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PURE BENDING OF THE ORTHOTROPIC ELASTIC RECTANGLE

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In this contribution, a new method for obtaining the solution to the linear elasticity problem is proposed. An idea of this method is based on certain special expansion of these displacement fields in finite trigonometric series with exponential coefficients. This approach leads to an equivalent problem which requires to derive the exact solution. The proposed method has a mixed analytic-numerical form and has been illustrated by the solution to the boundary value problem for a rectangular region subjected to bending by loads applied on the opposite sides of the region. The numerical results derived have been compared with solutions obtained in the framework of the Euler-Bernoulli beam theory.

1. Introduction

There is a number of methods for obtaining solutions to the 3D linear elasticity problems which are based on different expansions of the unknown fields in functional series. As a rule, the fundamental solutions to special problems are used. Many examples of these procedures can be found in the recent literature, see e.g. [1, 2].

An alternative method for solutions to 3D linear elasticity problems has been proposed in [3, 4]. The above-mentioned method is based on the displacement fields expanded in trigonometric series with certain special coefficients. In this way, some new general solutions have been obtained.

In this paper, the plane linear elasticity theory problems for orthotropic solids are considered. The functional coefficients in finite trigonometric series are assumed in the exponential form. In this way, the general solution constitutes a good starting point for the numerical analysis.

A similar approach is used in the known Lévy method [6]. Using that method, we can find the solutions for plates, which are supported on two edges and free on the other edges. In this paper we propose other constraints of displacements than those used in the method of Lévy. The representations of displacements satisfy exactly all the boundary conditions for bending of a plate to a cylindrical surface. In this manner we are able to reduce the functional coefficients which appear in expansions of displacements and we obtain the exact solution the problems considered.

Our earlier papers [3–5] concerned the plates, which were considered as 3D bodies. The unknown parameters in expansions of displacements were found from the condition of minimum of the potential energy. That method did not give the exact solutions of the boundary conditions and the equations of equilibrium.

2. Constraints imposed on displacements

We consider an elastic body, the reference configuration $\Omega \subset R^3$ of which in the Cartesian system of coordinates $(x_1, x_2, x_3) \in R^3$, is given by $\Omega = (-a, a) \times (-h, h) \times (-l.l)$ where $a, l, h > 0$, $l \gg \max(a, h)$, i.e. the origin of coordinates is located in the body center.

Assume that the part of the body boundary: $\partial_1\Omega = \{\pm a\} \times (-h, h)$ is subjected to the boundary tractions $\mathbf{p} = \mathbf{p}(x_1, x_2, x_3)$. The regular functions $\mathbf{u} = \mathbf{u}(x_1, x_2, x_3)$, (u_k), $k = 1, 2, 3$ satisfying the boundary conditions on the remaining part of the boundary $\partial_2\Omega$ represent the displacements, and $\mathbf{V}$ stands for the space of displacements $\mathbf{u}$. The elastic properties of the body are represented by components of the elastic moduli tensor B_{klmn}, which satisfy the well-known symmetry and positive – definiteness conditions.

The variational form of the equilibrium problem for the body under consideration can be stated as follows:

For the known boundary tractions $\mathbf{p}$ *find a displacement* $\mathbf{u}$ *such that*

$$(\forall \mathbf{v} \in V) \quad \left[\int_{\Omega} B_{klmn} u_{k,l} v_{m,n} dv = \int_{\partial_1\Omega} p_m v_m da \right]. \tag{2.1}$$

In the above equations the beam is treated as the 3D body, therefore a local form of the equations of equilibrium can be derived by means of the well-known variational procedure. When using this approach, the simplified models can be obtained as well, by imposing certain constraints upon the displacements, i.e., restricting the displacements space to a certain proper subset of V. Hereafter we will follow the procedure outlined in [5].

We restrict our considerations to the class of such boundary value problems, in which the surfaces of body $x_2 = \pm h$ and its lateral edges $x_3 = \pm l$ remain unloaded, while its longitudinal edges $x_1 = \pm a$ are subjected to the tractions $\mathbf{p} = [\pm P \sin(x_2/h), 0, 0]$ applied antisymmetrically about the thickness (Fig. 1).

