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STABILITY AND ULTIMATE LOAD OF THREE-LAYERED PLATES – A PARAMETRIC STUDY

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This work deals with the analysis of influence of some geometrical and material parameters of layers on the mass, critical stress and ultimate load of three-layered plates. The plates are built of metal outer layers and a composite core. To obtain the maximum load value, the analysis is carried out in the elasto-plastic range basing on the Tsai-Wu criterion and Prandtl-Reuss equations. The solution is obtained by an analytical-numerical method.

Key words: metal-composite-metal panels, stability analysis, ultimate load, lightweight.

1. Introduction

Three-layered plates and shells have been investigated for many years and viewed as advantageous ones because they can easily be shaped according to the requirements of lightweight as well as high strength. Nowadays such structures are widely used not only in the aircraft industry but also in civil engineering, automobile industry, etc.

A typical three-layered structure consists of two outer layers (faces) of high strength properties with a filling layer (core) inbetween, which provides the appropriate stiffness of a structure. The inner layer is usually made of a material much lighter than the material of faces but also of lower mechanical properties.

Referring to the properties of the core, one can divide the three-layered structures into two categories:

- "sandwich" structures with light core carrying only the transverse shear loads [15],
- composite laminate structures in which the core is able to take part in carrying the in-plane loading.

In the latter case the three-layered structures are analysed on the ground of the laminate theory (see [10]), which means that the assumption of linear distribution of stresses across the thickness of a cross-section is adopted.

The amount of literature concerning the behaviour of laminated plates under in-plane loadings is substantial. In most works the analysis has been restricted to the buckling problem (e.g.[4, 12, 13, 14, 17, 25, 26]). The papers [2, 19, 20, 21] are devoted to the elastic nonlinear analysis. There are few works dealing with the plastic buckling state (e.g. [22]).

Considering the material characteristics of composites it can be noticed that many of them behave as brittle ones, but there exist composites of characteristics showing the possibility of working in the plastic range. Therefore it is possible to conduct the analysis of strains and stresses in the elastic and elasto-plastic range and to draw a load-displacement curve for the laminated composite plate. This curve describes the pre-buckling state until a critical stress is reached, next the elastic and elasto-plastic state until maximum (ultimate) load is attained and finally, when the phase of failure occurs (Fig. 1). The following quantities can be determined: the buckling stress, the post-buckling stiffness of a structure, the ultimate load and the energy absorbed by the structure during loading (the area under the curve). The character of the curve in the phase of failure indicates whether the structure is ductile or brittle.

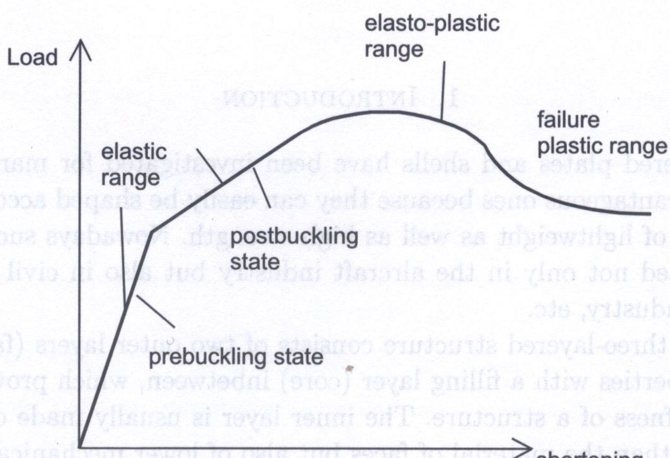


FIG. 1. Typical load-shortening curve for a plate structure.

When the analysis of post-buckling state is carried out in the elasto-plastic region the complexity of a problem (due to geometrical and physical non-linearity) is so great that a purely analytical solution is out of question. Undoubtedly the numerical methods (e.g. FEM) are the most powerful instruments but they are expensive and time-consuming. The analytical-numerical methods in which the elastic post-buckling state is described analytically and next the elasto-plastic state is dealt with on the basis of the theory of plasticity and predicted by iterative numerical procedure, are less general but they give quick and sufficiently accurate results.

