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BOUNDS FOR THE EFFECTIVE SHEAR MODULUS

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This paper deals with the uniform torsion of nonhomogeneus elastic beams. The concept of the effective shear modulus is deduced from torsional rigidity. Upper and lower bounds are derived for the effective shear modulus. It is proven that the effective shear modulus of a compound beam is between the weighted arithmetic and harmonic means of shear moduli of the beam components.

Key words: bounds, effective, shear modulus, nonhomogeneous, uniform torsion.

1. Introduction

The paper deals with the uniform (pure) torsion of isotropic nonhomogeneous elastic beam whose cross-section A may be simply or multiply connected bounded plane domain. The outer boundary curve of A is the closed curve c_0 and the inner boundary curves are the closed curves $c_1, c_2, \dots c_n$. The shear modulus G depends on the cross-sectional coordinates x and y, so that $G = G(x, y)$. It may be, that the considered beam is a composite of different homogeneous materials, G is piecewise constant on A. Types of these beams are compound beams and reinforced beams (see Arutjunjan, and Abramjan [1], Lekhnitskii [3], Muskhelishvili [6]).

According to the Saint–Venant theory of pure torsion of a nonhomogeneous elastic beam, equations

$$\nabla\cdot\left(\frac{1}{G}\nabla U\right) = -2 \quad \text{in} \quad A, \tag{1.1}$$

$$U = 0 \quad \text{on } c_0, \qquad U = K_i \quad \text{on } c_i, \tag{1.2}$$

$$\oint_{c_i} \frac{1}{G}\mathbf{n}\cdot\nabla U ds = 2A_i, \qquad (i = 1, 2, ..., n) \tag{1.3}$$

must be satisfied [3, 4]. In equations (1.1), (1.2), (1.3) ∇ is the two-dimensional del operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{e}_x + \frac{\partial}{\partial y}\mathbf{e}_y. \tag{1.4}$$

$\mathbf{e}_x$, $\mathbf{e}_y$ are unit vectors in the directions of axes x and y, respectively; U is the Prandtl's stress function; K_i is the value of U on the inner boundary curve c_i, (K_i= constant); $\mathbf{n}$ is the outer unit normal vector to the inner boundary curve c_i; A_i is the area enclosed by the curve c_i; dot denotes the scalar product of two vectors according to [5].

Knowing the elastic torsional stress function $U = U(x, y)$ we can determine the shearing stresses τ_{xz}, τ_{yz} and the torsional rigidity R by the following formulas according to ECSEDI [2], LEKHNITSKII [3], and LOMAKIN [4]

$$\tau_{xz} = \vartheta\frac{\partial U}{\partial y}, \qquad \tau_{yz} = -\vartheta\frac{\partial U}{\partial x}, \tag{1.5}$$

$$R = 2\left(\int_A U dA + \sum_{i=1}^{n} K_i A_i\right), \tag{1.6}$$

$$R = \int_A \frac{|\nabla U|^2}{G} dA. \tag{1.7}$$

The connection between the rate of twist ϑ and the applied torque T is $T = R\vartheta$.

We denote by $\Phi = \Phi(x, y)$ the warping function of the cross-section for the unit value of ϑ. Using the solution $U = U(x, y)$ of the boundary value problem (1.1), (1.2), (1.3) we can write according to LEKHNITSKII [3], and LOMAKIN [4]

$$\frac{\partial \Phi}{\partial x} = \frac{1}{G}\frac{\partial U}{\partial y} + y, \qquad \frac{\partial \Phi}{\partial y} = -\left(\frac{\partial U}{G\,\partial x} + x\right). \tag{1.8}$$

In the next section we will present two bounding relations for R and we will give the definition of the effective shear modulus.

2. Inequality relations, effective shear modulus

THEOREM 1: *With any continuous function $\tilde{\Phi} = \tilde{\Phi}(x, y)$ in the domain $\bar{A} = A \cup c \quad (c = \bigcup_{i=1}^{n} c_i)$ for which the integral*

$$I\left[\tilde{\Phi}\right] = \int_A G(x, y)\left[\left(\frac{\partial \tilde{\Phi}}{\partial x} - y\right)^2 + \left(\frac{\partial \tilde{\Phi}}{\partial y} + x\right)^2\right] dA \tag{2.1}$$

exists, the relation

(2.2) $$R \leq I\left[\tilde{\Phi}\right]$$

holds. Equality in (2.1) *is valid only if* $\tilde{\Phi} = \Phi + C$, *where* C *is an arbitrary constant.*

The proof of the upper bound formula (2.1) can be obtained from the principle of minimum of potential energy (see ECSEDI [2], and LOMAKIN [4] WEBER–GÜNTER [7]).