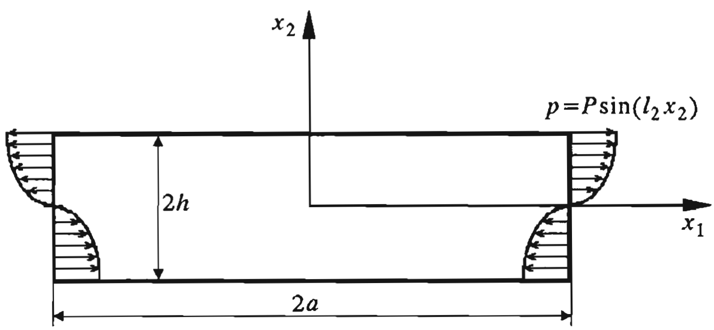

FIG. 1. Scheme of bending of a rectangular beam.

A two-dimensional vector field of displacements determines the strain tensor field:

$$\varepsilon_{\alpha\beta} = u_{(\alpha,\beta)} , \qquad \alpha, \beta = 1, 2 . \tag{2.2}$$

The body is assumed to be made of an elastic anisotropic material, so that the stress tensor is given by:

$$\sigma_{\alpha\beta} = B_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} , \tag{2.3}$$

where $B_{\alpha\beta\gamma\delta}$ are components of the elastic moduli tensor.

In a 3D case, the stiffness matrix $\mathbf{B}$ is the inverse of the matrix $\mathbf{A}$ of elastic moduli, determined by the Young moduli E_i, shear moduli G_{ij} and Poisson ratios ν_i, $i = 1, 2, 3$ in the following way:

$$A_{iijj} = \begin{cases} \dfrac{1}{E_i} & \text{if} \quad i = j \\ -\dfrac{\nu_i}{E_j} & \text{if} \quad i \neq j \end{cases}$$

$$A_{ijij} = \frac{1}{G_{ij}} \qquad \text{(no summation)}.$$

3. Formulation of the problem

We assume that the body under consideration is orthotropic. The equations of equilibrium $\sigma_{\alpha\beta,\beta} = 0$, $\alpha, \beta = 1, 2$ for the unknown displacements have the following form:

$$
\begin{aligned}
&B_{1111}u_{1,11} + (B_{1122} + B_{1212})u_{2,12} + B_{1212}u_{1,22} = 0\,,\\
&B_{2222}u_{2,22} + (B_{1122} + B_{1212})u_{1,12} + B_{1212}u_{2,11} = 0\,.
\end{aligned}
\tag{3.1}
$$

Now, by virtue of Eq. (2.2) we look for a displacement vector (u_α), $\alpha = 1, 2$ in the following form:

$$
\begin{aligned}
u_1(x_\alpha) &= \sum_m \sum_\beta f^m_{1\beta}(x_\beta)\sin(d^m_{3-\beta}x_{3-\beta})\,,\\
u_2(x_\alpha) &= \sum_m \sum_\beta f^m_{2\beta}(x_\beta)\cos(d^m_{3-\beta}x_{3-\beta})\,,
\end{aligned}
\tag{3.2}
$$

where $m = 1, 2, ..., m_0$, m_0 is a certain natural number and

$$
d_1 = \frac{(2m-1)\pi}{2\alpha}\,, \qquad d_2 = \frac{(2m-1)\pi}{2h}\,.
$$

The functions $f^m_{\alpha\beta}(x_\beta)$ are unknown and depend exclusively on the x_β coordinate, and the trigonometric functions $\sin(\cdot)$, $\cos(\cdot)$ depend on the variable $x_{3-\beta}$, $\beta = 1, 2$ only.

Substitution of Eq. (3.2) into the equations of equilibrium (3.1) yields:

$$
\sum_\omega (C^m_{\alpha\beta\omega}\partial^\omega f^m_{1\beta} + D^m_{\alpha\beta\omega}\partial^\omega f^m_{2\beta}) = 0\,, \quad \alpha, \beta = 1, 2\,, \quad \omega = 0, 1, 2\,,
\tag{3.3}
$$

where $\partial^\omega f^m_{\alpha\beta}$ is the derivative of order ω of the functions $f_{\alpha\beta}(\cdot)$, and where we have denoted:

$$
\begin{aligned}
&C^m_{112} = B_{1111}, && C^m_{110} = -B_{1212}(d^m_2)^2, && D^m_{111} = -(B_{1122} + B_{1212})d^m_2,\\
&C^m_{122} = B_{1212}, && C^m_{120} = -B_{1111}(d^m_1)^2, && D^m_{121} = -(B_{1122} + B_{1212})d^m_1,\\
&C^m_{211} = -D^m_{111}, && D^m_{210} = -B_{2222}(d^m_2)^2, && D^m_{212} = C^m_{122},\\
&C^m_{221} = -D^m_{121}, && D^m_{220} = -B_{1212}(d^m_1)^2, && D^m_{222} = B_{2222}.
\end{aligned}
\tag{3.4}
$$

The remaining 12 parameters $C^m_{\alpha\beta i}$, $D^m_{\alpha\beta i}$ not mentioned in (3.4), are equal to zero.

