $a_i \neq a_j$. We introduce the projection on the plane V_{ij}; i.e.: $[\,\cdot\,]_{V_{ij}}$ only to reduce to 2 the number of dimensions of the matrices representing the tensors involved and to avoid the use of all the zeroes needed in the directions normal to V_{ij}.

By definition, the tensor in (4.10) has in V_{ij} the following representation:

$$\begin{bmatrix} a_i & \epsilon \\ \epsilon & a_j \end{bmatrix}. \tag{4.14}$$

The eigenvectors and eigenvalues of (4.10) are the following:

$$\begin{aligned} [\bar{\mathbf{a}}_i]_{V_{ij}} &= \begin{vmatrix} \frac{1}{2}[a_i - a_j + q] \\ \epsilon \end{vmatrix}, \qquad \bar{a}_i = \frac{1}{2}(a_i + a_j + q), \\ [\bar{\mathbf{a}}_j]_{V_{ij}} &= \begin{vmatrix} \frac{1}{2\epsilon}[a_i - a_j - q] \\ 1 \end{vmatrix}, \qquad \bar{a}_j = \frac{1}{2}(a_i + a_j - q), \end{aligned} \tag{4.15}$$

where $q = \sqrt{\left(a_i^2 + a_j^2 - 2a_ia_j + 4\epsilon^2\right)}$ and $[\bar{\mathbf{a}}_i]_{V_{ij}}$, $[\bar{\mathbf{a}}_j]_{V_{ij}}$ represent the projections on V_{ij} of the new eigenvectors $\bar{\mathbf{a}}_i$, $\bar{\mathbf{a}}_j$ of the tensor $[\mathbf{A}+\epsilon(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i)]$.

If we suppose that $a_i \neq a_j$, q can be expressed in the form

$$q = (a_i - a_j)\sqrt{1 + \left[\frac{2\epsilon}{a_i - a_j}\right]^2} = (a_i - a_j)\sqrt{1 + [2\alpha]^2}, \tag{4.16}$$

where $\alpha = \epsilon / (a_i - a_j)$. Using in (4.16) the first two terms of the following expansion:

$$\sqrt{1 + [2\alpha]^2} = 1 + \frac{1}{2}[2\alpha]^2 - \frac{1}{8}[2\alpha]^4 + \frac{1}{16}[2\alpha]^6 + O\left(a^6\right)$$

we have

$$q = [(a_i - a_j) + 2\alpha\epsilon] + o\left(\epsilon^2\right). \tag{4.17}$$

Using this value of q, the relations in (4.15) become

$$\begin{aligned} [\bar{\mathbf{a}}_i]_{V_{ij}} &= \begin{vmatrix} 1 \\ +\alpha \end{vmatrix} + \left|o\left(\epsilon^2\right)\right|, \qquad \bar{a}_i = a_i + \alpha\epsilon + o\left(\epsilon^2\right), \\ [\bar{\mathbf{a}}_j]_{V_{ij}} &= \begin{vmatrix} -\alpha \\ 1 \end{vmatrix} + \left|o\left(\epsilon^2\right)\right|, \qquad \bar{a}_j = a_j - \alpha\epsilon + o\left(\epsilon^2\right). \end{aligned} \tag{4.18}$$

From (4.18) we understand that the new eigenvectors are obtained from the old ones by means of a rotation equal to $\alpha = \epsilon / (a_i - a_j)$ in the plane V_{ij}.

From (4.18) we immediately obtain the new eigenbases

$$
\begin{aligned}
&[\bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_i]_{V_{ij}} = \begin{bmatrix} 1 & +\alpha \\ +\alpha & 0 \end{bmatrix} + [o(\epsilon^2)], \\
&[\bar{\mathbf{a}}_j \otimes \bar{\mathbf{a}}_j]_{V_{ij}} = \begin{bmatrix} 0 & -\alpha \\ -\alpha & 1 \end{bmatrix} + [o(\epsilon^2)].
\end{aligned} \tag{4.19}
$$

Using (4.19) we get the new expressions

$$
\begin{aligned}
&[G(\bar{a}_i)\,\bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_i + G(\bar{a}_j)\,\bar{\mathbf{a}}_j \otimes \bar{\mathbf{a}}_j]_{V_{ij}} \\
&\qquad = \begin{bmatrix} G(\bar{a}_i) & \alpha(G(\bar{a}_i) - G(\bar{a}_j)) \\ \alpha(G(\bar{a}_i) - G(\bar{a}_j)) & G(\bar{a}_j) \end{bmatrix} + [o(\epsilon^2)] \\
&[G(a_i)\,\mathbf{a}_i \otimes \mathbf{a}_i + G(a_j)\,\mathbf{a}_j \otimes \mathbf{a}_j]_{V_{ij}} = \begin{bmatrix} G(a_i) & 0 \\ 0 & G(a_j) \end{bmatrix}.
\end{aligned} \tag{4.20}
$$

