ENGINEERING TRANSACTIONS • *Engng. Trans.* • **64**, 4, 409–415, 2016
Polish Academy of Sciences • Institute of Fundamental Technological Research (IPPT PAN)
National Engineering School of Metz (ENIM) • Poznan University of Technology

Vibrations of Point-Supported Rectangular Thin Plate Subjected to a Moving Force

Filip ZAKĘŚ

Wroclaw University of Environmental and Life Sciences
The Faculty of Environmental Engineering and Geodesy
Plac Grunwaldzki 24, 50-365 Wrocław, Poland
e-mail: filip.zakes@up.wroc.pl

In this paper, the dynamic behaviour of a rectangular thin plate simply supported on all edges and point supported within its region is investigated. The problem is solved by replacing this type of structure with a simply supported plate subjected to a given moving load and redundant forces situated in positions of intermediate point supports. Redundant forces are obtained by solving Volterra integral equations of the first order, which are compatibility equations corresponding to each redundant. Solutions for a simply supported plate loaded with a moving point force and concentrated time-varying force are given. Difficulties of solving Volterra integral equations analytically are bypassed by applying a simple numerical procedure. Finally, a numerical example of a plate with two point supports is presented in order to show the effectiveness of the presented method.

Key words: thin plate, vibrations, moving load, Volterra integral equations.

1. Introduction

The problem of dynamic behavior of a structure subjected to a moving load is both important and interesting from the theoretical point of view and its practical application in structural designing. For many years, this problem has been analyzed by many authors with different types of structures as well as various types of moving loads taken into account [1–4].

This work is focused on the case of a rectangular thin plate simply supported on all edges and point-supported within its area. The number of point supports and their locations are arbitrary. The analyzed plate is loaded with a concentrated force of constant magnitude that is moving with a constant speed along one of the edges

2. Vibrations of simply supported thin plate

In the first step, we shall concentrate on a rectangular thin plate of dimensions $B \times L$ simply supported on all edges subjected to a dynamic load $p(x, y, t)$. The equation of motion describing undamped vibrations of the plate has the form:

$$D\left[\frac{\partial^4 w(x,y,t)}{\partial x^4} + 2\frac{\partial^4 w(x,y,t)}{\partial x^2 y^2} + \frac{\partial^4 w(x,y,t)}{\partial y^4}\right] + \rho h \frac{\partial^2 w(x,y,t)}{\partial t^2} = p(x,y,t), \tag{2.1}$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ is the flexural rigidity of the plate (E is the Young's modulus and ν is the Poisson's ratio), ρ is the mass density, and h is the thickness of the plate. In the following chapters, the cases of moving constant force and concentrated time varying force will be analyzed.

2.1. Case of a moving constant force

Let us consider a plate subjected to a point force of constant magnitude P moving with constant velocity v along the axis y_0, that is, parallel to one of the edges (see Fig. 1). The vibrations $w^P(x, y, t)$ of the plate are described by the following equation:

$$D\left[\frac{\partial^4 w^P(x,y,t)}{\partial x^4} + 2\frac{\partial^4 w^P(x,y,t)}{\partial x^2 y^2} + \frac{\partial^4 w^P(x,y,t)}{\partial y^4}\right] + \rho h \frac{\partial^2 w^P(x,y,t)}{\partial t^2} = P\delta(x - vt)\delta(y - y_0), \tag{2.2}$$

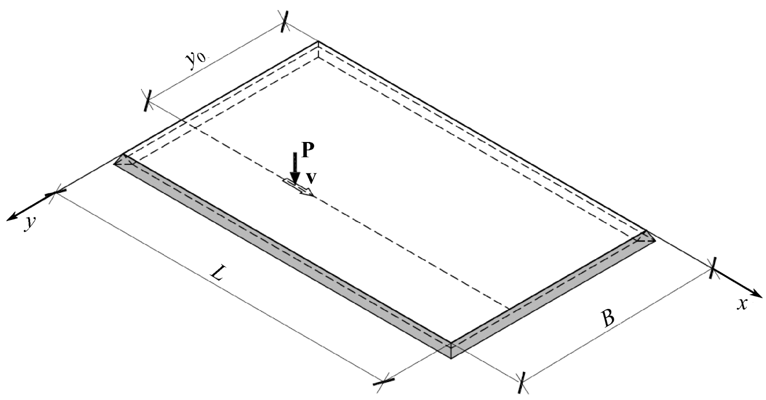

Fig. 1. Simply supported plate subjected to a moving point force.

