ENGINEERING TRANSACTIONS • *Engng. Trans.* • **50**, 1-2, 43–54, 2002
Polish Academy of Sciences • Institute of Fundamental Technological Research
10.24423/engtrans.507.2002

NONLINEAR MATHIEU EQUATION AND ITS APPROXIMATION WITHOUT A SMALL PARAMETER

Ji-Huan He

Department of Mathematics,
Shanghai Donghua University,
No. 1882 Yan'an xi Lu Road, Shanghai 200051, China
jhhe@mail.shu.edu.cn

In this paper, the linear and nonlinear Mathieu equations without a small parameter are considered, which cannot be solved by the perturbation techniques. However, using the variational iteration method, their periodic solutions can be readily obtained with high accuracy. In addition, some special cases have been discussed, where the perturbation solutions are meaningless even when there exists a small parameter.

1. Introduction

In this paper we will use the variational iteration method [1–5] to study the following linear Mathieu equation [6, 7]

$$\begin{cases} \dfrac{d^2u}{dt^2} + (\omega^2 + g(t))u = 0, \\ u(0) = A, \qquad u'(0) = 0, \end{cases} \tag{1.1}$$

and the nonlinear Mathieu equation with cubic nonlinearity

$$\begin{cases} \dfrac{d^2u}{dt^2} + (\omega^2 + g(t))u + \beta u^3 = 0, \\ u(0) = A, \qquad u'(0) = 0, \end{cases} \tag{1.2}$$

where $g(t)$ is a periodic function.

The above equations play an important role in astrophysics, radio-engineering and automatic control, and their stability periodic solution interests not only engineers, but also mathematicians. Various perturbation techniques can be applied to the linear Mathieu equation with a small parameter, resulting, of course, in limited accuracy and applicability of the solution. In the case when $\omega = 1, 2, 3, \cdots$, the perturbation methods will lose its power to find its approximate solution. It is even more difficult to study the nonlinear Mathieu equation (1.2) by perturbation techniques. In this paper, the variational iteration method proposed by the present author will be applied to the above mentioned problems, the results reveal that even its first approximations exhibit high accuracy.

2. Linear Mathieu equation and its approximation

The basic idea of the variational iteration method is to construct a correction functional. For linear Mathieu equation (1.1), the correction functional can be expressed as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \left[\frac{d^2 u_n(\tau)}{d\tau^2} + \omega^2 u_n(\tau) + g(\tau)\tilde{u}_n(\tau) \right] d\tau, \tag{2.1}$$

where λ is a general Lagrange multiplier [8], $\tilde{u}_n$ is considered as a restricted variable [9], i.e. $\delta\tilde{u}_n$=0.

Making the correction functional (2.1) stationary with respect to u_n

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda \left[\frac{d^2 u_n(\tau)}{d\tau^2} + \omega^2 u_n(\tau) + g(\tau)\tilde{u}_n(\tau) \right] d\tau$$

$$= \delta u_n(t) + \lambda(\tau)\delta u_n'(\tau)\big|_{\tau=t} - \lambda'(\tau)\delta u_n(\tau)\big|_{\tau=t} + \int_0^t (\lambda'' + \omega^2\lambda)\delta u_n d\tau = 0$$

yields the following stationary conditions

$$\begin{cases} \lambda''(\tau) + \omega^2\lambda(\tau) = 0, \\ \lambda(\tau)\big|_{\tau=t} = 0, \\ 1 - \lambda'(\tau)\big|_{\tau=t} = 0. \end{cases} \tag{2.2}$$

The multiplier, therefore, can be determined as follows:

$$\lambda = \frac{1}{\omega} \sin\omega(\tau - t). \tag{2.3}$$

Substitution of (2.3) in (2.1) results in

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega} \int_0^t \sin\omega(\tau - t) \left[\frac{d^2 u_n(\tau)}{d\tau^2} + \omega^2 u_n(\tau) + g(\tau)u_n(\tau) \right] d\tau. \tag{2.4}$$

To compare it with the perturbation solution [6, 7], we also set

$$g(t) = 2\varepsilon \cos 2t. \tag{2.5}$$

Suppose that its initial approximation has the form:

$$u_0(t) = A \cos \alpha t, \tag{2.6}$$

where $\alpha = \alpha(\varepsilon)$ is an unknown function of ε with $\alpha(0) = \omega$.