In the present work the three-layered plates built of two metal layers with a middle layer of a composite (MCM plates – see [1]) are investigated. The plates are subjected to compression in one direction but the unloaded edges are restrained from pulling in, which implies bi-directional compression. The variations of some properties that may be interesting for designers such as: mass, bending stiffness, critical stress and ultimate load with respect to the g/h ratio (see Fig. 2) are determined for the assumed material and geometrical properties of the layers and of the plate as a whole.

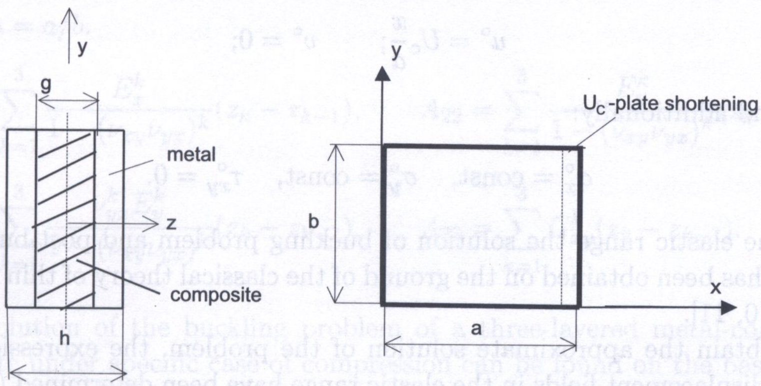


FIG. 2. Geometry and loading of a plate.

It should be underlined that for appropriate prediction of the behaviour of thin-walled columns built of rectangular multi-layered plates, it is crucial to determine the behaviour of the individual element (wall) because on this ground it is possible to estimate (approximately as a lower bound) the ultimate load of the whole structure.

2. Formulation of the problem

Thin three-layered plates subjected to compression are considered. The loading is applied in such a way that during the analysis, the response of the plate to the increment of displacements of corners (Fig. 2) is traced.

The plates are initially flat and stress free. It is assumed that the plate edges are simply supported and remain straight during loading. Additionally the unloaded edges are restrained from pulling in.

The plates consist of two identical isotropic layers (faces) that cover the middle layer (a core) made of different material. When mechanical properties of all layers are of the same range, Kirchhoff's hypothesis can be applied for the entire cross-section.

The material of the middle layer is treated here as orthotropic, with principal axes of orthotropy parallel to the plate edges. Therefore in this case neither shear nor twist coupling nor bending-extension coupling exists [10, 13].

The elastic material properties are determined by the following independent constants:

- for the outer layers: E, ν;
- for the middle layer (orthotropic) – $E_x, E_y, \nu_{yx}, G_{xy}$.

The pre-buckling displacement and stress fields of a plate are described by:

- its displacements in the x and y directions

$$u^\circ = U_c \frac{x}{a}; \qquad v^\circ = 0; \tag{2.1}$$

- and additionally:

$$\sigma_x^\circ = \text{const}, \quad \sigma_y^\circ = \text{const}, \quad \tau_{xy}^\circ = 0. \tag{2.2}$$

In the elastic range the solution of buckling problem and post-buckling behaviour has been obtained on the ground of the classical theory of thin laminated plates [10, 11].

To obtain the approximate solution of the problem, the expressions representing displacement fields in the elastic range have been determined (a detailed description of the method is given in [8, 16]). The results are reported below.

The deflection function w assumes the form:

$$w = f \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}, \tag{2.3}$$

where f denotes a free parameter.

The in-plane displacements $u$and v have been obtained in the following forms:

$$u = u^\circ + f^2\left(C_1 \sin\frac{2\pi x}{a} + B_1 \sin\frac{2\pi x}{a} \cos\frac{2\pi y}{b}\right); \tag{2.4}$$

$$v = v^\circ + f^2\left(C_2 \sin\frac{2\pi y}{b} + B_2 \sin\frac{2\pi y}{b} \cos\frac{2\pi x}{a}\right); \tag{2.5}$$

where C_1, C_2, B_1, B_2 are constants depending on the material and geometrical properties of the layers.