4. General solution to the problem

In order to obtain the general solution to the problem under consideration, we assume the functional coefficients in (3.2) to the form:

$$f^m_{\alpha\beta}(x_\beta) = R^m_{\alpha\beta} \exp(l_\beta \lambda^m_\beta x_\beta) \,, \tag{4.1}$$

where $l_1 = 1/a$, $l_2 = 1/h$ and $R^m_{\alpha\beta}$, λ^m_β are unknown complex parameters:

$$R^m_{\alpha\beta} = S^m_{\alpha\beta} + iT^m_{\alpha\beta} \,, \qquad \lambda^m_\beta = \psi^m_\beta + i\kappa^m_\beta \,. \tag{4.2}$$

Upon substituting Eqs. (4.1) and (3.4) into Eqs. (3.3), we obtain the system of algebraic equations for unknown parameters $R^m_{\alpha\beta}$:

$$\begin{aligned}
&B_{1111}(l_1\lambda^m_1)^2 R^m_{11} - (B_{1122} + B_{1212}) d^m_2 l_1 \lambda^m_1 R^m_{21} - B_{1212}(d^m_2)^2 R^m_{11} = 0 \,,\\
&B_{1212}(l_1\lambda^m_1)^2 R^m_{21} + (B_{1122} + B_{1212}) d^m_2 l_1 \lambda^m_1 R^m_{11} - B_{2222}(d^m_2)^2 R^m_{21} = 0 \,,\\
&B_{1212}(l_2\lambda^m_2)^2 R^m_{12} - (B_{1122} + B_{1212}) d^m_1 l_2 \lambda^m_2 R^m_{22} - B_{1111}(d^m_1)^2 R^m_{12} = 0 \,,\\
&B_{2222}(l_2\lambda^m_2)^2 R^m_{22} - (B_{1122} + B_{1212}) d^m_1 l_2 \lambda^m_2 R^m_{12} - B_{1212}(d^m_1)^2 R^m_{22} = 0 \,.
\end{aligned} \tag{4.3}$$

The $2m_0$ unknown parameters R^m_{11}, R^m_{21} appear in the first two equations (4.3), and R^m_{12}, R^m_{22} are unknown parameters in the remaining two equations. The two systems of equations obtained above are uniform and have nontrivial solutions only if their determinants are equal to zero. Therefore, we obtain the following characteristic equations for unknown parameters λ^m_α which appear in (4.2):

$$\begin{aligned}
&B_{1111}B_{1212}(\Lambda^m_1)^4 - [(B_{1122} + B_{1212})^2 - B_{1111}B_{2222} - B^2_{1212}](\Lambda^m_1)^2 + \\
&\qquad\qquad + B_{1212}B_{2222} = 0 \,,\\
&B_{2222}B_{1212}(\Lambda^m_2)^4 - [(B_{1122} + B_{1212})^2 - B_{1111}B_{2222} - B^2_{1212}](\Lambda^m_2)^2 + \\
&\qquad\qquad + B_{1212}B_{1111} = 0 \,,
\end{aligned} \tag{4.4}$$

where $\Lambda^m_\alpha = \dfrac{l_\alpha \lambda^m_\alpha}{d^m_{3-\alpha}}$.

Thus, we have arrived at the fourth-order algebraic equations with the real coefficients which contain only even powers of the parameter λ_α. It can be shown that there exist four real roots of Eqs. (4.4), (two positive and two negative) so that $\kappa^m_\alpha = 0$ holds in Eq. (4.2). Consequently, the parameters $R^m_{\alpha\beta}$ will be also real and $T^m_{\alpha\beta} = 0$ in Eq. (4.2).