THEOREM 2: *With any function* $\tilde{U} = \tilde{U}(x, y)$ *being continuous in the domain* $\bar{A} = A \cup c$ $(c = \bigcup_{i=1}^{n} c_i)$ *and satisfying the boundary conditions*

(2.3) $$\tilde{U} = 0 \quad \text{on } c_0, \qquad \tilde{U} = \tilde{K}_i = \text{constant on } c_i \ (i=1,\ 2,\ \ldots,\ n),$$

the inequality relation

(2.4) $$R \geq \frac{4\left(\int_A \tilde{U} dA + \sum_{i=1}^{n} \tilde{K}_i A_i\right)^2}{L\left[\tilde{U}\right]}$$

is true, assuming that the integral

(2.5) $$L\left[\tilde{U}\right] = \int_A \frac{\left|\nabla \tilde{U}\right|^2}{G} dA$$

exists and is positive. Equality in the relation (2.4) *holds only if* $\tilde{U} = \lambda U$, *where* λ *is a constant different from zero.*

The proof of the inequality relation (2.4) is based on the principle of minimum of the complementary energy (see ECSEDI [2], and LOMAKIN [4] WEBER–GÜNTER [7]).

We denote the stress function by U_0, the warping function by Φ_0, the torsional rigidity by R_0 if the shear modulus has a unit value; in this case the beam is homogeneous. The concept of the effective shear modulus is based on the torsional rigidity of nonhomogeneous, isotropic, linear elastic beam. The effective shear modulus G_e for a beam is defined by the equation

(2.6) $$G_e = \frac{R}{R_0}$$

The aim of the present paper is to give upper and lower bounds for the effective shear modulus.

It is evident for compound beams that due to ARUTJUNJAN, ABRAMJAN [1], ECSEDI [2], LOMAKIN [4].

$$R = \sum_{j=1}^{p} \frac{1}{G_j} \int_{A_j} |\nabla U|^2 \, dA, \quad L\left[\tilde{U}\right] = \sum_{j=1}^{p} \frac{1}{G_j} \int_{A_j} \left|\nabla \tilde{U}\right|^2 dA, \tag{2.7}$$

$$I\left[\tilde{\Phi}\right] = \sum_{j=1}^{p} G_j \int_{A_j} \left[\left(\frac{\partial \tilde{\Phi}}{\partial x} - y\right)^2 + \left(\frac{\partial \tilde{\Phi}}{\partial y} + x\right)^2\right] dA, \tag{2.8}$$

where p is the number of the phases forming the beam, the whole cross-section is $A = \bigcup_{j=1}^{p} A_j$ and the shear modulus of the homogeneous material in the domain A_j is denoted by G_j.

Here, we note that the functions $U = U(x, y)$, $\tilde{U} = \tilde{U}(x, y)$, $\Phi = \Phi(x, y)$ and $\tilde{\Phi} = \tilde{\Phi}(x, y)$ must satisfy some fitting conditions on the common boundary curve of the regions A_i and A_j. These conditions mean that [1, 2, 3, 4, 6]

a) the stresses acting on the surfaces separating different materials, are equal in magnitude and opposite in direction,

b) the displacements remain continuous across the common boundary of the regions A_i and A_j (because different parts of the beam are joined together by perfect bonds).

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The upper and lower bounds of the effective shear modulus will be formulated in terms of Prandtl's stress function $U_0 = U_0(x, y)$ of the homogeneous beam.

THEOREM 3: *The two-sided bounding formula*

$$\frac{\int_A G(x,y) \left|\nabla U_0\right|^2 dA}{\int_A \left|\nabla U_0\right|^2 dA} \geq G_e \geq \frac{\int_A \left|\nabla U_0\right|^2 dA}{\int_A \frac{\left|\nabla U_0\right|^2}{G(x,y)} dA} \tag{3.1}$$

holds.

THEOREM 4: *For a compound beam we have*

$$\sum_{j=1}^{p} \alpha_j G_j \geq G_e \geq \frac{1}{\sum_{j=1}^{p} \frac{\alpha_j}{G_j}}, \tag{3.2}$$

where

$$\alpha_j = \frac{\int\limits_{A_j} |\nabla U_0|^2 \, dA}{\int\limits_{A} |\nabla U_0|^2 \, dA} \qquad (j = 1, 2, ..., p) \tag{3.3}$$

and

$$\sum_{j=1}^{p} \alpha_j = 1. \tag{3.4}$$

P r o o f. The validity of Theorem 3 and 4 follows from the inequality relations (2.2) and (2.4). Putting $\tilde{U} = U_0$ to the lower bound expression (2.4) and using the formula