Thus, because of (4.13) and (4.20) the projection of the first variation is

$$
\begin{aligned}
&\left[d\mathbf{G}(\mathbf{A})_{\sqrt{2}\hat{\mathbf{A}}_{ij}}\right]_{V_{ij}} \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \begin{bmatrix} G(a_i + \alpha\epsilon) - G(a_i) & [G(a_i + \alpha\epsilon) - G(a_j - \alpha\epsilon)]\alpha \\ [G(a_i + \alpha\epsilon) - G(a_j - \alpha\epsilon)]\alpha & G(a_j - \alpha\epsilon) - G(a_j) \end{bmatrix}
\end{aligned} \tag{4.21}
$$

what gives

$$
\left[d\mathbf{G}(\mathbf{A})_{\sqrt{2}\hat{\mathbf{A}}_{ij}}\right]_{V_{ij}} = \begin{bmatrix} 0 & \dfrac{G(a_i) - G(a_j)}{(a_i - a_j)} \\ \dfrac{G(a_i) - G(a_j)}{(a_i - a_j)} & 0 \end{bmatrix}. \tag{4.22}
$$

So we can state that the value of the derivative along the shear base direction $(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i)$ is still of a shear deformation type of value $\dfrac{G(a_i) - G(a_j)}{(a_i - a_j)}$. Hence we can write

$$
d\mathbf{G}(\mathbf{A})_{(\sqrt{2}\hat{\mathbf{A}}_{ij})} = \frac{G(a_i) - G(a_j)}{(a_i - a_j)}\,(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i). \tag{4.23}
$$

CASE 2. $a_i = a_j$ for $i \neq j$. Let us consider now the case $a_i = a_j$. We can use as new eigenvalues and eigenvectors the following set

$$[\bar{\mathbf{a}}_i]_{V_{ij}} = \begin{vmatrix} -\dfrac{1}{\sqrt{2}} \\ +\dfrac{1}{\sqrt{2}} \end{vmatrix}, \quad \bar{a}_i = a_i - \epsilon$$

(4.24)

$$[\bar{\mathbf{a}}_j]_{V_{ij}} = \begin{vmatrix} +\dfrac{1}{\sqrt{2}} \\ +\dfrac{1}{\sqrt{2}} \end{vmatrix}, \quad \bar{a}_j = a_i + \epsilon$$

Consequently the eigenbases are

$$[\bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_i]_{V_{ij}} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

(4.25)

$$[\bar{\mathbf{a}}_j \otimes \bar{\mathbf{a}}_j]_{V_{ij}} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

We can now obtain the new expressions

$$[G(\bar{a}_i)\, \bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_i]_{V_{ij}} = \frac{1}{2} \begin{bmatrix} +G(a_i - \epsilon) & -G(a_i - \epsilon) \\ -G(a_i - \epsilon) & +G(a_i - \epsilon) \end{bmatrix},$$

(4.26)

$$[G(\bar{a}_j)\, \bar{\mathbf{a}}_j \otimes \bar{\mathbf{a}}_j]_{V_{ij}} = \frac{1}{2} \begin{bmatrix} +G(a_i + \epsilon) & +G(a_i + \epsilon) \\ +G(a_i + \epsilon) & +G(a_i + \epsilon) \end{bmatrix}.$$

From (4.13) and (4.26) we obtain

(4.27) $$\left[d\mathbf{G}(\mathbf{A})_{\sqrt{2}\hat{\mathbf{A}}_{ij}}\right]_{V_{ij}} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \begin{bmatrix} G(a_i + \epsilon) + G(a_i - \epsilon) - 2G(a_i) & G(a_i + \epsilon) - G(a_i - \epsilon) \\ G(a_i + \epsilon) - G(a_i - \epsilon) & G(a_i + \epsilon) + G(a_i - \epsilon) - 2G(a_i) \end{bmatrix}.$$

Relation (4.27) leads to

(4.28) $$\left[d\mathbf{G}(\mathbf{A})_{\sqrt{2}\hat{\mathbf{A}}_{ij}}\right]_{V_{ij}} = \begin{bmatrix} 0 & G'(a_i) \\ G'(a_i) & 0 \end{bmatrix}.$$

Introducing explicitly the shear base $(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i)$, we get from (4.28)

$$d\mathbf{G}(\mathbf{A})_{\sqrt{2}\hat{\mathbf{A}}_{ij}} = G'(a_i)(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i). \tag{4.29}$$