$$C_1 = \frac{\pi}{16a}\left(\frac{A_{12}}{A_{11}}\lambda^2 - 1\right), \quad C_2 = \frac{\pi\lambda}{16a}\left(\frac{A_{12}}{A_{22}}\frac{1}{\lambda^2} - 1\right),$$

$$B_1 = \frac{\widehat{e}\,\widehat{b} - \widehat{c}\,\widehat{d}}{\widehat{a}\,\widehat{d} - \widehat{b}^2}, \qquad B_2 = \frac{\widehat{c}\,\widehat{b} - \widehat{e}\,\widehat{a}}{\widehat{a}\,\widehat{d} - \widehat{b}^2},$$

$$\widehat{a} = \frac{1}{a^2}(A_{11} + A_{33}\lambda^2), \quad \widehat{b} = (A_{12} + A_{33})\frac{\lambda}{a^2},$$

$$\widehat{c} = \frac{-\pi}{16a^3}\left[-A_{11} + (A_{12} + 2A_{33})\lambda^2\right],$$

$$\widehat{d} = \frac{1}{a^2}(A_{22}\lambda^2 + A_{33}), \quad \widehat{e} = \frac{-\pi\lambda}{16a^3}\left[A_{22}\lambda^2 + (A_{12} + 2A_{33})\right],$$

Here $\lambda = a/b$.

$$A_{11} = \sum_{k=1}^{3} \frac{E_x^k}{1 - (\nu_{xy}\nu_{yx})^k}(z_k - z_{k-1}), \qquad A_{22} = \sum_{k=1}^{3} \frac{E_y^k}{1 - (\nu_{xy}\nu_{yx})^k}(z_k - z_{k-1}),$$

$$A_{11} \sum_{k=1}^{3} \frac{\nu_{yx}^k E_y^k}{1 - (\nu_{xy}\nu_{yx})^k}(z_k - z_{k-1}), \qquad A_{33} = \sum_{k=1}^{3} G_{xy}^k (z_k - z_{k-1}).$$

The solution of the buckling problem of a three-layered metal-composite-metal plate under specific case of compression can be found on the basis of the solution of isotropic single-layered plate reported by TIMOSHENKO and GERE [23]. Applying the energy method to the considered three-layered plate, after some transformations, we have:

$$\text{(2.6)} \qquad e_{\mathrm{cr}} = \frac{(U_c)_{\mathrm{cr}}}{a} = \frac{\pi^2}{48\lambda}\frac{h^2}{a^2}\frac{L_1 + L_2\lambda^4 + 2L_3\lambda^2}{(1+\nu)L_4}, \qquad \sigma_{\mathrm{cr}} = \frac{E}{1-\nu^2}e_{\mathrm{cr}}L_5,$$

where

$$L_1 = 1 - (1 - \alpha_1)\left(\frac{g}{h}\right)^3, \qquad L_2 = 1 - (1 - \alpha_2)\left(\frac{g}{h}\right)^3,$$

$$L_3 = 1 - (1 - \alpha_3)\left(\frac{g}{h}\right)^3,$$

$$L_4 = 1 - (1 - \alpha_1 \frac{1+\nu_{yx}}{1+\nu})\frac{g}{h}, \qquad L_5 = 1 - (1 - \alpha_1)\frac{g}{h},$$

$$\alpha_1 = \frac{E_x/E}{(1-\nu_{xy}\nu_{yx})/(1-\nu^2)}, \qquad \alpha_2 = \frac{E_y/E}{(1-\nu_{xy}\nu_{yx})/(1-\nu^2)},$$

$$\alpha_3 = \nu_{yx}\alpha_1 + \frac{2G_{xy}(1-\nu)}{E}.$$

Once displacements u, v, w are given the elastic stresses can be determined in any point of a three-layered plate (using the geometrical relations between strains and displacements and Hooke's law for orthotropic and/or isotropic material).

To determine the ultimate load, the analysis of the post-buckling state has to be carried out in the elasto-plastic range. In the plastic range the following assumptions are made:

- the material properties of layers are independent and known in the whole range of loading up to and beyond the yield limit,
- the appropriate yield criterion is applied according to the characteristic of the considered materials,
- all assumptions of the nonlinear von Kármán plate theory still hold,
- the forms of displacement functions are the same in the elastic and elasto-plastic ranges but their amplitude "f" can vary arbitrarily,
- according to the plastic flow theory, the increments of plastic strains are described by Prandtl–Reuss equations.