Substituting (2.6) into (1.1) results in the following residual

$$R(t) = A(-\alpha^2 + \omega^2) \cos \alpha t + 2\varepsilon A \cos 2t \cos \alpha t. \tag{2.7}$$

The unknown α can be determined by the methods of weighted residuals, for example, the least squares method, the Galerkin method and the collocation method. Hereby the collocation method will be applied by setting

$$R(0) = A(-\alpha^2 + \omega^2) + 2\varepsilon A = 0, \tag{2.8}$$

which leads to the result:

$$\alpha = \sqrt{\omega^2 + 2\varepsilon} \quad \text{or} \quad \omega = \sqrt{\alpha^2 - 2\varepsilon}. \tag{2.9}$$

By the iteration formula (2.4), we have

$$\begin{aligned} u_1(t) &= A \cos \alpha t \\ &\quad + \frac{1}{\omega} \int_0^t \sin \omega(\tau - t) \big[A(-\alpha^2 + \omega^2) \cos \alpha\tau + 2\varepsilon A \cos 2\tau \cos \alpha\tau\big] d\tau \\ &= A \cos \alpha t + \frac{1}{\omega} \int_0^t \sin \omega(\tau - t) \big[A(-\alpha^2 + \omega^2) \cos \alpha\tau\big] d\tau \\ &\quad + \frac{\varepsilon A}{\omega} \int_0^t \sin \omega(\tau - t) [\cos(2+\alpha)\tau + \cos(2-\alpha)\tau] d\tau \\ &= A \cos \alpha t - A(\cos \alpha t - \cos \omega t) + \frac{\varepsilon A}{\omega^2 - (2+\alpha)^2} [\cos \omega t - \cos(2+\alpha)t] \\ &\quad + \frac{\varepsilon A}{\omega^2 - (2-\alpha)^2} [\cos \omega t - \cos(2-\alpha)t] \\ &= A \cos \omega t + \frac{\varepsilon A}{\omega^2 - (2+\alpha)^2} [\cos \omega t - \cos(2+\alpha)t] \\ &\quad + \frac{\varepsilon A}{\omega^2 - (2-\alpha)^2} [\cos \omega t - \cos(2-\alpha)t], \end{aligned} \tag{2.10}$$

with α defined by (2.9).

In Ref. [6], the following case has been systematically studied:

$$\omega^2 = n^2 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + \cdots \; (n{=}1,2,3,\ldots). \tag{2.11}$$

To compare with the perturbation solution, we will also study the following cases:

In case n=1, setting $\omega = \sqrt{1-2\varepsilon}$ and $\alpha = 1$, we have

$$u_1(t) = A\cos(\sqrt{1-2\varepsilon})t - \frac{\varepsilon A}{8+2\varepsilon}\left[\cos(\sqrt{1-2\varepsilon})t - \cos 3t\right] - \frac{A}{2}\left[\cos(\sqrt{1-2\varepsilon})t - \cos t\right]. \tag{2.12}$$

For small ε, we have

$$u_1(t) = A\cos t - \frac{\varepsilon A}{8}(\cos t - \cos 3t), \tag{2.13}$$

which can be also obtained by the perturbation methods.

It should be specially pointed out that the perturbation solution (2.13) is valid only for small ε, while (2.12) is valid not only for small ε, but also for very large ε.

