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STRESS CONTROLLED SHAPE OPTIMIZATION OF 2D ELASTIC STRUCTURES

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A simple and effective method of stress-controlled shape optimization of 2D linear elastic structures is presented. The main elements of the method are: adaptive FE grids fitting well to the structure shape at each iteration step of the proposed simple optimization algorithm and the concept of stress level factor, controlling directly the design variables being the grid parameters. Several examples of beam-type and plate-type 2D structures are investigated. A few iterations only are needed in order to reach a nearly optimal solution.

1. Introduction

In the last three decades a lot of attention has been paid to the shape optimization. Already the survey by Haftka and Grandhi [1] presents a large number of papers devoted to this problem. A more recent bibliography of the problem is given in the book [2] by Hassani and Hinton. Moreover, the last results in sensitivity analysis and adaptive fine meshes allow to introduce some new computational techniques. An example of such approach are the papers [3, 4] by Gutkowski and Zawidzka, where the analytical-numerical methods of satisfying the Kuhn–Tucker conditions are applied to optimize plane structures using the automatically adaptive FE grids. In these grids the node coordinates are determined directly by the shape design geometrical variables.

However, the shape optimization still remains one of the most complex problems for designers who are looking for algorithms relatively simple and easy to apply in the practice. Such an evolutionary optimization algorithm based on the FE stress distribution is proposed by Li *et al.* [5].

The aim of the present paper is elaborating and testing of a simple and effective stress-based algorithm for shape optimization of 2D linear elastic structures using FE adaptive grids, in which the design variables are directly responsible for the shape geometry and are controlled by the stress distribution.

2. The concept of stress-controlled shape optimization

Let us consider a linear elastic 2D body B in its initial shape B_0 with given boundary conditions, subjected to some loads P_k and to some constraints for stresses. Some bounds for the body dimensions resulting from technical reasons, like the minimum depth value for beam-type structures, will be also taken into account.

In the case of 2D structures we can distinguish usually their more or less stressed parts. With the given serviceability requirements it is rather impossible to find a shape assuring a uniform strength in the whole structure. Therefore we will search for a shape that enables a uniform strength distribution in the most stressed parts of the structure. As the strength measure, the equivalent stress will be assumed following the Huber–Mises criterion for the plane stress state:

$$\sigma_e = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\sigma_{xy}^2}, \tag{2.1}$$

where σ_{xx}, σ_{yy}, σ_{xy} denote the stress components.

Let σ_{ref} denote a certain reference stress, e.g. an admissible stress. We introduce the stress level SL as:

$$SL \equiv \frac{\sigma_e}{\sigma_{\text{ref}}}. \tag{2.2}$$

The proposed stress-controlled optimization process consists in covering the initial body shape by a special kind of an adaptive FE grid and then in the iterative change of this grid induced by the stress level, (2.2), in order to reach a possible uniform stress level distribution $SL \cong 1$ in the most stressed parts.

3. The adaptive FE grids

Let us consider a body to be optimized covered by a FE grid as shown in the Fig. 1. The grid consists of R "Columns" and P "Layers" with $(P+1)(R+1)$ nodes. Within this mesh, an additional division into triangular elements is done what gives the total number of FE equal to $2P * R$.

Straight lines S_r separating the "Columns" will remain immovable during each iteration. The distances between the nodes lying on such a line are constant for this line and equal to h_r. The values of h_r, $r = 1,...,R$, are design variables. They are controlled by the maximum stress level values SL in the right-hand neighbor "Columns" (hence we assume $h_{R+1} = h_R$).