In the present work it has been assumed that the material characteristics of isotropic and orthotropic layers are elastic-perfectly plastic. Therefore the following material properties in the plastic range have to be known:

- for isotropic material (faces) – σ_Y – yield limit;
- for orthotropic material (a core) – T_1, C_1, T_2, C_2 – yield limit in tensile and compression tests in x and y direction, respectively. S represents the yield stress in pure shear.

For orthotropic materials Tsai and Wu [24] proposed the yield (failure) criterion that takes into account the difference in strengths due to positive and negative stresses. In case of a plane stress state, the Tsai–Wu criterion is formulated as follows:

$$F = k_1\sigma_x + k_2\sigma_y + k_3\tau_{xy} + k_{11}\sigma_x^2 + k_{22}\sigma_y^2 - k_{12}\sigma_x\sigma_y + 3k_{33}\tau_{xy}^2 = 1, \tag{2.7}$$

where parameters k_1, k_2, k_3 and $k_{11}, k_{22}, k_{12}, k_{33}$ are determined by tensile, compressive and shear tests as given below:

$$k_1 = \frac{1}{T_1} - \frac{1}{C_1}, \qquad k_{11} = \frac{1}{T_1 C_1},$$

$$k_2 = \frac{1}{T_2} - \frac{1}{C_2}, \qquad k_{22} = \frac{1}{T_2 C_2}, \tag{2.8}$$

$$k_3 = 0, \qquad k_{33} = \frac{1}{3S^2}.$$

The last unknown parameter k_{12} in (2.7) is related to the interaction of two stress components σ_x and σ_y. This parameter can be determined in many ways: by an infinite number of combined stresses or by simplified assumptions. In this paper it is assumed that the magnitude of interaction term k_{12} results from the following inequality [24]:

$$k_{11}k_{22} - \frac{1}{4}k_{12}^2 \geq 0, \tag{2.9}$$

so the parameter k_{12} can be expressed as:

$$k_{12} = \pm 2\sqrt{\frac{1}{T_1 C_1}\frac{1}{T_2 C_2}}, \tag{2.10}$$

or (see Ref. [27]):

$$k_{12} \approx \frac{2}{T_1 T_2 + C_1 C_2}. \tag{2.11}$$

It is easy to notice that both the Hill yield criterion and Huber–Mises criterion can be obtained from the Eq. (2.7). The associated flow rule for a given yield criterion can be expressed by the Prandtl–Reuss equations [9] as follows:

$$d\varepsilon_{ij}^p = \Lambda S_{ij}, \qquad i,j = 1,2,3\,, \tag{2.12}$$

where

$$S_{ij} = \frac{1}{3}\frac{\partial F}{\partial \sigma_{ij}}, \qquad i,j = 1,2,3\,.$$

In the analysis of elasto-plastic plates undergoing large deformations, the infinitesimal increments in (2.10) have to be replaced by the finite ones (denoted by Δ). Then the relations between stress and strain increments in the elasto-plastic range are described by the Prandtl–Reuss equations in a form:

$$\begin{aligned}
\Delta\sigma_x &= \frac{E_x}{(1-\nu_{xy}\nu_{yx})}\left[\Delta\varepsilon_x + \nu_{xy}\Delta\varepsilon_y - \Lambda(S_{xx} + \nu_{xy}S_{yy})\right],\\
\Delta\sigma_y &= \frac{E_y}{(1-\nu_{xy}\nu_{yx})}\left[\Delta\varepsilon_y + \nu_{yx}\Delta\varepsilon_x - \Lambda(S_{yy} + \nu_{yx}S_{xx})\right],\\
\Delta\tau_{xy} &= G_{xy}(\Delta\gamma_{xy} - \Lambda S_{xy}),
\end{aligned} \tag{2.13}$$

where S_{xx}, S_{yy}, S_{xy} are defined as:

$$S_{xx} = \frac{TC}{3}(k_1 + 2k_{11}\sigma_x - k_{12}\sigma_y),$$

(2.14)
$$S_{yy} = \frac{TC}{3}(k_2 + 2k_{22}\sigma_y - k_{12}\sigma_x),$$

$$S_{xy} = 2TCk_{33}\tau_{xy}.$$

T and C denote the values of yield (failure) stress in tension and compression, respectively, selected as the reference quantities (see [18]).