4.3. Derivative of $\mathbf{G}(\mathbf{A})$ *as a fourth-order tensor*

Making the following assumptions:

$$G_{ij} = \begin{cases} \dfrac{\partial G}{\partial a_i} & \text{if} \quad i = j, \\ \dfrac{\partial G}{\partial a_i} & \text{if} \quad i \neq j \quad \text{and} \quad a_i = a_j, \\ \dfrac{\mathbf{G}(a_i) - \mathbf{G}(a_j)}{(a_i - a_j)} & \text{if} \quad i \neq j \quad \text{and} \quad a_i \neq a_j, \end{cases} \tag{4.30}$$

from the results given in (4.5), (4.23) and (4.29), see for example [16], we can state that the linear transformation $\dfrac{\partial \mathbf{G}(\mathbf{A})}{\partial \mathbf{A}} : \mathrm{Sym}(V_n) \to \mathrm{Sym}(V_n)$ has the eigenvalues G_{ij} with the relative eigenvectors $(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i)$.

We are now ready to write the derivative of $\mathbf{G}(\mathbf{A})$ in its spectral form

$$\begin{aligned} D\mathbf{G}(\mathbf{A}) &= \frac{\partial \mathbf{G}}{\partial \mathbf{A}}(\mathbf{A}) \\ &= \frac{1}{4}\sum_{i,j=1} G_{ij}(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i) \otimes (\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i). \end{aligned} \tag{4.31}$$

Note that in (4.31,) the factor $\dfrac{1}{2}$ appears twice. The first one is due to the orthonormal base involved, see (4.4), the second one depends on the symmetry of the indices (i, j) and (j, i) in (4.31).

Reminding (2.38), derivative in (4.31) can also be written in a more compact form

$$D\mathbf{G}(\mathbf{A}) = \frac{1}{2}\sum_{i,j=1} G_{ij} \hat{\mathbf{A}}_{ij} \otimes \hat{\mathbf{A}}_{ij}. \tag{4.32}$$

Moreover note that the fourth-order tensor $D\mathbf{G}(\mathbf{A})_{ijhk}$ presents the major symmetry

$$D\mathbf{G}(\mathbf{A})_{ijhk} = D\mathbf{G}(\mathbf{A}.)_{hkij}.$$

In a 3-dimensional space and in the case of distinct eigenvalues, from (4.32) we get

$$DG(\mathbf{A}) = \begin{cases} + G'(a_1)\hat{\mathbf{A}}_1 \otimes \hat{\mathbf{A}}_1 + G'(a_2)\hat{\mathbf{A}}_2 \otimes \hat{\mathbf{A}}_2 + G'(a_3)\,\hat{\mathbf{A}}_3 \otimes \hat{\mathbf{A}}_3 \\ + \dfrac{G(a_2) - G(a_3)}{2(a_2 - a_3)} \hat{\mathbf{A}}_{23} \otimes \hat{\mathbf{A}}_{23} + \dfrac{G(a_3) - G(a_1)}{2(a_3 - a_1)} \hat{\mathbf{A}}_{13} \otimes \hat{\mathbf{A}}_{13} \\ + \dfrac{G(a_1) - G(a_2)}{2(a_1 - a_2)} G_{12} \hat{\mathbf{A}}_{12} \otimes \hat{\mathbf{A}}_{12} \end{cases} \tag{4.33}$$

If we want a matrix representation between the differential and the derivative, we can write

$$[d\mathbf{G}(\mathbf{A})[\mathbf{X}]]_{6,1} = [D\mathbf{G}(\mathbf{A})]_{6,6}\,[\mathbf{X}]_{6,1} \tag{4.34}$$

where, for example, by $[D\mathbf{G}(\mathbf{A})]_{6,6}$ we have denoted the matrix representation of the derivative in the Voigt notation. In extended form we have

$$\left| \begin{matrix} dG_{11} \\ dG_{22} \\ dG_{33} \\ dG_{23} \\ dG_{31} \\ dG_{12} \end{matrix} \right|_{[A,\mathbf{X}]} = \begin{bmatrix} G_{1111} & G_{1122} & G_{1133} & G_{1123} & G_{1131} & G_{1112} \\ G_{2211} & G_{2222} & G_{2233} & G_{2223} & G_{2231} & G_{2212} \\ G_{3311} & G_{3322} & G_{3333} & G_{3323} & G_{3331} & G_{3312} \\ G_{2311} & G_{2322} & G_{2333} & G_{2323} & G_{2331} & G_{2312} \\ G_{3111} & G_{3122} & G_{3133} & G_{3123} & G_{3131} & G_{3112} \\ G_{1211} & G_{1222} & G_{1233} & G_{1223} & G_{1231} & G_{1212} \end{bmatrix}_A \left| \begin{matrix} X_{11} \\ X_{22} \\ X_{33} \\ 2X_{23} \\ 2X_{31} \\ 2X_{12} \end{matrix} \right| . \tag{4.35}$$