where $\delta(\,.\,)$ denotes the Dirac delta. This case was analyzed by Fryba in [1]. The solution of Eq. (2.2) has the form of double sine series and can be obtained by using the orthogonality method. Assuming zero initial conditions, the vibrations of the plate can be expressed as

$$(2.3)\quad w^P(x,y,t) = \frac{4P}{\rho h B L}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{\sin\dfrac{m\pi v t}{L}\sin\dfrac{n\pi y_0}{B}\sin\dfrac{m\pi x}{L}\sin\dfrac{n\pi y}{B}}{\omega_{mn}^2-\left(\dfrac{m\pi v}{L}\right)^2} - \frac{\dfrac{m\pi v}{L}\sin\omega_{mn}t\sin\dfrac{n\pi y_0}{B}\sin\dfrac{m\pi x}{L}\sin\dfrac{n\pi y}{B}}{\omega_{mn}\left[\omega_{mn}^2-\left(\dfrac{m\pi v}{L}\right)^2\right]}\right],$$

where

$$\omega_{mn}^2 = \frac{D}{\rho h}\left[\left(\frac{m\pi}{L}\right)^4 + 2\left(\frac{m\pi}{L}\right)^2\left(\frac{n\pi}{B}\right)^2 + \left(\frac{n\pi}{B}\right)^4\right].$$

2.2. Case of a concentrated time-varying force

In the next step, we consider the vibrations of a plate due to a time varying force $X_i(t)$ concentrated at point (x_i, y_i) – see Fig. 2. The equation of motion has the form:

$$(2.4)\quad D\left[\frac{\partial^4 w^X(x,y,t)}{\partial x^4} + 2\frac{\partial^4 w^X(x,y,t)}{\partial x^2 y^2} + \frac{\partial^4 w^X(x,y,t)}{\partial y^4}\right] + \rho h\frac{\partial^2 w^X(x,y,t)}{\partial t^2} = X_i(t)\delta(x-x_i)\delta(y-y_i)$$

and can be solved similarly to the previous case, that is, by assuming solution in the form of double sine series and using the orthogonality method.

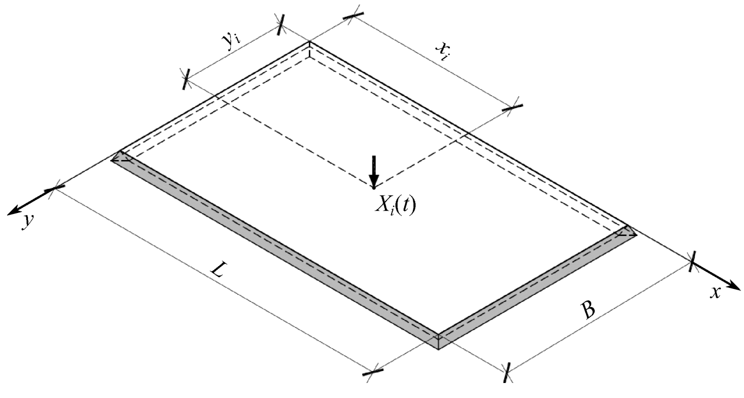

FIG. 2. Simply supported plate subjected to a concentrated time-varying force.

Introducing the impulse response function $h_{mn}(t) = \frac{1}{\omega_{mn}} \sin \omega_{mn} t$ vibrations of the plate due to force $X_i(t)$ can be presented in the convolution form:

$$w^{Xi}(x,y,t) = \frac{4}{\mu BL} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x_i}{L} \sin \frac{n\pi y_i}{B} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{B} \cdot \int_0^t h_{mn}(t-\tau) X_i(\tau) d\tau. \tag{2.5}$$