In case n=2, setting $\omega = \sqrt{4-2\varepsilon}$ and $\alpha = 2$, we have

$$u_1(t) = A\cos(\sqrt{4-2\varepsilon})t - \frac{\varepsilon A}{2\varepsilon+12}\left[\cos(\sqrt{4-2\varepsilon})t - \cos 4t\right] + \frac{\varepsilon A}{4-2\varepsilon}\left[\cos(\sqrt{4-2\varepsilon})t - 1\right]. \tag{2.14}$$

For small ε, we have

$$u_1(t) = A\cos 2t - \frac{\varepsilon A}{12}(\cos 2t - \cos 4t) + \frac{\varepsilon A}{4}(\cos 2t - 1). \tag{2.15}$$

In this paper, we are interested in the approximate solution in cases of $\omega = 1, 2, 3, \cdots$, since, under such conditions, the perturbation solution is meaningless.

In case $\omega = 1$ or $\alpha = \sqrt{1+2\varepsilon}$, we obtain

$$u_1(t) = A\cos t + \frac{\varepsilon A}{1-(2+\sqrt{1+2\varepsilon})^2}\left[\cos t - \cos(2+\sqrt{1+2\varepsilon})t\right] + \frac{\varepsilon A}{1-(2-\sqrt{1+2\varepsilon})^2}\left[\cos \omega t - \cos(2-\sqrt{1+2\varepsilon})t\right]. \tag{2.16}$$

In case $\omega = 2$ or $\alpha = \sqrt{4+2\varepsilon}$, we have

$$(2.17)\quad u_2(t) = A\cos 2t + \frac{\varepsilon A}{4-(2+\sqrt{4+2\varepsilon})^2}\left[\cos 2t - \cos(2+\sqrt{4+2\varepsilon})t\right] + \frac{\varepsilon A}{4-(2-\sqrt{4+2\varepsilon})^2}\left[\cos 2t - \cos(2-\sqrt{4+2\varepsilon})t\right].$$

The approximate solutions (2.16) and (2.17), which are valid not only for small ε, but also for very large parameter ε, can not be obtained by perturbation techniques.

3. Nonlinear Mathieu equation with cubic nonlinearity

For simplicity, in this section we discuss only the case $\omega = 1$ and $g(t)$ defined by (2.5). The nonlinear Mathieu equation is a coupled equation to the Duffing equation ($\varepsilon = 0$) and the linear Mathieu equation ($\beta = 0$).

The correction functional for Eq. (1.2) can be written down as follows:

$$(3.1)\quad u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left[\frac{d^2u_n(\tau)}{d\tau^2} + u_n(\tau) + g(\tau)\tilde{u}_n(\tau) + \beta\tilde{u}_n^3\right]d\tau.$$

The Lagrange multiplier can be readily identified as above, and the following iteration formula can be obtained:

$$(3.2)\quad u_{n+1}(t) = u_n(t) + \int_0^t \sin(\tau - t)\left[\frac{d^2u_n(\tau)}{d\tau^2} + u_n(\tau) + g(\tau)u_n(\tau) + \beta u_n^3\right]d\tau.$$

We also assume that its initial approximation has the form

$$(3.3)\quad u_0(t) = A\cos\alpha t,$$

where $\alpha{=}\alpha(\varepsilon)$ is an unknown function of ε with $\alpha(0) = 1$.

Substituting (3.3) into (1.2) results in the following residual

$$(3.4)\quad \begin{aligned} R(t) &= A(-\alpha^2+1)\cos\alpha t + 2\varepsilon A\cos 2t\cos\alpha t + \beta A^3\cos^3\alpha t \\ &= A(-\alpha^2+1)\cos\alpha t + \varepsilon A[\cos(2+\alpha)t + \cos(2-\alpha)t] \\ &\quad + \frac{1}{4}\beta A^3(3\cos\alpha t + \cos 3\alpha t). \end{aligned}$$

Applying the Galerkin method to identify the unknown α, i.e. setting

$$\int_0^{2\pi/\alpha} \cos\alpha t\, R(t)dt = 0 \tag{3.5}$$

leads to

$$\alpha = \sqrt{1 + \frac{3}{4}\beta A^2}. \tag{3.6}$$

By the iteration formula (3.2), we have

$$u_1(t) = A\cos\alpha t + \frac{1}{\omega}\int_0^t \sin\omega(\tau - t)R(\tau)d\tau = A\cos\alpha t$$
$$+ \frac{A(1-\alpha^2+\frac{3}{4}\beta A^2)}{\alpha^2-1}(\cos\alpha t - \cos t) + \frac{\varepsilon A}{1-(2+\alpha)^2}[\cos t - \cos(2+\alpha)t]$$
$$+ \frac{\varepsilon A}{1-(2-\alpha)^2}[\cos t - \cos(2-\alpha)t] + \frac{\beta A}{4(1-9\alpha^2)}[\cos t - \cos 3\alpha t]. \tag{3.7}$$