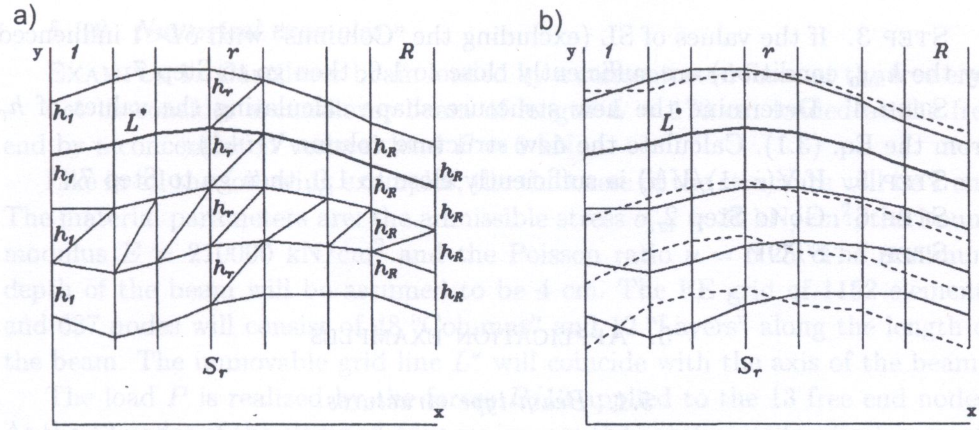

FIG. 1. An adaptive FE grid (a) and its iterative change (b).

The way of controlling the design variables is as follows: at each iteration step i the equivalent stresses σ_e are calculated for each FE, then its maximum value for each "Column" is chosen and the maximum stress level SL is calculated according to the Eq. (2.2). The next iteration will be performed with the new values of h_r:

$$h_r(i+1) = h_r(i) f(SL(i)) \geq h_{\min}, \quad r = 1, ..., R, \tag{3.1}$$

where $f(SL)$ denotes a certain function of the stress level that we call the stress level factor and $h_{\min}$ results from a bound for the minimum structure dimension. The form of the function $f(SL)$ depends on the particular class of problems and will be discussed in detail way in Sec. 5.

It is important to note that one of the lines dividing the structure into "Layers" (line L^* in the Fig. 1) has to remain immovable during the whole iteration process. This is imposed by the main outline of the designed structure (e.g. an axis of a beam or of an arc, an edge of a plate, etc.).

For some particular problems polar-type adaptive grids can be applied. In such cases "Columns" are lying between the radii and "Layers" are "circumferential".

4. The optimization algorithm

STEP 1. Determine the initial shape of the structure to be optimized and cover it with an adaptive FE grid.

STEP 2. Perform the FE stress analysis and calculate the maximum stress level SL for each column. Calculate the structure volume $V(i)$, i being the iteration number.

STEP 3. If the values of SL (excluding the "Columns" with $SL<1$ influenced by the $h_{\min}$ condition) are sufficiently close to 1.0, then go to Step 7.

STEP 4. Determine the new structure shape calculating the values of h_r from the Eq. (3.1). Calculate the new structure volume $V(i+1)$.

STEP 5. If $V(i+1)/V(i)$ is sufficiently close to 1.0, then go to Step 7.

STEP 6. Go to Step 2.

STEP 7. STOP.

5. APPLICATION EXAMPLES

5.1. *Beam-type structures*

5.1.1. The stress level factor. In order to specify the stress level factor $f(SL)$ we will refer to the classical beam theory.

Let us consider a segment of an isostatic beam having an arbitrary elastic section modulus distribution $W_o(x)$ in its initial shape and subjected to some bending moment distribution $M(x)$. Let $W_u(x)$ denote the elastic section modulus distribution for the uniform strength beam with the admissible reference stress σ_{ref}. Because the beam is an isostatic one, we have:

(5.1) $$|M(x)| = W_o(x)\,|\sigma_{xx}(x)| = W_u(x)\sigma_{\text{ref}},$$

where $|\sigma_{xx}|$ means the maximum normal stress for the initial shape of the beam. The stress level for the initial shape is $SL(x) = |\sigma_{xx}|/\sigma_{\text{ref}}$; hence in view of (5.1), the elastic section modulus distribution $W_u(x)$ assuring the uniform strength is:

(5.2) $$W_u(x) = W_o(x)SL(x).$$

For beams with rectangular cross-section of the depth h and width b we have the elastic section modulus $W = bh^2/6$, therefore the depth h_u of a uniform-strength isostatic beam of constant width b will be:

(5.3) $$h_u(x) = h_o(x)\sqrt{SL(x)}.$$

Let us note that in the frame of the beam theory, Eq. (5.3) yields the immediate solution in the isostatic case, and in the hyperstatic case it can be used for some iterative procedure.