Thus the matrix representation of the derivative in (4.33) can be written as

$$[D\mathbf{G}(\mathbf{A})]_{6,6} = \begin{bmatrix} \dfrac{\partial G}{\partial a_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{\partial G}{\partial a_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\partial G}{\partial a_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{G(a_2) - G(a_3)}{2(a_2 - a_3)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{G(a_3) - G(a_1)}{2(a_3 - a_1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{G(a_1) - G(a_2)}{2(a_1 - a_2)} \end{bmatrix} . \tag{4.36}$$

For example, if $\mathbf{G}(\mathbf{A}) = \sqrt{\mathbf{A}}$, the non-zero terms of the derivative are

$$(4.37)\qquad \begin{aligned} G_{iiii} &= D\,(a_i)^{1/2} = \frac{1}{2\sqrt{a_i}}\,, \\ G_{ijij} &= \frac{G(a_i) - G(a_j)}{(a_i - a_j)} = \frac{\sqrt{a_i} - \sqrt{a_j}}{(a_i - a_j)} = \frac{1}{\sqrt{a_i} + \sqrt{a_j}}\,. \end{aligned}$$

Note that even if $a_i = a_j$, the formula in $(4.37)_2$ is still well defined and gives the same result of $(4.37)_1$ and (4.28).

Moreover, we have seen that the eigenprojectors $(\mathbf{a}_i \otimes \mathbf{a}_i)$ can be obtained by the Sylvester formula in (3.49) without calculating the eigenvectors $\mathbf{a}_i$. Unfortunately there exists no equivalent formula for the shear base directions $(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i)$. For this reason, according to some authors, see [41], an explicit basis-free formulation can be computationally convenient.

Then, by virtue of (2.8) and (4.31), see XIAO [16], we can write

$$(4.38)\qquad \begin{aligned} D\mathbf{G}(\mathbf{A})[\mathbf{X}] &= \frac{1}{4}\sum_{i,j=1} G_{ij}\,(\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i) \otimes (\mathbf{a}_i \otimes \mathbf{a}_j + \mathbf{a}_j \otimes \mathbf{a}_i)\,\mathbf{X} \\ &= \frac{1}{2}\sum_{i,j=1} G_{ij}\Big[(\mathbf{a}_i \otimes \mathbf{a}_i)\,\mathbf{X}(\mathbf{a}_j \otimes \mathbf{a}_j) + (\mathbf{a}_j \otimes \mathbf{a}_j)\mathbf{X}(\mathbf{a}_i \otimes \mathbf{a}_i)\Big] \\ &= \sum_{i,j=1} G_{ij}\,(\mathbf{a}_i \otimes \mathbf{a}_i)\,\mathbf{X}\,(\mathbf{a}_j \otimes \mathbf{a}_j) = \sum_{i,j=1} G_{ij}\,(\mathbf{A}_i \boxtimes \mathbf{A}_j)\,\mathbf{X} \end{aligned}$$

$\mathbf{X} \in \mathrm{Sym}$.

It is important to note that through the $\boxtimes$ operator and using the same coefficients G_{ij}, we have obtained in (4.38) a basis-free expression for the derivative.

4.4. *Derivative of Isotropic Lagrangean Tensors with respect to* $\mathbf{F}$

Choosing $\mathbf{A} = \mathbf{F}^T\mathbf{F}$ we have

$$a_i = \lambda_i^2, \qquad \mathbf{a}_i = \mathbf{e}_i^L$$

and (4.32) becomes

$$(4.39)\qquad D\mathbf{G}_L\left(\mathbf{F}^T\mathbf{F}\right) = \frac{\partial \mathbf{G}_L}{\partial\left(\mathbf{F}^T\mathbf{F}\right)} = \frac{1}{2}\sum_{i,j} G_{ij}\,\hat{\mathbf{E}}_{ij}^L \otimes \hat{\mathbf{E}}_{ij}^L,$$

where

$$(4.40)\qquad \hat{\mathbf{E}}_{ij}^L = \frac{1}{\sqrt{2 + 2\delta_{ij}}}\left(\mathbf{e}_i^L \otimes \mathbf{e}_j^L + \mathbf{e}_j^L \otimes \mathbf{e}_i^L\right).$$

From (2.40) we have

$$\frac{\partial\left(\mathbf{F}^T\mathbf{F}\right)}{\partial\mathbf{F}}[\mathbf{H}]=\left(\mathbf{F}^T\mathbf{H}+\mathbf{H}^T\mathbf{F}\right)=2\,\widehat{\mathbf{F}^T\mathbf{H}}. \tag{4.41}$$