Let $\Psi^m_{\nu\alpha}$, $\nu = 1,2,3,4$ be the solutions of Eqs. (4.4). From the definition of Λ^m_α it follows that $\psi^{\nu m}_\alpha = \Psi^m_{\nu\alpha} d^m_{3-\alpha}/l_\alpha$, while $S^{\nu m}_{\alpha\beta}$ results from Eqs. (4.3). Thus, a general solution of (3.3) has the form:

$$f^m_{\alpha\beta}(x_\beta) = \sum_\nu S^m_{\nu\alpha\beta} \exp(l_\beta \psi^m_{\nu\beta} x_\beta) \,. \tag{4.5}$$

Bearing in mind (4.3) we conclude that the first two equations (4.3) are linearly dependent so that the coefficients $S^m_{\nu\alpha 1}$ are also linearly dependent; hence $S^m_{\nu 21} = K^m_{\nu 1} S^m_{\nu 11}$, where

$$K^m_{\nu 1} = \frac{B_{1111} l_1 \psi^m_{\nu 1} - B_{1212} d^m_2}{B_{1122} + B_{1212}} \,.$$

Similarly for $S^m_{\nu\alpha 2}$ (Eq. (4.3)) we can write $S^m_{\nu 22} = K^m_{\nu 2} S^m_{\nu 12}$, where

$$K^m_{\nu 2} = \frac{B_{1212} l_2 \psi^m_{\nu 2} - B_{1111} d^m_1}{B_{1122} + B_{1212}} \,.$$

Substituting the above formulae into Eqs. (3.2) we obtain the following formulae for the displacement field:

$$\begin{aligned} u_1(x_\alpha) &= \sum_m \sum_\beta \sum_\nu S^m_{\nu 1\beta} \exp(l_\beta \psi^m_{\nu\beta} x_\beta) \sin(d^m_{3-\beta} x_{3-\beta}) \,, \\ u_2(x_\alpha) &= \sum_m \sum_\beta \sum_\nu K_{\nu\beta} S^m_{\nu 1\beta} \exp(l_\beta \psi^m_{\nu\beta} x_\beta) \cos(d^m_{3-\beta} x_{3-\beta}) \,. \end{aligned} \tag{4.6}$$

By virtue of Eq. (2.3), the strains of the beam are:

$$\begin{aligned} \varepsilon_{11}(x_\alpha) &= \sum_m \sum_\nu \Big[S^m_{\nu 11} (l_1 \psi^m_{\nu 1}) \exp(l_1 \psi^m_{\nu 1} x_1) \sin(d^m_2 x_2) \\ &\qquad + S^m_{\nu 12} d^m_1 \exp(l_2 \psi^m_{\nu 2} x_2) \cos(d^m_1 x_1) \Big] \,, \\ \varepsilon_{22}(x_\alpha) &= \sum_m \sum_\nu \Big[- K^m_{\nu 1} S^m_{\nu 11} d_2 \exp(l_1 \psi^m_{\nu 1} x_1) \sin(d^m_2 x_2) \\ &\qquad + K^m_{\nu 2} S^m_{\nu 12} l_2 \psi^m_{\nu 2} \exp(l_2 \psi^m_{\nu 2} x_2) \cos(d^m_1 x_1) \Big] \,, \\ \varepsilon_{12}(x_\alpha) &= \frac{1}{2} \sum_m \sum_\nu \Big[(S^m_{\nu 11} d^m_2 + K^m_{\nu 1} S^m_{\nu 11} l_1 \psi^m_{\nu 1}) \exp(l_1 \psi^m_{\nu 1} x_1) \cos(d^m_2 x_2) \\ &\qquad + (S^m_{\nu 12} l_2 \psi^m_{\nu 2} - K^m_{\nu 2} S^m_{\nu 12} d^m_1) \exp(l_2 \psi^m_{\nu 2} x_2) \sin(d^m_1 x_1) \Big] \,. \end{aligned}$$

Substitution of the displacements (4.6) into (3.1) yields the stress distributions in the beam given by:

$$\sigma_{11} = \sum_m \sum_\nu \Big[S^m_{\nu 11} d^m_2 (B_{1111} l_1 \psi^m_{\nu 1} - B_{1133} K^m_{\nu 1}) \exp(l_1 \psi^m_{\nu 1} x_1) \sin(d^m_2 x_2)$$
$$+ S^m_{\nu 12} d^m_1 (B_{1111} - B_{1133} K^m_{\nu 1} l_2 \psi^m_{\nu 2}) \exp(l_2 \psi^m_{\nu 2} x_2) \cos(d^m_1 x_1) \Big] ,$$
$$\sigma_{22} = \sum_m \sum_\nu \Big[S^m_{\nu 11} d^m_2 (B_{1133} l_1 \psi^m_{\nu 1} - B_{3333} K^m_{\nu 1}) \exp(l_1 \psi^m_{\nu 1} x_1) \sin(d^m_2 x_2)$$
$$+ S^m_{\nu 12} d^m_1 (B_{1133} - B_{3333} K^m_{\nu 2} l_2 \psi^m_{\nu 2}) \exp(l_2 \psi^m_{\nu 2} x_2) \cos(d^m_1 x_1) \Big] ,$$
$$\sigma_{12} = \frac{1}{2} B_{1313} \sum_m \sum_\nu \Big[S^m_{\nu 11} d^m_2 (1 + K^m_{\nu 1} l_1 \psi^m_{\nu 1}) \exp(l_1 \psi^m_{\nu 1} x_1) \cos(d^m_2 x_2)$$
$$+ S^m_{\nu 12} d^m_1 (l_2 \psi^m_{\nu 2} - K^m_{\nu 2}) \exp(l_2 \psi^m_{\nu 2} x_2) \sin(d^m_1 x_1) \Big] .$$