$$R_0 = 2\left(\int\limits_A U_0 dA + \sum_{i=1}^{n} K_{0i} A_i\right) = \int\limits_A |\nabla U_0|^2 \, dA \tag{3.5}$$

we obtain the lower bounds of G_e formulated in two-sided bounding formulae (3.1) and (3.2). Application of the inequality relation (2.1) yields the result

$$R \leq \int\limits_A G(x,y) \left[\left(\frac{\partial \Phi_0}{\partial x} - y\right)^2 + \left(\frac{\partial \Phi_0}{\partial y} + x\right)^2\right] dA \tag{3.6}$$

$$= \int\limits_A G(x,y) \, |\nabla U_0|^2 \, dA.$$

In the derivation of formula (3.6) we have used the following equations:

$$\tilde{\Phi} = \Phi_0, \qquad \frac{\partial \Phi_0}{\partial x} - y = \frac{\partial U_0}{\partial y}, \qquad \frac{\partial \Phi_0}{\partial y} + x = -\frac{\partial U_0}{\partial x}. \tag{3.7}$$

From inequality (3.6) by the use of formula (3.5) we get the upper bounds of G_e formulated in the two-sided bounding formulae (3.1), (3.2). □

4. Examples

4.1. Example 1

Let us consider a solid cross-section bounded by a circle whose radius is a, the centre of the cross-section being the origin of the cross-sectional coordinate system x, y. We introduce the polar coordinates r, φ by the definition

$$x = r\cos\varphi, \qquad y = r\sin\varphi \quad (0 \leq \varphi \leq 2\pi, \quad 0 \leq r \leq a).$$

The shear modulus is a given function of r and φ, that is $G = G(r, \varphi)$. Application of the bounding formula (3.1) leads to the following result:

$$\frac{2\int_0^{2\pi}\int_0^{a} G(r,\varphi) r^3 dr d\varphi}{a^4\pi} \geq G_e \geq \frac{a^4\pi}{2\int_0^{2\pi}\int_0^{a} \frac{r^3}{G(r,\varphi)} dr d\varphi}. \tag{4.1}$$

4.2. Example 2

The cross-section in this example defined by:

$$A = \{(x,y)| -a \leq x \leq a \text{ and } 0 \leq y \leq b\},$$
$$A_1 = \{(x,y)| -a \leq x \leq 0 \text{ and } 0 \leq y \leq b\},$$
$$A_2 = \{(x,y)| 0 \leq x \leq a \text{ and } 0 \leq y \leq b\}.$$

The shear modulus in the region A_i is G_i $(i = 1, 2)$. The considered cross section is a composite rectangular cross-section.

In the present case we have

$$\int_{A_1} |\nabla U_0|^2 dA = \int_{A_2} |\nabla U_0|^2 dA = \frac{1}{2}\int_{A} |\nabla U_0|^2 dA.$$

By the use of the two-sided bounding relation (3.2) we obtain

$$\frac{1}{2}(G_1 + G_2) \geq G_e \geq \frac{2}{\frac{1}{G_1} + \frac{1}{G_2}}. \tag{4.2}$$

4.3. Example 3

Consider a circular tube with outer and inner radii a and b, respectively. Let it be reinforced by a ring of rods, made of different material, each of radius δ. The centers of the rods are spaced uniformly on a concentric circle of radius ρ, as shown in Fig. 1. The origin of the cross-sectional coordinate system is taken at the center of the tube. The number of inclusions is q. The tube is made of an elastic material with shear modulus G_1 and the elastic material of the inclusion has shear modulus G_2.

Application of the relation (3.2) gives the result

$$(1-\alpha) G_1 + \alpha G_2 \geq G_e \geq \frac{1}{\frac{1-\alpha}{G_1} + \frac{\alpha}{G_2}}, \tag{4.3}$$

where

$$\alpha = \frac{q}{\left(\frac{a}{\delta}\right)^4 - \left(\frac{b}{\delta}\right)^4} \left[1 + 2\left(\frac{\rho}{\delta}\right)^2\right]. \tag{4.4}$$

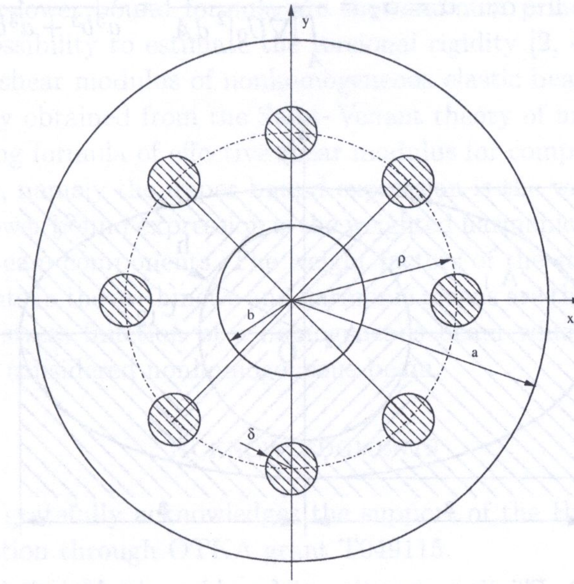

FIG. 1. Circular tube having a ring of circular inclusions.