Because the FE grid parameter h_r for the beam-type structures is responsible for the structure dimension having the depth character, we will refer to the Eq. (5.3) assuming the stress level factor in Eq. (3.1) as:

(5.4) $$f(SL) = \sqrt{SL}.$$

5.1.2. Numerical examples

EXAMPLE 1. Cantilever beam loaded by a concentrated force at the free end.

Let us consider a cantilever beam of length $L = 170$ cm loaded at the free end by a concentrated vertical load $P = 6$ kN.

Like in all the following examples, the thickness of the structure will be 1 cm. The material parameters are: the admissible stress $\sigma_{\text{ref}} = 21$ kN/cm^2, the Young modulus $E = 210000$ kN/cm^2 and the Poisson ratio $\nu = 0.25$. The minimum depth of the beam will be assumed to be 4 cm. The FE grid of 1152 elements and 637 nodes will consist of 48 "Columns" and 12 "Layers" along the length of the beam. The immovable grid line L^* will coincide with the axis of the beam.

The load P is realized by the forces $P/13$ applied to the 13 free end nodes. At the 13 nodes of the clamped edge we impose the constraints for displacements $u_x = u_y = 0$.

As it is shown in the Fig. 2, already one iteration step leads to the known parabolic solution resulting from the beam theory.

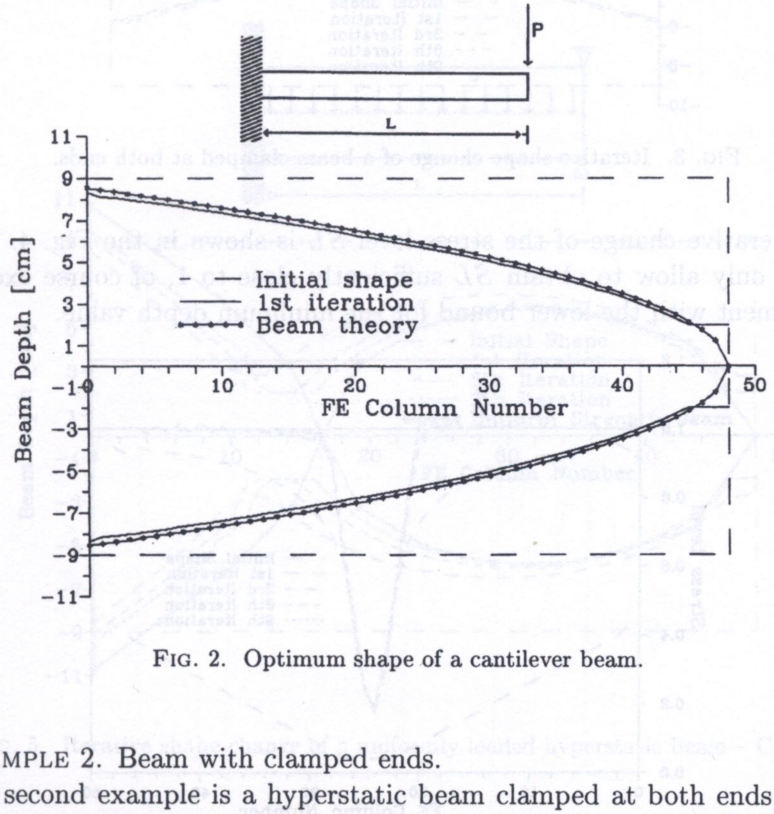

FIG. 2. Optimum shape of a cantilever beam.

EXAMPLE 2. Beam with clamped ends.

The second example is a hyperstatic beam clamped at both ends of length $L = 340$ cm, loaded in the span center by a concentrated load $P = 12$ kN. In view of the symmetry we can consider one half of the beam only. The conditions at the clamped edge are similar to the first example. At the 13 nodes of the edge being the symmetry axis we assume $u_x = 0$ and we apply the forces $P/26$.

In Fig. 3 the iterative change of the shape of the beam is shown. It is easy to see that 3 iterations only are sufficient to reach a stable shape.