The unknowns $S^{\nu m}_{\alpha\beta}$ are determined from the boundary conditions at the opposite edges of the beam:

$$\text{(4.7)} \qquad \sigma_{11}(x_2)\Big|_{x_1=\pm a} = P \sin(l_2 x_2) , \qquad \sigma_{12}(x_2)\Big|_{x_1=\pm a} = 0 ,$$

and on its lateral surfaces:

$$\text{(4.8)} \qquad \sigma_{21}(x_1)\Big|_{x_2=\pm h} = 0 , \qquad \sigma_{22}(x_1)\Big|_{x_2=\pm h} = 0 .$$

Multiplying the first equation of (3.1) by $\sin(d^n_2 x_2)$ and $\cos(d^n_2 x_2)$ and then integrating it over $[-a_2, a_2]$, we arrive at the algebraic equations for the unknown coefficients $S^{\nu m}_{\alpha\beta}$:

$$\sum_\nu S^{\nu m}_{\alpha\beta} = \sum_\nu S^m_{\nu 11} d^m_2 (B_{1111} l_1 \psi^m_{\nu 1} - B_{1133} K^m_{\nu 1}) \exp(\psi^m_{\nu 1}) = P_{11(n)} ,$$

where $P_{11(n)}$ stand for the Fourier coefficients:

$$P_{11(n)} = \frac{1}{h} \int_{-h}^{h} P \sin(l_2 x_2) \sin(d^n_2 x_2) dx_2 .$$

It should be noted that the orthogonality of trigonometric functions has been also utilised.

By analogy, for the Eqs. (4.8) we have:

$$\sum_m \sum_\nu \Big[S^m_{\nu 11} d^m_2 (1 + K^m_{\nu 1} l_1 \psi^m_{\nu 1}) \exp(\psi^m_{\nu 1}) + S^m_{\nu 12} d^m_1 (l_2 \psi^m_{\nu 2} - K^m_{\nu 2}) I^n_2 \sin(d^m_1 a) \Big] = 0 ,$$

$$\sum_\nu \Big[S^m_{\nu 12} d^m_1 (l_2 \psi^m_{\nu 2} - K^m_{\nu 2}) \exp(\psi^m_{\nu 2}) \Big] = 0 ,$$

$$\sum_m \sum_\nu \Big[S^m_{\nu 11} d^m_2 (B_{1133} l_1 \psi^m_{\nu 1} - B_{3333} K^m_{\nu 1}) \exp(l_1 \psi^m_{\nu 1} x_1) \sin(d^m_2 x_2) + S^m_{\nu 12} d^m_1 (B_{1133} - B_{3333} K^m_{\nu 2} l_2 \psi^m_{\nu 2}) \exp(\psi^m_{\nu 2}) \Big] ,$$

with the following notations:

$$I^{mn}_1 = \frac{1}{h} \int_{-h}^{h} \exp(l_1 \psi^m_{\nu 1} x_1) \cos(d^n_1 x_1) dx_1 ,$$

$$I^{mn}_2 = \frac{1}{h} \int_{-h}^{h} \exp(l_2 \psi^m_{\nu 2} x_2) \cos(d^n_2 x_2) dx_2 .$$

5. Analysis of the results

Let a rectangular beam be made of the orthotropic material determined by the Young moduli $E_1 = 5.7 \cdot 10^{10}\,\mathrm{N/m^2}$, $E_2 = 1.4 \cdot 10^{10}\,\mathrm{N/m^2}$, $E_3 = 1.4 \cdot 10^{10}\,\mathrm{N/m^2}$, the shear modulus $G_{12} = 0.57 \cdot 10^{10}\,\mathrm{N/m^2}$, and the Poisson ratio $\nu_{12} = 0.068$, and be subjected to the boundary tractions shown in Fig. 1. Using the procedure proposed above, the boundary value problem has been solved.