Now $\dfrac{\partial\mathbf{G}_L}{\partial\mathbf{F}}[\mathbf{H}]$ can be obtained just by applying the chain rule in $(4.2)_1$

$$\frac{\partial\mathbf{G}_L}{\partial\mathbf{F}}[\mathbf{H}]=\frac{1}{2}\sum_{i,j}G_{ij}\hat{\mathbf{E}}_{ij}^L\otimes\hat{\mathbf{E}}_{ij}^L\left(\mathbf{H}^T\mathbf{F}+\mathbf{F}^T\mathbf{H}\right). \tag{4.42}$$

Note that in (4.42) we have found the value of the derivative for the increment $\mathbf{H}$, but this value is not given as a product of a fourth-order tensor and $\mathbf{H}$. But, reminding relation $(2.7)_1$, Eq. (4.42) can be also written as

$$\frac{\partial\mathbf{G}_L}{\partial\mathbf{F}}[\mathbf{H}]=\left[\sum_{i,j}G_{ij}\hat{\mathbf{E}}_{ij}^L\otimes\mathbf{F}\hat{\mathbf{E}}_{ij}^L\right][\mathbf{H}]. \tag{4.43}$$

Thus we finally get the simplified expression

$$\frac{\partial\mathbf{G}_L}{\partial\mathbf{F}}=\left[\sum_{i,j}G_{ij}\hat{\mathbf{E}}_{ij}^L\otimes\mathbf{F}\hat{\mathbf{E}}_{ij}^L\right]. \tag{4.44}$$

Moreover, from the expression of $\mathbf{F}$ in (3.41) we have

$$\mathbf{F}\hat{\mathbf{E}}_{ij}^L=\frac{1}{\sqrt{2+2\delta_{ij}}}\left[\sum_h\lambda_h\mathbf{e}_h^E\otimes\mathbf{e}_h^L\right]\mathbf{e}_i^L\otimes\mathbf{e}_j^L=\frac{\lambda_i}{\sqrt{2+2\delta_{ij}}}\mathbf{e}_i^E\otimes\mathbf{e}_j^L \tag{4.45}$$

so that (4.44) can be also written as

$$\frac{\partial\mathbf{G}_L}{\partial\mathbf{F}}=\left[\sum_{i,j}G_{ij}\lambda_i\hat{\mathbf{E}}_{ij}^L\otimes\hat{\mathbf{E}}_{ij}^{EL}\right], \tag{4.46}$$

where

$$\hat{\mathbf{E}}_{ij}^{EL}=\frac{1}{\sqrt{2+2\delta_{ij}}}\mathbf{e}_i^E\otimes\mathbf{e}_j^L. \tag{4.47}$$

Using relations (4.38) and (4.41) in $(4.2)_1$ we derive the following basis-free expression:

$$D\mathbf{G}_L(\mathbf{F})[\mathbf{H}]=\sum_{i,j}G_{ij}\mathbf{E}_i^L\boxtimes\mathbf{E}_j^L\left(\mathbf{H}^T\mathbf{F}+\mathbf{F}^T\mathbf{H}\right) \tag{4.48}$$

Using for the obvious relation

$$G_{ij}\mathbf{E}_i^L\mathbf{H}^T\mathbf{F}\mathbf{E}_j^L = G_{ji}\mathbf{E}_j^L\mathbf{F}^T\mathbf{H}\mathbf{E}_i^L \tag{4.49}$$

we can also write (4.48) in the equivalent forms

$$D\mathbf{G}_L(\mathbf{F})[\mathbf{H}] = \begin{cases} 2\sum_{i,j} G_{ij}\mathbf{E}_i^L\mathbf{F}^T\boxtimes\mathbf{E}_j^L, \\ 2\sum_{i,j} G_{ij}\mathbf{E}_i^L\boxtimes^T\mathbf{E}_j^L\mathbf{F}^T. \end{cases} \tag{4.50}$$

Reminding (3.41) we have the simplified expressions

$$D\mathbf{G}_L(\mathbf{F})[\mathbf{H}] = \begin{cases} 2\sum_{i,j=1}^{3} G_{ij}\lambda_i\mathbf{E}_i^{LE}\boxtimes\mathbf{E}_j^L, \\ 2\sum_{i,j=1}^{3} G_{ij}\lambda_j\mathbf{E}_i^L\boxtimes^T\mathbf{E}_j^{LE}. \end{cases} \tag{4.51}$$

4.5. Derivative of Isotropic Eulerian Tensors with respect to **F**

From (4.32) we get

$$\frac{\partial\mathbf{G}_E}{\partial(\mathbf{F}\mathbf{F}^T)}[\mathbf{H}] = \frac{1}{2}\sum_{i,j} G_{ij}\hat{\mathbf{E}}_{ij}^E\otimes\hat{\mathbf{E}}_{ij}^E, \tag{4.52}$$

where

$$\hat{\mathbf{E}}_{ij}^E = \frac{1}{\sqrt{2+2\delta_{ij}}}\left(\mathbf{e}_i^E\otimes\mathbf{e}_j^E + \mathbf{e}_j^E\otimes\mathbf{e}_i^E\right). \tag{4.53}$$