For an orthotropic material with the elastic-perfectly plastic characteristics the parameter Λ (which is a scalar, positive definite) is [18]:

$$\Lambda = \frac{(S_{xx} + \nu_{yx}\eta S_{yy})\Delta\varepsilon_x + \eta(S_{yy} + \nu_{yx}S_{xx})\Delta\varepsilon_y + G^* S_{xy}\Delta\gamma_{xy}}{S_{xx}^2 + 2\eta\nu_{yx}S_{yy}S_{xx} + \eta S_{yy}^2 + G^* S_{xy}^2}, \tag{2.15}$$

where:

$$\eta = \frac{E_y}{E_x}, \qquad G^* = G_{xy}(1 - \nu_{yx}\nu_{xy})/E_x.$$

3. Method of solution

The Rayleigh–Ritz variational method is applied to the elasto-plastic problem. It has been proved by Graves-Smith [5] that it is possible to apply the variational method to the plates undergoing finite deflections (see also [6–8]).

The potential energy at any point of a plate is a sum of elastic and plastic components. The plastic strain energy existing prior to the current strain increment bears no direct relation to the current state of stresses. For the purposes of minimization, this energy may arbitrarily be assumed to be zero and only further changes of the strain energy have been taken into account.

$$\Delta W = \int_V \left[\left(\sigma_x + \frac{1}{2}\Delta\sigma_x\right)\Delta\varepsilon_x + \left(\sigma_y + \frac{1}{2}\Delta\sigma_y\right)\Delta\varepsilon_y + \left(\tau_{xy} + \frac{1}{2}\Delta\tau_{xy}\right)\Delta\gamma_{xy}\right] dxdydz, \tag{3.1}$$

where V is the volume of the plate, $\sigma_x, \sigma_y, \tau_{xy}$ denote the stresses before the loading increment is applied and $\Delta\sigma_x, \Delta\sigma_y, \Delta\tau_{xy}, \Delta\varepsilon_x, \Delta\varepsilon_y, \Delta\gamma_{xy}$ denote the stress and strain increments produced by the increment of shortening ΔU_c.

In the elasto-plastic range the current state of stresses depends on the path of loading, so the solution of the problem can only be reached numerically. Therefore the numerical solution starts from the evaluation of the energy increment (3.1). In order to accomplish this, every layer is divided equally into $i \times j \times k$ appropriate 3D elements. The energy values calculated in each of elements are summed for a whole structure.

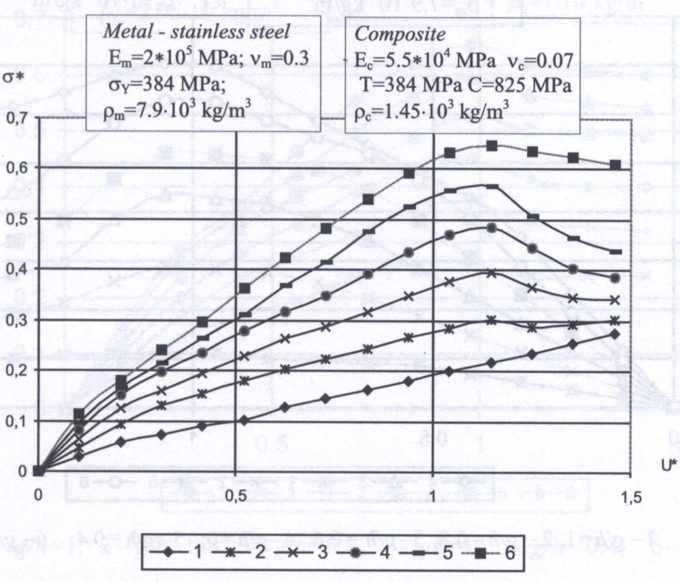
Next, the numerical minimisation of the energy functional with respect to the independent parameter f is performed. The average stresses corresponding directly to the load applied to a considered structure are subsequently computed.

In each step of calculations the active, passive and neutral processes and also the reduction of stress to the yield surface are taken into account.

4. Case studies

We consider square plates ($\lambda = 1$) built of two identical metal layers with a middle layer of a light material. For different material properties of the layers, the variation of mass, critical stress and ultimate load has to be determined as a function of ratio g/h (thickness of a core related to the total plate thickness).