Figures 2-7 show the diagrams of the displacements and stresses obtained for $m_0 = 25$, and for the ratio of beam thickness to its length $h/a = 1/20$.

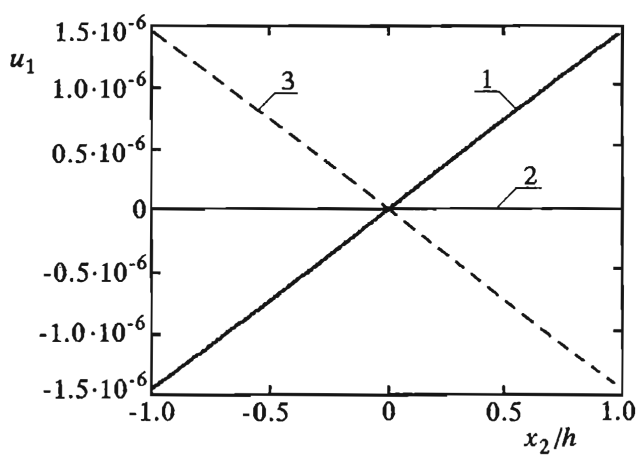

FIG. 2. Distribution of the displacement u_1 over the thickness of a beam.

In Fig. 2, a line segment No. 1 represents the displacement u of the right-hand side edge of the beam, while Nos. 2 and 3 represent displacements of the middle section and left-hand side edge of the beam, respectively. The displacements in longitudinal sections of the beam, i.e., for x_2 = const, are of the same nature. Therefore, it can be clearly seen that the displacement distributions u_1 over both the beam thickness and length are linear.

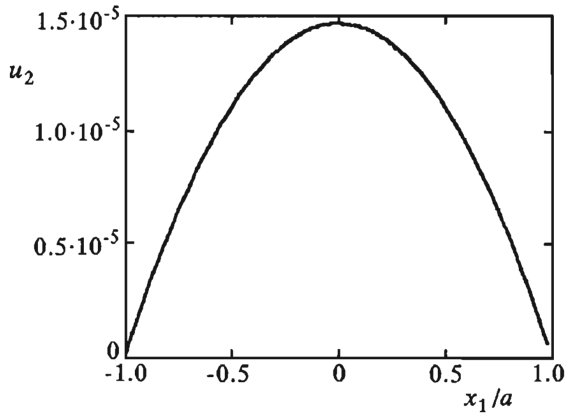

FIG. 3. Change of the deflections of a beam in its longitudinal sections.

From Fig. 3 it can be seen that the distribution of beam deflection over its length (x_2 = const) has the parabolic form, reaching its maximal value at $x_2 = 0$. The deflection u_2 remains unchanged over the beam thickness, i.e., its diagrams for the upper, middle and bottom surfaces coincide.

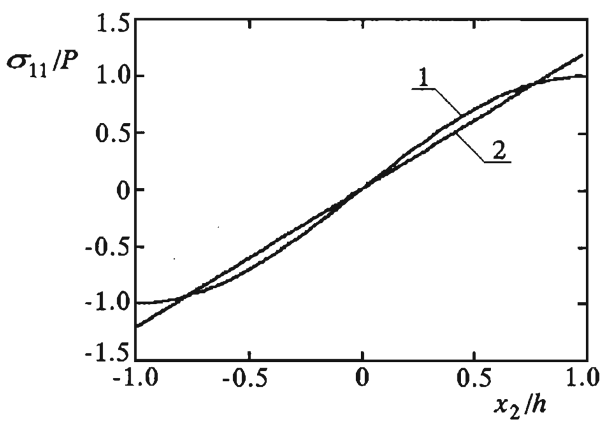

FIG. 4. Distribution of the normal stress σ_{11} over the thickness of a beam.

As it can be seen from Fig. 4, the normal stress distribution σ_{11} over the beam thickness is linear almost everywhere except near the boundary layer, where the boundary conditions have to be satisfied. On the other hand, the diagram shown in Fig. 5 shows that the stress σ_{11} remains unchanged over almost the whole beam bottom surface, except for the boundary layer, where it tends to the value of the external tractions applied. Therefore, it can be concluded that the considered beam deforms uniformly over its length.