From (2.40) we obtain

$$\frac{\partial(\mathbf{F}\mathbf{F}^T)}{\partial\mathbf{F}}[\mathbf{H}] = \mathbf{H}\mathbf{F}^T + \mathbf{F}\mathbf{H}^T = 2\widehat{\mathbf{F}\mathbf{H}^T}. \tag{4.54}$$

Then the value of the derivative in $(4.2)_2$ is given by

$$\frac{\partial\mathbf{G}_E}{\partial\mathbf{F}}[\mathbf{H}] = \sum_{i,j} G_{ij}\hat{\mathbf{E}}_{ij}^E\otimes\hat{\mathbf{E}}_{ij}^E\widehat{\mathbf{F}\mathbf{H}^T}. \tag{4.55}$$

Reminding $(2.7)_2$, Eq. (4.55) can also be written:

$$\frac{\partial\mathbf{G}_E}{\partial\mathbf{F}}[\mathbf{H}] = \left[\sum_{i,j} G_{ij}\hat{\mathbf{E}}_{ij}^E\otimes\hat{\mathbf{E}}_{ij}^E\,\mathbf{F}\right]\mathbf{H}. \tag{4.56}$$

Hence the derivative is

$$\frac{\partial \mathbf{G}_E}{\partial \mathbf{F}} = \sum_{i,j} G_{ij} \hat{\mathbf{E}}_{ij}^E \otimes \hat{\mathbf{E}}_{ij}^E \, \mathbf{F}. \tag{4.57}$$

Moreover, because of (4.47) and (3.41) we have

$$\hat{\mathbf{E}}_{ij}^E \mathbf{F} = \sum_{i,j} \frac{1}{\sqrt{2+2\delta_{ij}}} \mathbf{e}_i^E \otimes \mathbf{e}_j^E \left[\sum_h \lambda_h \mathbf{e}_h^E \otimes \mathbf{e}_h^L \right] = \sum_{i,j} \frac{\lambda_i}{\sqrt{2+2\delta_{ij}}} \mathbf{e}_i^E \otimes \mathbf{e}_j^L \tag{4.58}$$

so that (4.57) becomes

$$\frac{\partial \mathbf{G}_E}{\partial \mathbf{F}} [\mathbf{H}] = \sum_{i,j} G_{ij} \lambda_i \hat{\mathbf{E}}_{ij}^E \otimes \hat{\mathbf{E}}_{ij}^{EL} \,. \tag{4.59}$$

In order to obtain a basis-free expression of the derivative, we can substitute relations (4.38) and (4.54), in $(4.2)_2$

$$D\mathbf{G}_E(\mathbf{F})[\mathbf{H}] = \sum_{i,j} G_{ij} \mathbf{E}_i^E \boxtimes \mathbf{E}_j^E \left(\mathbf{H}\mathbf{F}^{\mathbf{T}} + \mathbf{F}\mathbf{H}^T \right). \tag{4.60}$$

Derivative in (4.60) has the two equivalent formulations:

$$D\mathbf{G}_E(\mathbf{F})[\mathbf{H}] = 2\sum_{i,j} G_{ij} \mathbf{E}_i^E \boxtimes \mathbf{E}_j^E \mathbf{F} = 2\sum_{i,j} G_{ij} \mathbf{E}_i^E \, \mathbf{F} \boxtimes^T \mathbf{E}_j^E. \tag{4.61}$$

The above derivative can also be written as

$$D\mathbf{G}_E(\mathbf{F})[\mathbf{H}] = 2\sum_{i,j} G_{ij} \lambda_j \mathbf{E}_i^E \boxtimes \mathbf{E}_j^{EL} = 2\sum_{i,j} G_{ij} \lambda_i \mathbf{E}_i^{EL} \boxtimes^T \mathbf{E}_j^E \tag{4.62}$$

The basis-free expressions in (4.61) and (4.62) are the Eulerian counterparts of the ones in (4.50) and (4.51).

4.6. Derivative of **R** *with respect to* **F**

For the sake of completeness we consider the variation of **R** with respect to **F**.