Two metal outer layers are considered: steel (Figs. 3–6) and aluminium (Figs. 7, 8) and also two composite (but isotropic) cores: composite denoted by CFS003/LTM25 [3] (Figs. 3, 4) and epoxy resin (Figs. 5–8).



1-g/h = 1, 2-g/h = 0.8, 3-g/h = 0.6, 4-g/h =0.4, 5-g/h = 0.2, 6-g/h = 0

Fig. 3. Load-shortening curves for plates of $a/h = 100$.

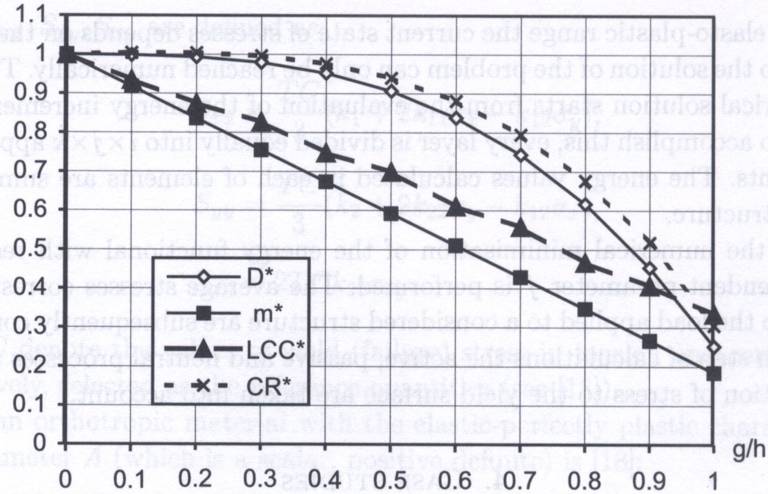
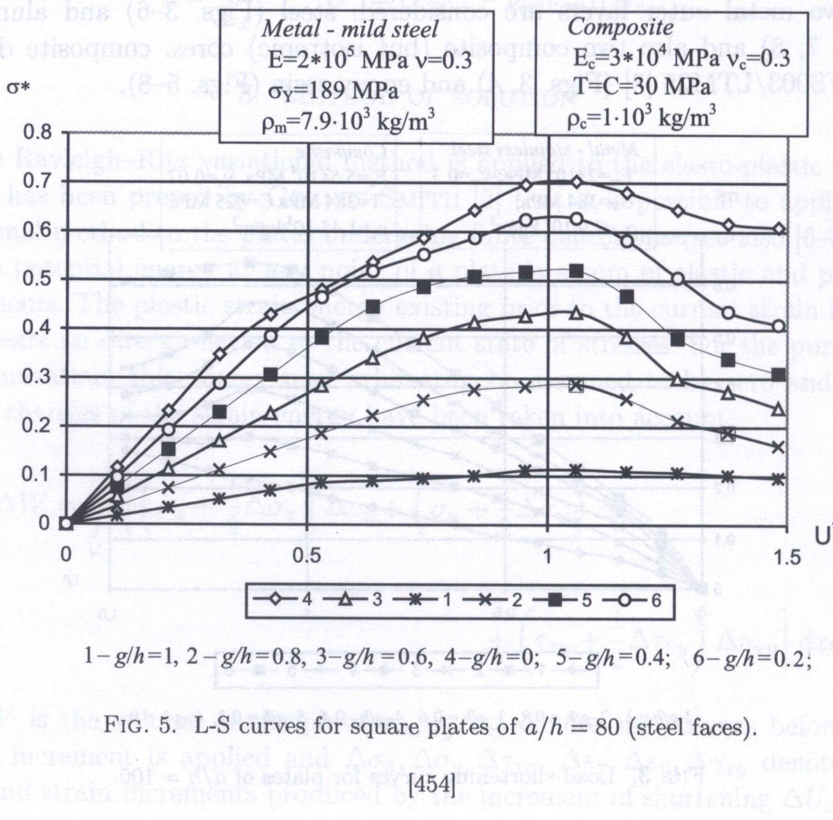


FIG. 4. Variation of dimensionless mass, critical stress and ultimate load for plates of $a/h = 100$ (material parameters given in Fig. 3).