In case of $\beta = 0$, Eq. (1.2) becomes the linear Mathieu equation, and its corresponding approximate solution reads

$$u_1(t) = A\cos t + \frac{\varepsilon A}{1-(2+\alpha)^2}[\cos t - \cos(2+\alpha)t]$$
$$+ \frac{\varepsilon A}{1-(2-\alpha)^2}[\cos t - \cos(2-\alpha)t]. \tag{3.8}$$

In case of $\varepsilon = 0$, Eq. (1.2) becomes the well-known Duffing equation, and its approximate solution is expressed as follows:

$$u_1(t) = A\cos\alpha t + \frac{A(1-\alpha^2+\frac{3}{4}\beta A^2)}{\alpha^2-1}(\cos\alpha t - \cos t)$$
$$+ \frac{\beta A}{4(1-9\alpha^2)}[\cos t - \cos 3\alpha t]. \tag{3.9}$$

All approximate solutions (3.7), (3.8) and (3.9) for the nonlinear Mathieu equation, the linear Mathieu equation and the Duffing equation, respectively, have high accuracy. For example, the exact period of the Duffing equation can be written down as follows:

$$T_{ex} = \frac{4}{\sqrt{1+\beta A^2}} \int_0^{\pi/2} \frac{dx}{\sqrt{1-k\sin^2 x}} k = \frac{\beta A^2}{2(1+\beta A^2)}. \tag{3.10}$$

The period of Eq. (3.9) can be expressed as follows:

$$T = 2\pi / \sqrt{1+\frac{3}{4}\beta A^2}, \tag{3.11}$$

while its perturbation solution reads

$$T = 2\pi(1 - \frac{3}{8}\beta A^2). \tag{3.12}$$

It is obvious that (3.12) is the approximation of (3.11) in case of $\beta << 1$. That means that the approximate period obtained by the variational iteration method is also valid for a small parameter β. However, for large parameter β, the perturbation solution (3.12) will be no longer valid, while (3.11) will be valid even in the case $\beta \to \infty$:

$$\lim_{\varepsilon\to\infty} \frac{T_{ex}}{T} = \lim_{\varepsilon\to\infty} \left\{ \frac{\sqrt{1+\frac{3}{4}\varepsilon A^2}}{2\pi} \frac{4}{\sqrt{1+\varepsilon A^2}} \int_0^{\pi/2} \frac{dx}{\sqrt{1-k\sin^2 x}} \right\}$$

$$= \frac{2\sqrt{3/4}}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1-0.5\sin^2 x}} = \frac{2\sqrt{3/4}}{\pi} \times 1.68575 = 0.93.$$

So the approximate solutions obtained by the variational iteration method are of very high accuracy.

4. Convergence and further discussion

The rate of convergence of the proposed method heavily depends upon the accuracy of the identification. If the Lagrange multiplier can be exactly identified, then its exact solution can be obtained by only one step. For nonlinear equations and some linear equations (such as the Mathieu equation), in order to identify the Lagrange multiplier in such a simple way as possible, restricted variation has to be applied; as a result, their exact solutions can be obtained by iteration.