3. Vibrations of a point supported plate

Using the solutions for the cases of load presented above, the vibrations of a plate simply supported on all edges and having k point supports in its region can be presented as vibrations of a simply supported plate subjected to a given moving load and k redundant forces acting at the locations of point supports (see Fig. 3):

$$w(x,y,t) = \sum_{i=1}^{k} w^{Xi}(x,y,t) + w^P(x,y,t). \tag{3.1}$$

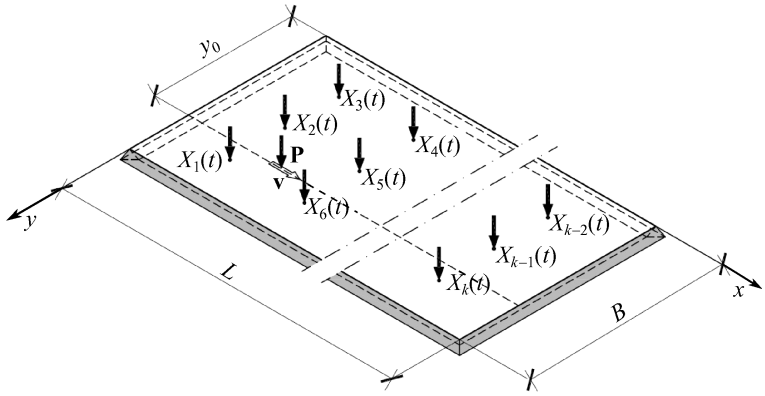

Fig. 3. Simply supported plate subjected to a moving force P and k redundant point forces.

3.1. Volterra integral equations

The vibrations of the plate at the position of "i" redundant point support (x_i, y_i) resulting from the force $X_j(t)$ acting at the position of "j" point support (x_j, y_j) can be expressed as

$$w^{Xj}(x_i, y_i, t) = \int_0^T d_{ij}(t-\tau) X_j(\tau) d\tau, \tag{3.2}$$

where

$$(3.3) \qquad d_{ij}(t) = \frac{4}{\rho h B L} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \omega_{mn} t \sin \dfrac{m\pi x_i}{L} \sin \dfrac{n\pi y_i}{B} \sin \dfrac{m\pi x_j}{L} \sin \dfrac{n\pi y_j}{B}}{\omega_{mn}}.$$

The expression $d_{ij}(t)$ was obtained from Eq. (2.5). Since we know that at the positions of point supports, the deflections of the plate are equal to 0 we can write a set of compatibility equations in the form of Volterra integral equations of the first order:

$$(3.4) \qquad \sum_{i=1}^{k} \int_{0}^{T} d_{ij}(t-\tau) X_j(\tau) d\tau + \Delta_{iP}(t) = 0, \qquad j = 1, 2, \ldots, k,$$

where $\Delta_{iP}(t) = w^P(x_i, y_i, t)$.

3.2. Numerical procedure

Because Volterra integral equations (3.4) are difficult to solve analytically, we shall apply a simple numerical procedure similar to the one used in [4] to describe vibrations of multi-span beams, because of its simplicity and satisfying efficiency. In the first step, the time of force movement along the plate $t = L/v$ is divided to N equal time segments. Then, we assume the collocation points τ_R at the middle of each time segment and the values of support reactions $X_j(\tau_r)$ as the unknowns. This allows us replace a set of integral equations (3.4) with algebraic equations by using the midpoint method:

$$(3.5) \qquad \sum_{j=1}^{k} \sum_{r=1}^{R} d_{ij}(t_R - \tau_r) X_j(\tau_r) \Delta\tau + \Delta_{iP}(t_R) = 0, \qquad i = 1, 2, \ldots, k,$$

where $t_R = R\Delta\tau$, $\tau_r = (r - 0,5)\Delta\tau$, $r = 1, 2, \ldots, R$, $R = 1, 2, \ldots, N$, $\Delta\tau = L/(Nv)$.