$1-g/h=1$, $2-g/h=0.8$, $3-g/h=0.6$, $4-g/h=0$; $5-g/h=0.4$; $6-g/h=0.2$;

FIG. 5. L-S curves for square plates of $a/h = 80$ (steel faces).

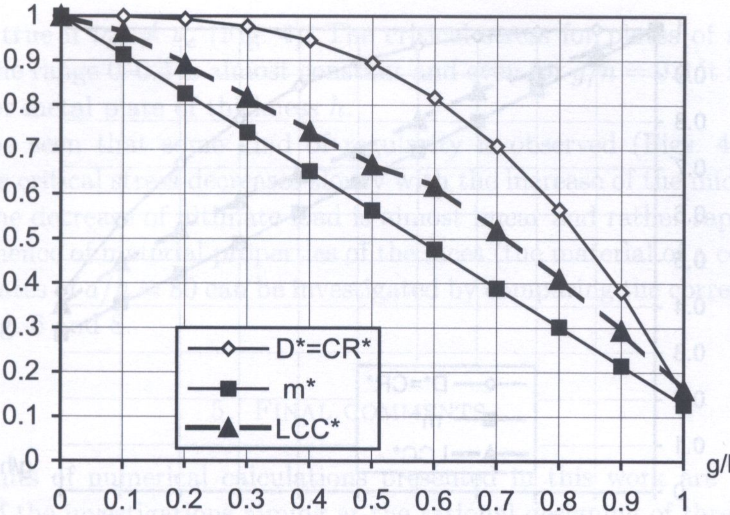
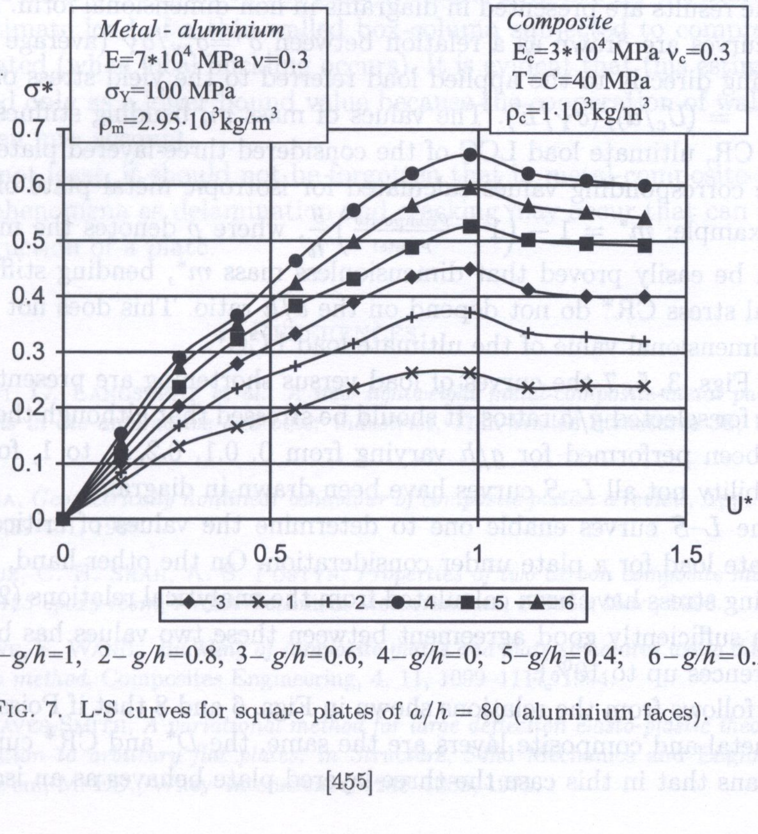


FIG. 6. Variation of dimensionless mass, critical stress and ultimate load for plates of $a/h = 80$ – steel faces (material parameters given in Fig. 5).



1–g/h=1, 2– g/h=0.8, 3– g/h=0.6, 4– g/h=0; 5–g/h=0.4; 6–g/h=0.2

FIG. 7. L-S curves for square plates of $a/h = 80$ (aluminium faces).