Hereby a very simple example will be illustrated. Consider the following linear equation

$$u'' + \omega^2 u = f(t). \tag{4.1}$$

Its correction functional can be written as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\{u_n''(\tau) + \omega^2 u_n(\tau) - f(\tau)\}d\tau. \tag{4.2}$$

The Lagrange multiplier can be readily obtained, and the following iteration formula can be obtained

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega}\int_0^t \sin\omega(\tau - t)\{u_n''(\tau) + \omega^2 u_n(\tau) - f(\tau)\}d\tau. \tag{4.3}$$

If we use its complementary solution $u_0 = C_1 \cos\omega t + C_2 \sin\omega t$ with suitable constants C_1 and C_2 as an initial approximation, using the iteration formula (4.3), we get

$$u_1(t) = C_1 \cos\omega t + C_2 \sin\omega t - \frac{1}{\omega}\int_0^t \sin\omega(\tau - t) f(\tau)d\tau. \tag{4.4}$$

It is easy to prove that (4.4) is the general solution of (4.1).

However, if we apply a restricted variation to the correction functional (4.2), then its exact solution can be obtained only by successive iterations. Considering a homogenous equation of (4.1), i.e. $f(x)$=0, we rewrite the correction functional as follows:

$$u_{n+1}(t) = y_n(t) + \int_0^t \lambda\{u_n''(\tau) + \omega^2 \tilde{u}_n(\tau)\}d\tau. \tag{4.5}$$

Herein $\tilde{u}_n$ is considered as a restricted variation; under this condition, its stationary conditions of the above correction functional (4.5) can be expressed as follows:

$$\begin{cases} \lambda''(\tau) = 0, \\ \lambda(\tau)|_{\tau=t} = 0, \\ 1 - \lambda'(\tau)|_{\tau=t} = 0. \end{cases} \tag{4.6}$$

The Lagrange multiplier, therefore, can be easily identified

$$\lambda = \tau - t, \tag{4.7}$$

leading to the following iteration formula:

$$u_{n+1}(t) = u_n(t) + \int_0^t (\tau - t)\{u_n''(\tau) + \omega^2 u_n(\tau)\}d\tau. \tag{4.8}$$

If, for example, the initial conditions are $u(0) = 1$ and $u'(0) = 0$, we begin with $u_0(t) = u(0) = 1$. Using the above iteration formula (4.8), we have the following approximate solutions:

$$\begin{aligned} u_1(t) &= 1 + \omega^2 \int_0^t (\tau - t)d\tau = 1 - \frac{1}{2!}\omega^2 t^2, \\ u_2(t) &= 1 - \frac{1}{2!}\omega^2 t^2 + \int_0^t (\tau - t)\left\{-\omega^2 + \omega^2 - \frac{1}{2!}\omega^4 \tau^2\right\}d\tau \\ &= 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4, \\ u_n(t) &= 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4 + \cdots + (-1)^n \frac{1}{(2n)!}\omega^{2n} t^{2n}. \end{aligned} \tag{4.9}$$

From the above solution procedure, we can see clearly that the approximate solutions converge to its exact solution $\cos \omega t$ relatively slowly due to the approximate identification of the multiplier. It should be specially pointed out that the more accurate is the identification of the multiplier, the faster the approximations will converge to its exact solution. If the Lagrange multiplier is approximated by

$$\lambda = \frac{1}{\omega} \sin \omega(\tau - t) \approx \tau - t - \frac{1}{3!}\omega^2 (\tau - t)^3, \tag{4.10}$$

then we have the following iteration formula:

$$u_{n+1}(t) = u_n(t) + \int_0^t \left\{\tau - t - \frac{1}{3!}\omega^2(\tau - t)^3\right\}\{u_n''(\tau) + \omega^2 u_n(\tau)\}d\tau. \tag{4.11}$$

We also begin with $u_0(t) = 1$; using the same approach, we have

$$u_1(t) = 1 + \int_0^t \left\{\tau - t - \frac{1}{3!}\omega^2(\tau - t)^3\right\}\omega^2 d\tau = 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4, \tag{4.12}$$

(4.12) [cont.]
$$u_2(t) = u_1(t) + \int_0^t \left\{\tau - t - \frac{1}{3}\omega^2(\tau - t)^3\right\}\left\{\frac{1}{4!}\omega^6\tau^4)\right\}d\tau$$
$$= 1 - \frac{1}{2!}\omega^2t^2 + \frac{1}{4!}\omega^4t^4 - \frac{1}{6!}\omega^6t^6 + \frac{1}{8!}\omega^8t^8.$$

So, it can be seen that the approximations obtained from (4.11) converge to its exact solution faster than those obtained from the iteration formula (4.8).