Following for example of PADOVANI [19], from the right polar decomposition in (3.5) we get

$$\frac{\partial \mathbf{R}}{\partial \mathbf{F}} [\mathbf{H}] = \mathbf{H}\mathbf{U}^{-1} + \mathbf{F} \frac{\partial \mathbf{U}^{-1}}{\partial \mathbf{F}} [\mathbf{H}]. \tag{4.63}$$

By variation of the identity $\mathbf{U}\mathbf{U}^{-1} = \mathbf{I}$ we also have

$$\frac{\partial \mathbf{U}^{-1}}{\partial \mathbf{F}}[\mathbf{H}] = -\mathbf{U}^{-1}\frac{\partial \mathbf{U}}{\partial \mathbf{F}}[\mathbf{H}]\mathbf{U}^{-1}. \tag{4.64}$$

Substituting the above expression of $\dfrac{\partial \mathbf{U}^{-1}}{\partial \mathbf{F}}$ in (4.63) we get

$$\frac{\partial \mathbf{R}}{\partial \mathbf{F}}[\mathbf{H}] = \mathbf{H}\mathbf{U}^{-1} - \mathbf{F}\mathbf{U}^{-1}\frac{\partial \mathbf{U}}{\partial \mathbf{F}}[\mathbf{H}]\,\mathbf{U}^{-1}. \tag{4.65}$$

If in (4.44) we choose $\mathbf{A} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{G}_L = \mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$, then 4.65) becomes

$$\frac{\partial \mathbf{R}}{\partial \mathbf{F}}[\mathbf{H}] = \mathbf{H}\mathbf{U}^{-1} - \left[\mathbf{R}\sum_{i,j} U_{ij}\hat{\mathbf{E}}_{ij}^{L}\mathbf{U}^{-1}\otimes\mathbf{F}\hat{\mathbf{E}}_{ij}^{L}\right]\mathbf{H} \tag{4.66}$$

which finally leads to the formula

$$\frac{\partial \mathbf{R}}{\partial \mathbf{F}} = \mathbf{1}\boxtimes\mathbf{U}^{-1} - \left[\sum_{i,j} U_{ij}\mathbf{R}\hat{\mathbf{E}}_{ij}^{L}\mathbf{U}^{-1}\otimes\mathbf{F}\hat{\mathbf{E}}_{ij}^{L}\right]. \tag{4.67}$$

If we remind (3.41) we get

$$\frac{\partial \mathbf{R}}{\partial \mathbf{F}} = \mathbf{1}\boxtimes\mathbf{U}^{-1} - \left[\sum_{i,j} U_{ij}\lambda_i\lambda_j^{-1}\hat{\mathbf{E}}_{ij}^{EL}\otimes\hat{\mathbf{E}}_{ij}^{EL}\right]. \tag{4.68}$$

Using the basis-free expression of $\dfrac{\partial \mathbf{U}}{\partial \mathbf{F}}$ in $(4.50)_1$, Eq. (4.65) becomes

$$\frac{\partial \mathbf{R}}{\partial \mathbf{F}}[\mathbf{H}] = \mathbf{H}\mathbf{U}^{-1} - \left[2\mathbf{R}\sum_{i,j} U_{ij}\mathbf{E}_i^L\mathbf{F}^T\boxtimes\mathbf{E}_j^L\mathbf{U}^{-1}\right]\mathbf{H}. \tag{4.69}$$

The expression of the derivative is therefore

$$\frac{\partial \mathbf{R}}{\partial \mathbf{F}} = \mathbf{1}\boxtimes\mathbf{U}^{-1} - \left[2\mathbf{R}\sum_{i,j} U_{ij}\mathbf{E}_i^L\mathbf{F}^T\boxtimes\mathbf{E}_j^L\mathbf{U}^{-1}\right]. \tag{4.70}$$

Using again (3.41) we can have a simplified expression of the derivative

$$\frac{\partial \mathbf{R}}{\partial \mathbf{F}} = \mathbf{1}\boxtimes\mathbf{U}^{-1} - \left[2\sum_{i,j} U_{ij}\lambda_i\lambda_j^{-1}\mathbf{E}_i^E\boxtimes\mathbf{E}_j^L\right]. \tag{4.71}$$

In the Eulerian frame, following the same reasoning, we could easily get the above derivatives as functions of $\mathbf{V}$.

4.7. *Time derivatives of Isotropic Tensors*

At this point it is quite obvious that we can evaluate the time derivatives just using the following formulae

$$\dot{\mathbf{G}}_L = \frac{\partial \mathbf{G}_L}{\partial \left(\mathbf{F}^T\mathbf{F}\right)} \frac{\partial \left(\mathbf{F}^T\mathbf{F}\right)}{\mathbf{F}} \dot{\mathbf{F}}, \qquad \dot{\mathbf{G}}_E = \frac{\partial \mathbf{G}_E}{\partial \left(\mathbf{F}\mathbf{F}^T\right)} \frac{\partial \left(\mathbf{F}\mathbf{F}^T\right)}{\mathbf{F}} \dot{\mathbf{F}}. \tag{4.72}$$