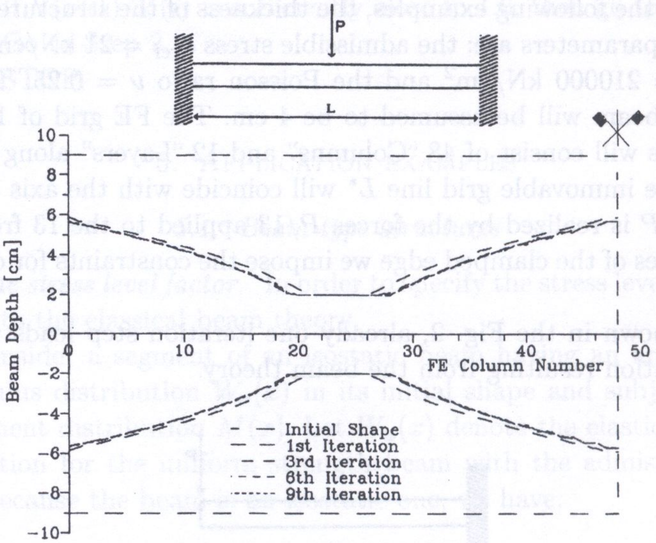

FIG. 3. Iterative shape change of a beam clamped at both ends.

The iterative change of the stress level SL is shown in the Fig. 4. Again 3 iterations only allow to obtain SL sufficiently close to 1, of course except the beam segment with the lower bound for the minimum depth value.

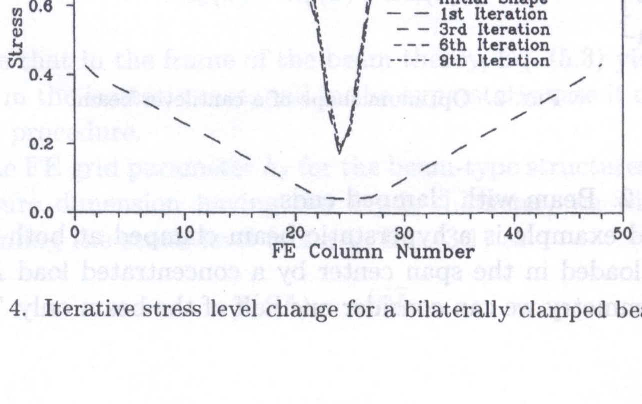

FIG. 4. Iterative stress level change for a bilaterally clamped beam.

Let us note that in view of the symmetry, not all features of the hyperstatic behavior of the beam could be manifested. Therefore the next example seems to be much more interesting.

EXAMPLE 3. Hyperstatic beam with one clamped and one simply supported end under uniform load.

The beam and loading parameters are: the length $L = 240$ cm, the uniformly distributed load $q = 16$ kN/m. Because we intend to compare our results with the uniform strength shape given by WIERZBICKI [6], in the frame of the beam theory, an idealized case will be considered first.

CASE 1. In this case we try to assure the support and loading conditions of our 2D structure in the manner possibly close to the one-dimensional beam case. The displacement conditions for the clamped end nodes are like in the previous examples, but the simple support is realized by the condition $u_y = 0$ for the other end nodes. The load q is applied to the nodes of the beam longitudinal axis in the form of concentrated vertical loads $P = qL/48 = 0.8$ kN.

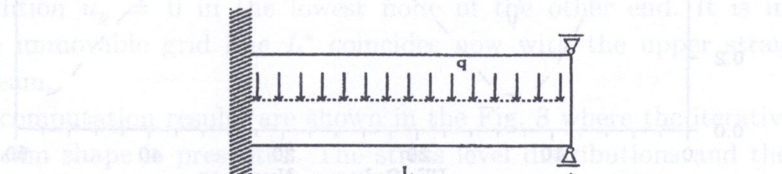

FIG. 5. Iterative shape change of a uniformly loaded hyperstatic beam – Case 1.