5. Appendix: a physical illustration of Eq. (2.1).

Consider an algebraic equation

(A.1) $$f(x) = 0.$$

Suppose that its initial approximate root is x_n, i.e.

(A.2) $$f(x_n) \neq 0.$$

In order to improve the accuracy, we correct x_n by the following expression:

(A.3) $$x_{n+1} = x_n + \lambda f(x_n),$$

where λ is a generalized Lagrange multiplier [8]. The multiplier can be identified in view of the stationary condition with respect to x_n, that is

(A.4) $$\frac{\partial x_{n+1}}{\partial x_n} = 0,$$

which leads to the results $\lambda = -1/f'(x_n)$. Substituting the identified multiplier into Eq. (A3), we obtain the well-known Newton iteration formula:

(A.5) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

For differential equations, Eq. (A.3) is replaced by Eq. (2.1), and the multiplier is determined by the functional stationary condition.

In Eq. (2.1) the concept of restricted variable is used. That means that the variable $\tilde{u}$ should be considered as a known function in the procedure of identification of the multiplier. To illustrate its basic idea, we consider the following algebraic equation:

(A.6) $$x^2 + bx + c + x^3 = 0.$$

We consider x^3 as a restricted variable, i.e.

$$x^2 + bx + c + \tilde{x}^3 = 0. \tag{A.7}$$

This means that $\tilde{x}$ can be considered as a known number. Solving Eq. (A7) results in

$$x = \frac{-b \pm \sqrt{b^2 - 4(c + \tilde{x}^3)}}{2}\,. \tag{A.8}$$

The value of $\tilde{x}$ can be calculated using the previous iteration value, so we can obtain the following iteration formula:

$$x_{n+1} = \frac{-b \pm \sqrt{b^2 - 4(c + x_n^3)}}{2}\,. \tag{A.9}$$

In Eq. (2.1) the restricted variable is used in order to simplify the procedure of identification of the multiplier.

ACKNOWLEDGEMENTS

The work is partly supported by the National Key Basic Research Special Fund of China (No. G1998020318), Shanghai Education Foundation for Young Scientists (98QN47) and National Natural Science Foundation of China.

REFERENCES

1. J.H. HE, *Approximate analytical solution for seepage flow with fractional derivatives in porous media*, Computer Methods in Applied Mech. and Engineering, **167**, 57-68, 1998.

2. J.H. HE, *Approximate solution for nonlinear differential equations with convolution product nonlinearities*, Computer Methods in Applied Mech. and Engineering, **167**, 69-73, 1998.

3. J.H. HE, *Variational iteration method: a kind of nonlinear analytical technique: some examples*, International Journal of Nonlinear Mechanics, **34**, 4, 699-708, 1999.

4. J.H. HE, *Variational iteration method for autonomous ordinary differential system*, Applied Math. and Computer, **114**, 2/3, 115-123, 2000.

5. J.H. HE, *A review on some new recently developed nonlinear analytical technique*, International Journal of Nonlinear Sciences and Numerical Simulation, **1**, 1, 51-70, 2000.

6. A.H. NAYFEH, *Introduction to perturbation techniques*, Wiley & Son, 1981.

7. X.H. SHAO and X.ZH. WANG, *Free vibration of a class of Hill's equation having a small parameter*, Applied Math. and Mech., (English edition), **11**, 4, 355-361, 1990.

8. M. INOKUTI, H. SEKINE and T. MURA, *General use of the Lagrange multiplier in nonlinear mathematical physics*, [in:] Variational Method in the Mechanics of Solids, S. NEMAT-NASSER [Ed.], Pergamon Press, 156-162, 1978.

9. B.A. FINLAYSON, *The method of weighted residual and variational principles*, Acad. Press, 1972.

Received August 10, 2000; revised version December 18, 2000.