For example, reminding (4.44) and (4.46) from $(4.72)_1$, we get the material time derivative

$$\dot{\mathbf{G}}_L = \left[\sum_{i,j} G_{ij} \hat{\mathbf{E}}_{ij}^L \otimes \mathbf{F} \hat{\mathbf{E}}_{ij}^L\right] \dot{\mathbf{F}} = \left[\sum_{i,j} G_{ij} \lambda_i \hat{\mathbf{E}}_{ij}^L \otimes \hat{\mathbf{E}}_{ij}^{EL}\right] \dot{\mathbf{F}}. \tag{4.73}$$

Moreover, from (4.50) and (4.51) we obtain the basis-free expressions

$$\dot{\mathbf{G}}_L \left(\mathbf{F}\left(t\right)\right) = \dot{\mathbf{G}}_L = \begin{cases} 2\sum\limits_{i,j} G_{ij} \mathbf{E}_i^L \dot{\mathbf{F}}^T \mathbf{F} \mathbf{E}_j^L = 2\sum\limits_{i,j} G_{ij} \lambda_j \mathbf{E}_i^L \dot{\mathbf{F}}^T \mathbf{E}_j^{EL}, \\ 2\sum\limits_{i,j} G_{ij} \mathbf{E}_i^L \mathbf{F}^T \dot{\mathbf{F}} \mathbf{E}_j^L = 2\sum\limits_{i,j} G_{ij} \lambda_i \mathbf{E}_i^{LE} \dot{\mathbf{F}} \mathbf{E}_j^L. \end{cases} \tag{4.74}$$

In the Eulerian frame we have

$$\dot{\mathbf{G}}_E = \left[\sum_{i,j} G_{ij} \hat{\mathbf{E}}_{ij}^E \otimes \hat{\mathbf{E}}_{ij}^E \mathbf{F}\right] \dot{\mathbf{F}} \tag{4.75}$$

and the basis-free expressions

$$\dot{\mathbf{G}}_E = \begin{cases} 2\sum\limits_{i,j} G_{ij} \mathbf{E}_i^E \dot{\mathbf{F}} \mathbf{F}^{\mathbf{T}} \mathbf{E}_j^E, = 2\sum\limits_{i,j} G_{ij} \lambda_j \mathbf{E}_i^L \dot{\mathbf{F}}^T \mathbf{E}_j^{EL} \\ 2\sum\limits_{i,j} G_{ij} \mathbf{E}_i^E \mathbf{F} \dot{\mathbf{F}}^T \mathbf{E}_j^E. = 2\sum\limits_{i,j} G_{ij} \lambda_i \mathbf{E}_i^{LE} \dot{\mathbf{F}} \mathbf{E}_j^L. \end{cases} \tag{4.76}$$

Time derivatives of $\mathbf{R}$ follow respectively from (4.67) and (4.68)

$$\begin{aligned} \dot{\mathbf{R}} &= \dot{\mathbf{F}} \mathbf{U}^{-1} - \left[\sum_{i,j} U_{ij} \mathbf{R} \hat{\mathbf{E}}_{ij}^L \mathbf{U}^{-1} \otimes \mathbf{F} \hat{\mathbf{E}}_{ij}^L\right] \dot{\mathbf{F}} \\ &= \dot{\mathbf{F}} \mathbf{U}^{-1} - \left[\sum_{i,j} U_{ij} \lambda_i \lambda_j^{-1} \hat{\mathbf{E}}_{ij}^{EL} \otimes \hat{\mathbf{E}}_{ij}^{EL}\right] \dot{\mathbf{F}}. \end{aligned} \tag{4.77}$$

Finally, basis-free expressions are obtained by (4.70) and (4.71)

$$\dot{\mathbf{R}} = \dot{\mathbf{F}}\mathbf{U}^{-1} - 2\left[\mathbf{R}\sum_{i,j} U_{ij}\mathbf{E}_i^L\mathbf{F}^T \boxtimes \mathbf{E}_j^L\mathbf{U}^{-1}\right]\dot{\mathbf{F}}$$

$$= \dot{F}\mathbf{U}^{-1} - \left[2\sum_{i,j} U_{ij}\lambda_i\lambda_j^{-1}\mathbf{E}_i^E\dot{\mathbf{F}}\mathbf{E}_j^L\right].$$

5. Conclusions

We have introduced a class of isotropic tensor-valued tensor functions which includes, as particular cases, all the most commonly used Lagrangian and Eulerian strain measures. In the framework of the eigenprojection approach, we provided a new and simple demonstration of the well-known formula for the derivatives of these tensor functions.

Further we have presented formulas for the derivatives with respect to **F** of these generalized strain measures in both Lagrangian and Eulerian frames. For each derivative we gave two different expressions. The first in terms of eigenvectors, the second one in terms of eigenbases. Finally, in the same fashion, the equivalent formulas for the time derivatives are presented.

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