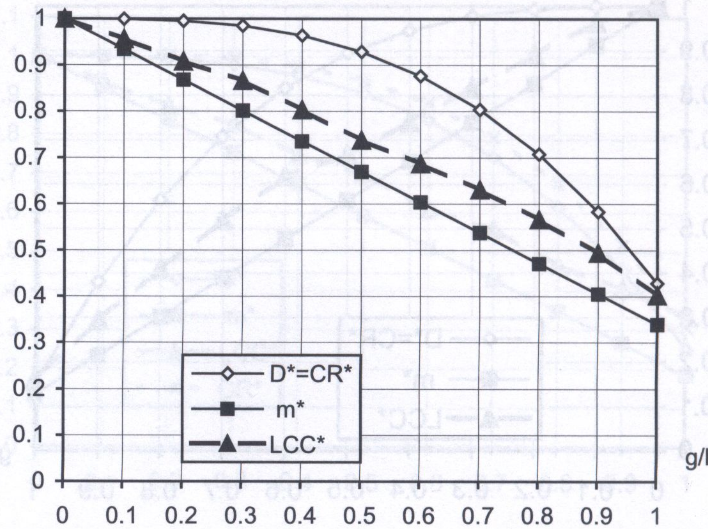


FIG. 8. Variation of dimensionless mass, critical stress and ultimate load for plates of $a/h = 80$ – aluminium faces (material parameters given in Fig. 7).

The results are presented in diagrams in non-dimensional form. Load-shortening curves are drawn as a relation between $\sigma^*=\sigma_{\mathrm{av}}/\sigma_Y$ (average stress corresponding directly to the applied load referred to the yield stress of metal face) and $U^* = (U_c/a)/(\sigma_Y/E)$. The values of mass m, bending stiffness D, critical stress CR, ultimate load LCC of the considered three-layered plate are referred to the corresponding values calculated for isotropic metal plate of thickness h (for example: $m^* = 1 - \left(1 - \frac{\rho_{\mathrm{composite}}}{\rho_{\mathrm{metal}}}\right)\frac{g}{h}$, where ρ denotes the mass density). It can be easily proved that dimensionless mass m^*, bending stiffness D^* and critical stress CR* do not depend on the a/h ratio. This does not apply to the non-dimensional value of the ultimate load LCC*.

In Figs. 3, 5, 7 the curves of load versus shortening are presented for MCM plates for selected g/h ratios. It should be stressed that although the calculations have been performed for g/h varying from 0, 0.1, 0.2, … to 1, for purpose of readability not all L–S curves have been drawn in diagrams.

The L–S curves enable one to determine the values of critical stress and ultimate load for a plate under consideration. On the other hand, the values of buckling stress have been calculated from the analytical relations (2.6). It is seen that a sufficiently good agreement between these two values has been achieved (differences up to 10%).

It follows from the relations shown in Figs. 6 and 8 that if Poisson's ratios of the metal and composite layers are the same, the D^* and CR* curves coincide. It means that in this case the three-layered plate behaves as an isotropic plate.

This is not true if $\nu_m \neq \nu_c$ (Fig. 4). The critical stress for plates of ratio g/h varying in the range 0–0.4 is almost constant and even for $g/h = 0.3$ it is greater than that for metal plate of thickness h.

It can be seen that some kind of regularity is observed (Figs. 4, 6, 8) – although the critical stress decreases slowly with the increase of the middle layer thickness, the decrease of ultimate load is almost linear and rather rapid.

The influence of material properties of the faces (the material of a core is the same) for plates of $a/h = 80$ can be investigated by comparing the corresponding curves in Fig. 6 and 8.

5. Final comments

The results of numerical calculations presented in this work are the mere beginning of the investigations aiming at the rational designing of three-layered (metal-composite-metal) plate structures. However, they can provide some practical advices for the design, enabling the selection of material and geometrical parameters (particularly the selection of the core thickness) in dependence of the required weight and strength properties.

It should be also noted that on the basis of results obtained for the individual plate, the ultimate load of a thin-walled box-column subjected to compression can be estimated (when local buckling occurs). It is evident that this estimation can be treated only as a lower bound value because the cooperation of walls has not been taken into account.

Last but not least, it should not be forgotten that in metal-composite-metal panels such phenomena as delamination and cracking may occur that can result in prior destruction of a plate.

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