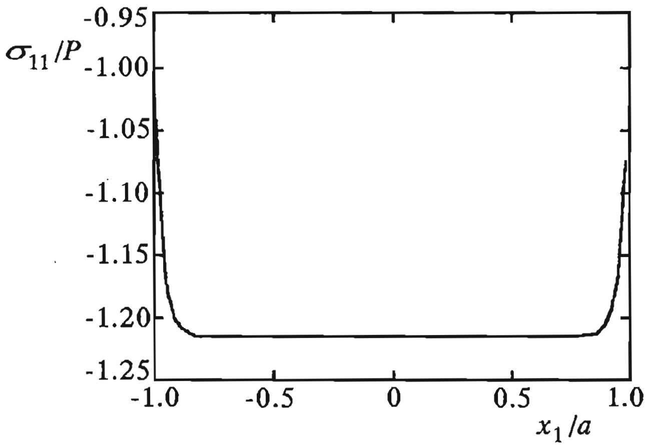

FIG. 5. Change of the normal stress σ_{11} on the lower surface of a beam.

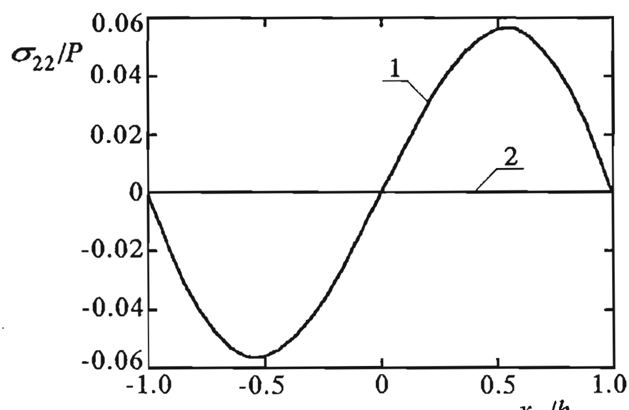

FIG. 6. Distribution of the normal stress σ_{22} over the thickness of a beam.

The normal stress σ_{22} equals zero on the upper, middle and bottom surfaces, respectively, of the beam reaching its maximal value at the distance $x_2 = \pm h/4$ from the extreme surfaces (see Fig. 6). Since the values of the stress σ_{22} are lower than those of the normal ones analysed above, it can be concluded that the condition for the plane stress is satisfied.

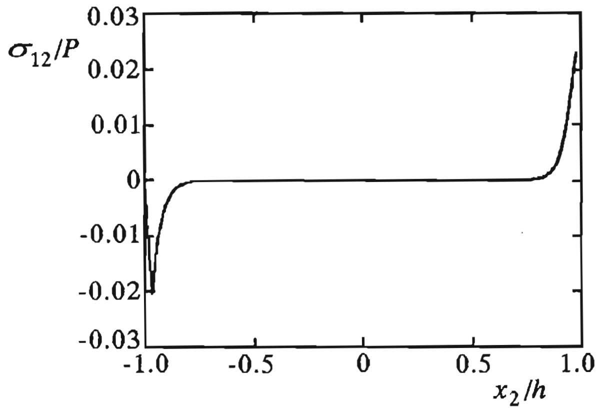

FIG. 7. Distribution of the tangential stresses σ_{12} over the middle surface.

The tangential stresses σ_{12} calculated within the framework of the plate bending theory reach their extreme values on the middle surface. From Fig. 7 it can be seen that the values of tangential stresses equal zero, except for the boundary layers, where they undergo a small jump of an order of 0.02.

Bearing in mind the above analysis it can be concluded that during bending of the sufficiently long orthotropic beams, the Kirchhoff theory assumptions hold true, i.e.:

- Distributions of stresses and displacements over the beam thickness are linear in the planes parallel to the middle surface.
- Beam deflection remains unchanged over the beam thickness.
- The stresses normal to the middle surface and the shear stresses are negligible.

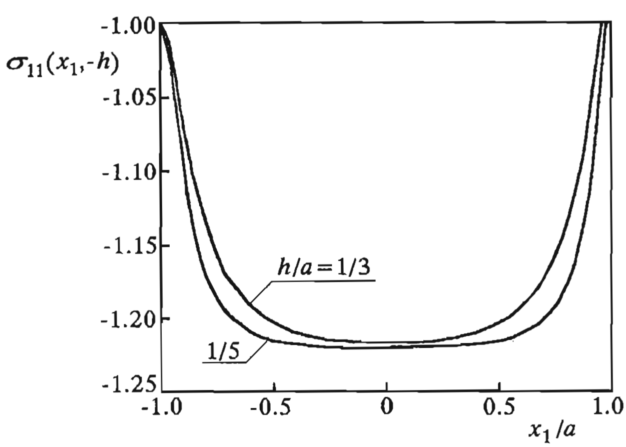

FIG. 8. Distribution of the normal stress over the bottom surface of a short beam.