The computation results are shown in Fig. 5. It is easy to see a good tendency of the iterative shape change in approaching the analytical solution [6]. The iterative stress level change is presented in Fig. 6. Some peaks occurring at the transition region to the constant minimum depth segment result from the FE deformation. Figure 7 shows the iterative volume change. It is easy to see that 3–4 iterations only are enough to reduce the structure volume. The volume reduction

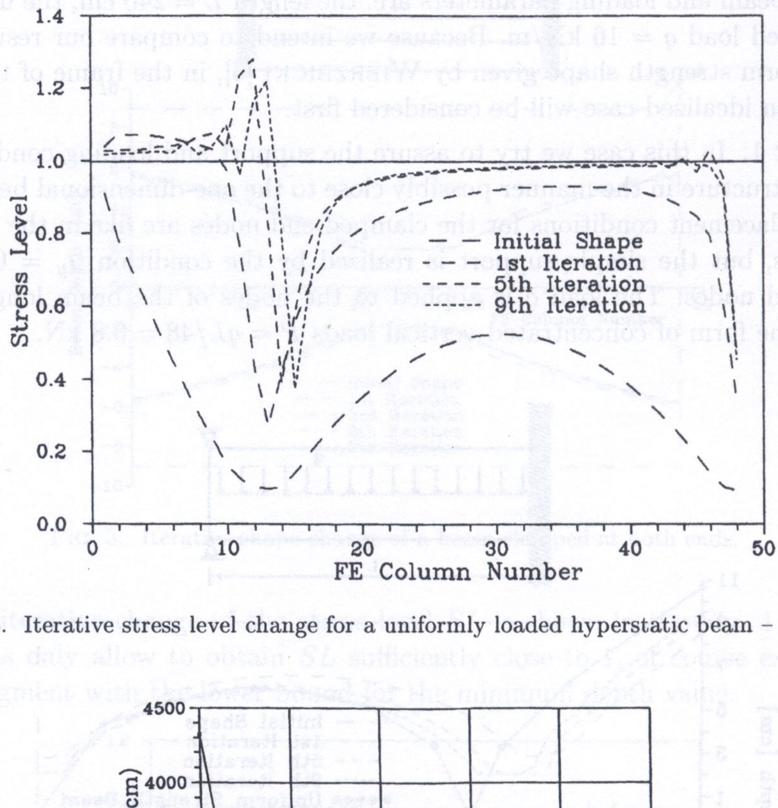

FIG. 6. Iterative stress level change for a uniformly loaded hyperstatic beam – Case 1.

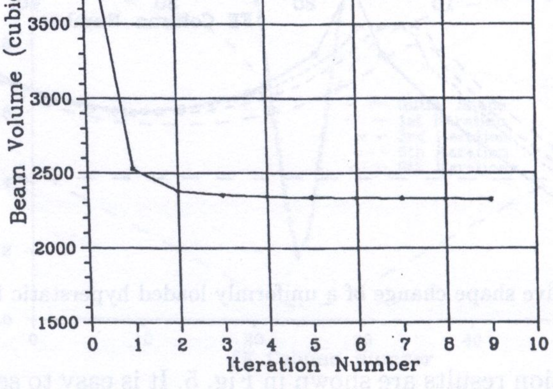

FIG. 7. Iterative volume change of a uniformly loaded hyperstatic beam – Case 1.

in the following iterations is less significant. However, looking at the Figs. 5 and 6 we note some discrepancies between the results of the 5-th and 9-th iterations. It does not mean any continuation of the optimization process but rather the redistribution of the same material volume due to the hyperstatic behavior of the beam, where the more rigid parts become more stressed. Hence 4 iterations are quite sufficient in order to get the optimum shape of the beam.

Let us note that the analytical solution [6] admits a zero depth of the beam for zero bending moment. In our solution however, we are obliged to keep some minimum depth of the beam because the number of finite elements in each column of the adaptive grid is constant during all iterations.

CASE 2. In view of certain serviceability requirements, sometimes we cannot admit any convexities of the upper edge of the beam and all the depth change has to be done by the outline of the lower edge. With the main data like in the previous case we consider a practical problem where the nodes of the upper edge are loaded by the forces $P = 0.8$ kN, the displacements conditions at the clamped end remain the same but the simple support is realized by the condition $u_y = 0$ in the lowest node of the other end. It is important that the immovable grid line L^* coincides now with the upper straight edge of the beam.