4. Numerical example

The presented example is of a rectangular thin plate simply supported on its four edges and point-supported at $x_1 = 10$ m, $y_1 = 10$ m and $x_2 = 30$ m, $y_2 = 10$ m (Fig. 4). The dimensions of the plate are equal to $B \times L = 20 \times 40$ m and its thickness $h = 0.4$ m. The plate is subjected to a force of constant magnitude $P = 10000$ N, moving with constant speed $v = 60$ m/s along the line $y_0 = 5$ m. The material properties of the plate are equal to $\rho = 2400$ kg/m^3, $E = 30 \cdot 10^9$ N/m^2 and $\upsilon = 0.2$. Figure 5 presents vertical vibrations of the

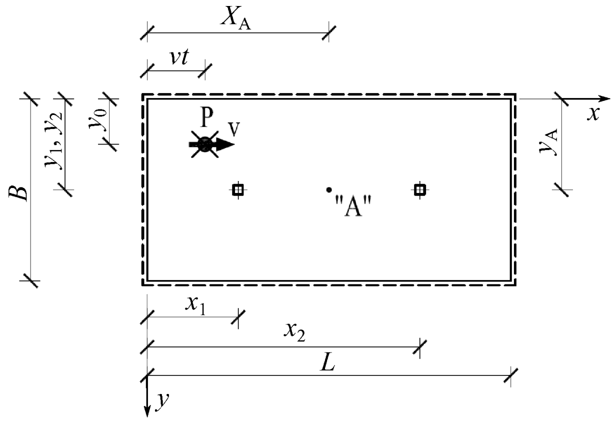

FIG. 4. Plate with two point supports subjected to a moving force

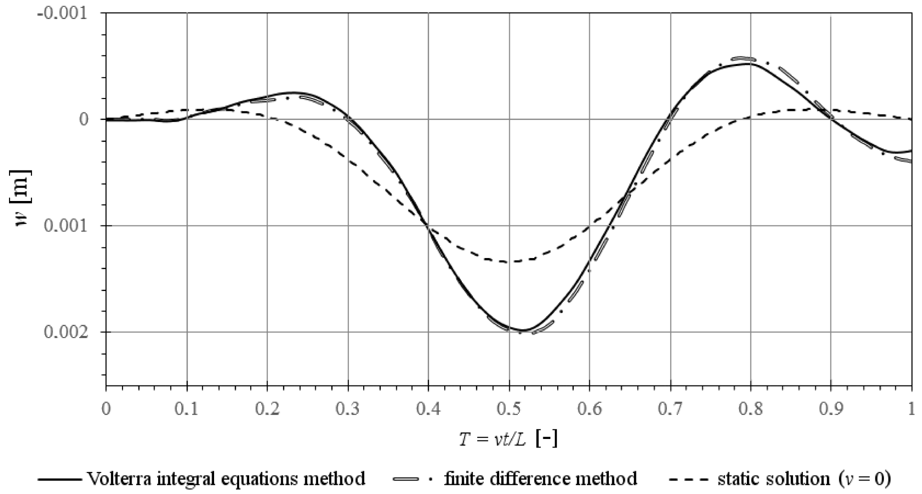

FIG. 5. Dynamic deflection of point "A"

point "A" ($x_A = 20$ m, $y_A = 10$ m), obtained by using the presented method compared with numerical results obtained by using the finite difference method. A very good agreement between these two methods was observed. Dashed line on the diagram marks the influence line of the static deflection. The results were obtained by dividing time of force movement into $N = 200$ time steps.

5. Conclusion

The presented method can be applied do describe the vibrations of point-supported thin plates. After the appropriate modification, this method can be expanded for the plates with different geometry and boundary conditions as well as for the plates subjected to different types of moving non-inertial load such as load distributed per unit length or per unit area. Using this method, we avoid spatial discretization of the structure – we discretize only the time of force movement. Applying the presented numerical procedure allows us to bypass difficulties of solving analytically integral equations and makes this task easy to solve by using simple computer programs. This method can be also used as verification for other numerical methods such as the finite element method.

References

1. Fryba L., *Vibrations of solids and structures under moving load*, Telford, London, 1999.
2. Leissa A.W., *Vibrations of plates*, US Government Printing Office, Washington DC, 1969.
3. Szcześniak W., *Selected aspects of the dynamics of plates* [in Polish], Oficyna Wydawnicza Politechniki Warszawskiej, Warszawa, 2000.
4. Zakęś F., Śniady P., *Application of Volterra integral equations in dynamics of multispan uniform continuous beams subjected to a moving load*, Shock and Vibration, vol. 2016, article ID 4070627, 2016. doi: 10.1155/2016/4070627

Received September 12, 2016; accepted version November 4, 2016.