The results presented above can be compared with those obtained from the analysis of normal and tangential stresses depending on the beam thickness. The two values of thickness ratio to its length have been considered: i.e., $h/a = 1/5$ and $h/a = 1/3$. It can be shown that the values of normal stresses, in these cases, are much smaller than the stresses for a long beam, cf. Fig. 8. On the contrary, the values of tangential stresses change considerably on the middle surface of the beam (see Fig. 9), being no longer equal to zero on the middle surface, as it was for $h/a = 1/20$.

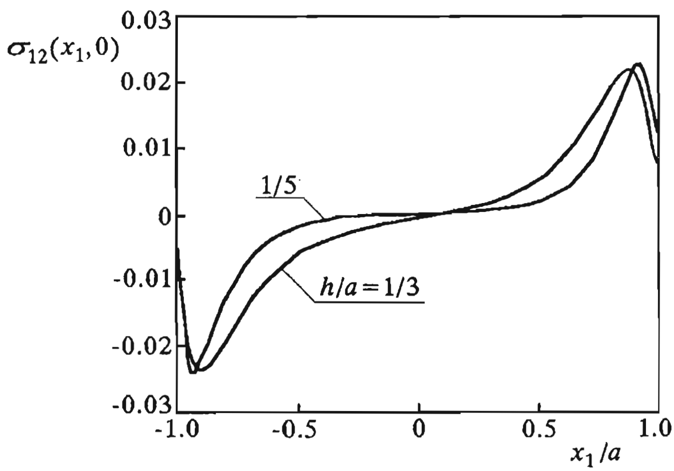

FIG. 9. Distribution of the tangential stress over the middle surface of a short beam.

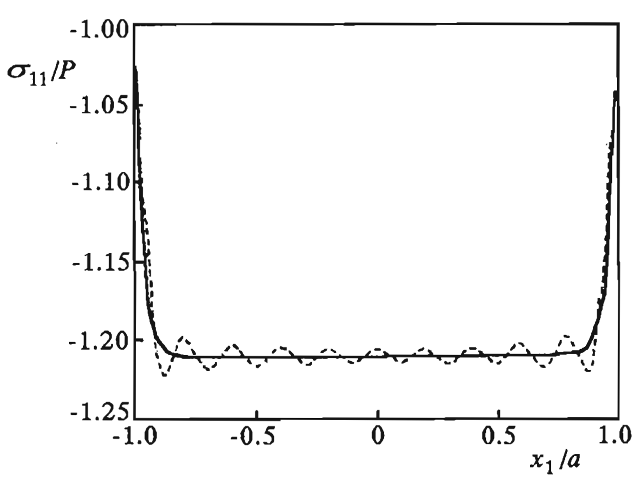

FIG. 10. Variation of the normal stress with the number of approximations.

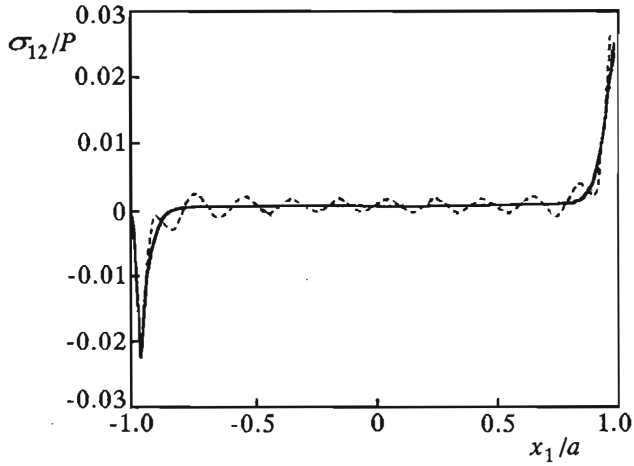

FIG. 11. Variation of the tangential stress with the number of approximations.

Figures 10 and 11 illustrate the influence of the number m_0, i.e. the number of terms in expansion of the displacements fields, on the value of the normal (Fig. 10) and tangential stresses (Fig. 11). The dashed line represents the case when $m_0 = 10$, while the solid one corresponds to $m_0 = 25$. It can be clearly seen that rising the number m_0 above 25 does not affect the results.

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