The computation results are shown in the Fig. 8 where the iterative change of the beam shape is presented. The stress level distributions and the volume

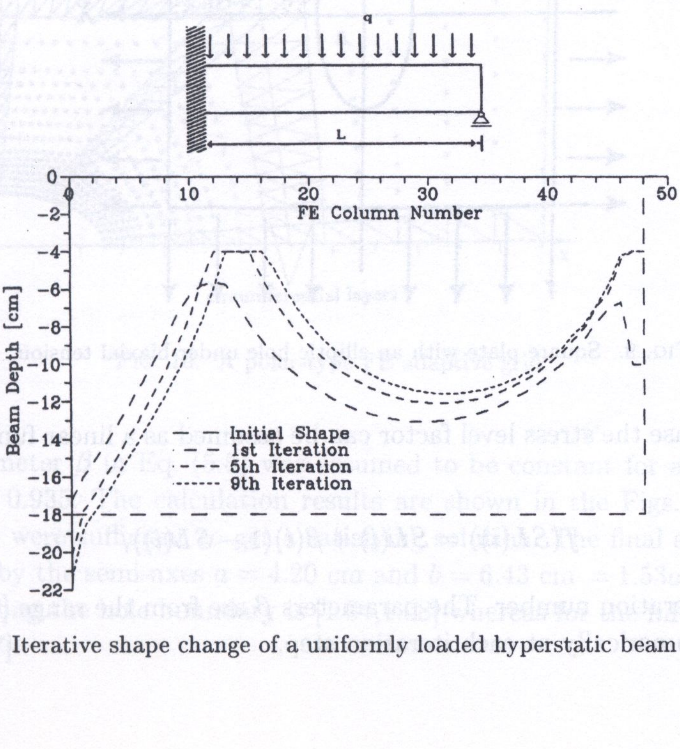

FIG. 8. Iterative shape change of a uniformly loaded hyperstatic beam – Case 2.

change are similar to those of Figs. 6 and 7. Also the conclusion concerning the practically sufficient iterations number remain the same as in the Case 1.

5.2. *Plate-type structures*

5.2.1. The stress level factor. For the plate-type structures, the problem of determining the stress level factor is much more complex because we can not refer to any simplified theory. Depending on the particular class of problems we have to determine this function by way of some numerical experiments.

In this section we deal with double symmetry square plates with elliptical hole under uniformly distributed bi-axial tension applied to the outer edges shown in Fig. 9. Our aim is optimization of the semi-axes of the hole in order to assure a possibly uniform stress level distribution around the hole.

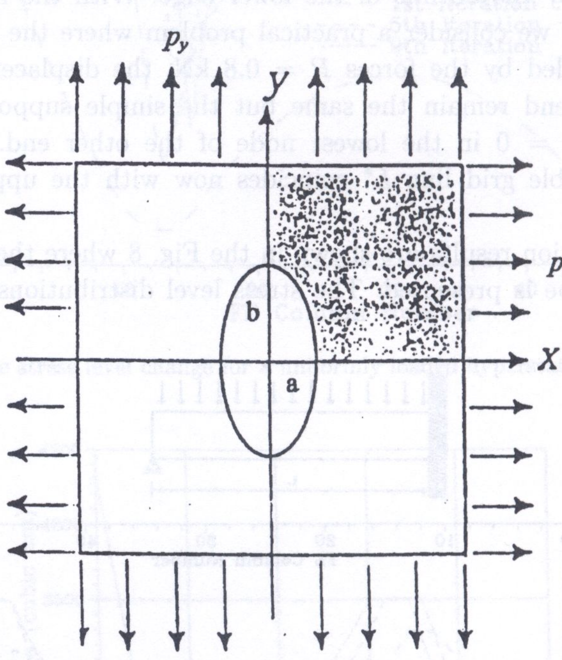

FIG. 9. Square plate with an elliptic hole under biaxial tension.

For this case the stress level factor can be assumed as a linear function of the stress level:

$$f(SL(i)) = SL(i) + \beta(i)(1 - SL(i)), \tag{5.5}$$

i being the iteration number. The parameters β are from the range [0,1] and are introduced numerically at each iteration step.