4.4. *Example 4*

The elliptical beam reinforced by a circular beam of a different material is analysed (Fig. 2). Let the boundary curve c_0 be given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4.5}$$

and let the radius of circular inclusion be h. The centres of boundary curve c_0 and the circular inclusion are the same point, the origin of the cross-sectional coordinate system xy (Fig. 2). The elliptical beam is of a from material with a shear modulus G_1 and the material of circular inclusion has the shear modulus G_2. In this case we have [1, 6, 7]

$$U_0(x,y) = \frac{a^2 b^2}{a^2 + b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right), \tag{4.6}$$

$$|\nabla U_0|^2 = \frac{4}{(a^2+b^2)^2}\left(b^4 x^2 + a^4 y^2\right), \tag{4.7}$$

$$(4.8)\qquad \int_A |\nabla U_0|^2\, dA = \frac{a^3 b^3}{a^2+b^2}\pi,$$

$$(4.9)\qquad \alpha_1 = 1-\alpha, \quad \alpha = \alpha_2 = \frac{\int_{A_2} |\nabla U_0|^2\, dA}{\int_A |\nabla U_0|^2\, dA} = \frac{\left(a^4+b^4\right) h^4}{a^5 b^3 + a^3 b^5}.$$

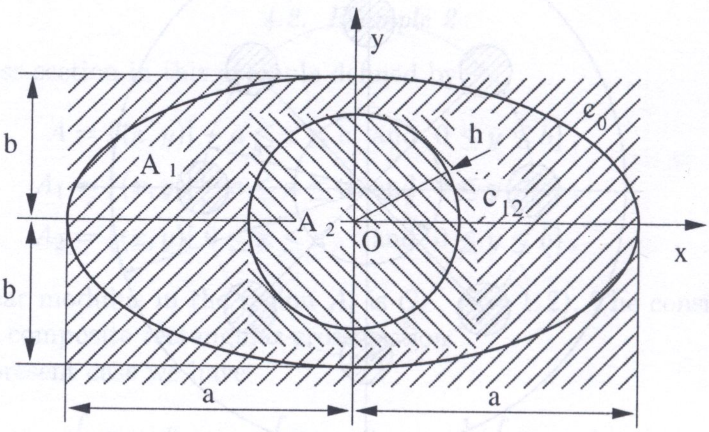

FIG. 2. Elliptic cross-section reinforced by a circular inclusion.

Here, we note $A = A_1 \cup A_2$ and A_1 is bounded by ellipse c_0 and the circle c_{12} whose inside is the domain A_2 (Fig. 2). The bounding formula for effective shear modulus is obtained from inequality (3.2) by means of the results derived above as

$$(4.10)\qquad (1-\alpha)\, G_1 + \alpha G_2 \geq G_e \geq \frac{1}{\dfrac{1-\alpha}{G_1} + \dfrac{\alpha}{G_2}}.$$

We remark that in (4.10):

- for the case $a > b = h$ we have

$$(4.11)\qquad \alpha = \frac{\dfrac{b}{a} + \left(\dfrac{b}{a}\right)^5}{1 + \left(\dfrac{b}{a}\right)^2},$$

- for the case $a = b = h$ we have

$$(4.12)\qquad \alpha = 1.$$

5. Conclusions

This paper deals with the pure (uniform) torsion of isotropic, non-homogeneous linear elastic beams. The formulation of the torsional problem is based on the Saint–Venant theory of uniform torsion [3, 4, 6, 7]. The roots of the presented upper-lower bound formula are the minimum principles of elasticity which give a possibility to estimate the torsional rigidity [2, 4, 7]. The concept of the effective shear modulus of nonhomogeneous elastic beam is based on the torsional rigidity obtained from the Saint–Venant theory of uniform torsion.

The bounding formula of effective shear modulus for compound beams has a simple meaning, namely the upper bound expression is the weighted arithmetic mean and the lower bound expression is the weighted harmonic mean of the shear moduli of the beam components. The weight factors of the shear moduli of the beam components in the arithmetic and harmonic means are the same, it depends on the Prandt stress function of a homogeneous beam which is geometrically identical to the considered nonhomogeneous beam.

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