5.2.2. Numerical example. Let us consider a square plate of side 80 cm subjected to bi-axial tensions $p_x = 9$ kN/cm and $p_y = 13.5$ kN/cm $= 1.5\ p_x$. In order to show the efficiency of the method, the initial shape of the elliptic hole is assumed to be quite different from the known theoretical solution (cf. the paper [7] by NAQIB *et al.*) where the axes of the ellipse are proportional to the loads applied in their directions. Hence the assumed values of the semi-axes are: $a = 8$ cm (0.1 of the side of the plate), and $b = 5.28$ cm $= a/1.5$.

For the double symmetry reasons it is enough to consider a quarter of the plate covered by an adaptive radial-type grid consisting of 24 "Columns" and 21 "Layers" with 1008 triangular elements and 550 nodes (Fig. 10). The immovable grid line L^* follows the outer edges of the plate.

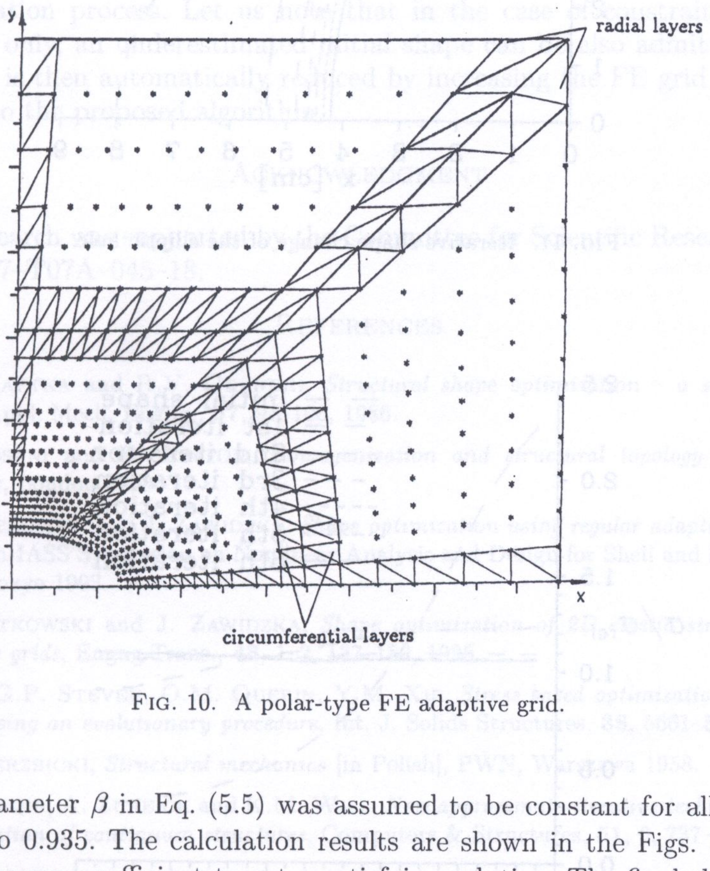

FIG. 10. A polar-type FE adaptive grid.

The parameter β in Eq. (5.5) was assumed to be constant for all iterations and equal to 0.935. The calculation results are shown in the Figs. 11 and 12. Six iterations were sufficient to get a satisfying solution. The final shape of the hole is given by the semi-axes $a = 4.20$ cm and $b = 6.43$ cm $= 1.53a$. The stress level range along the hole boundary is [1.04,1.08] whereas for the initial shape it was [0.30,2.27].

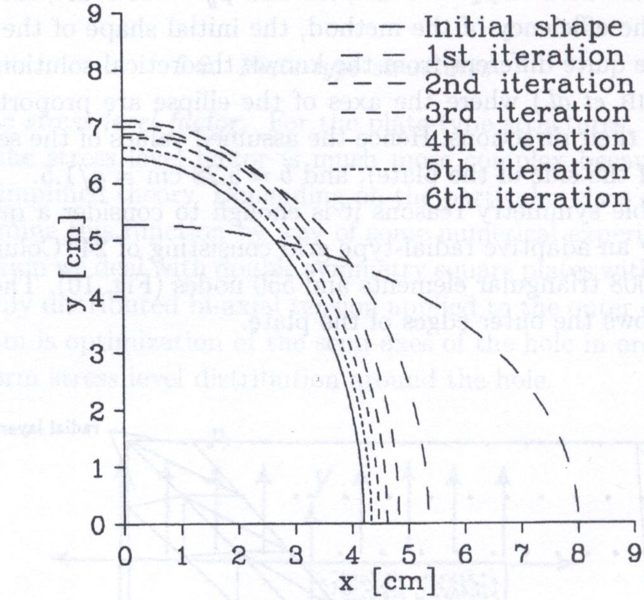

FIG. 11. Iterative shape change of the elliptic hole.

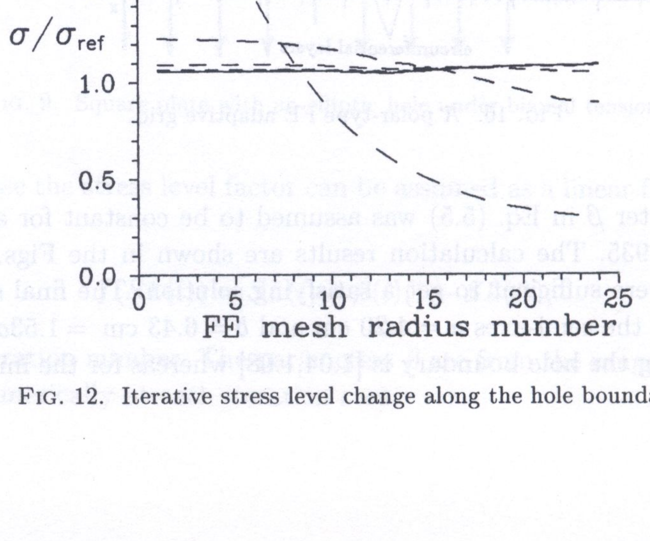

FIG. 12. Iterative stress level change along the hole boundary.

6. Concluding remarks

The presented stress-controlled optimization algorithm seems to be an efficient tool for optimizing the 2D linearly elastic, homogeneous structures in the small deformation range. The adaptive FE grids follow the structure shape at each iteration step. The number of iterations required to reach a nearly optimal solution is rather small and, depending on the problem, it is usually equal to 3–6, what is an unquestionable advantage of the proposed method. Moreover, already this number of iterations leads to stabilization of the structure volume on a certain minimum level.

In the present paper, a constraint on equivalent stress only was imposed. It is possible to take into account also some constraints on the displacements. In such a case however, the initial shape of the structure should be assumed as an overestimated one in order to remain within all the bounds at the beginning of the iteration process. Let us note that in the case of constraints imposed on stresses only, an underestimated initial shape can be also admitted and the stress level is then automatically reduced by increasing the FE grid parameters according to the proposed algorithm.

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References

1. R.T. Haftka and R.V. Grandhi, *Structural shape optimization – a survey*, Comp. Meth. Appl. Mech. Engng., **57**, 91–106, 1986.
2. B. Hassani and E. Hinton, *Homogenization and structural topology optimization*, Springer, London 1999.
3. W. Gutkowski and J. Zawidzka, *Shape optimization using regular adaptive grid*, Proc. of Seiken-IASS Symposion on Nonlinear Analysis and Design for Shell and Spatial Structures, Tokyo 1993.
4. W. Gutkowski and J. Zawidzka, *Shape optimization of 2D elastic structures using adaptive grids*, Engng.Trans., **43**, 1–2, 137–150, 1995.
5. Q. Li, G.P. Steven, O.M. Querin, Y.M. Xie, *Stress based optimization of torsional shafts using an evolutionary procedure*, Int. J. Solids Structures, **38**, 5661–5677, 2001.
6. W. Wierzbicki, *Structural mechanics* [in Polish], PWN, Warszawa 1958.
7. R.A. Naqib, A. Zureick and K.M. Will, *New approximate-iterative technique in shape optimization of continuum structures*, Computers & Structures, **51**, 6, 737–748, 1994.

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