ENGINEERING TRANSACTIONS • *Engng. Trans.* • **65**, 3, 423–445, 2017
Polish Academy of Sciences • Institute of Fundamental Technological Research (IPPT PAN)
National Engineering School of Metz (ENIM) • Poznan University of Technology

Research Paper

Analysis of Euler-Bernoulli Beams with Arbitrary Boundary Conditions on Winkler Foundation Using a B-Spline Collocation Method

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Structural beams are important parts of engineering projects. The structural analysis of beams is required to ensure that they provide the specifics needed to prevent and withstand failure. Therefore, the numerical solution to analyze an Euler-Bernoulli beam with arbitrary boundary conditions using sextic B-spline method is presented in this paper. A direct modeling technique is applied for modeling the Euler-Bernoulli beam with arbitrary boundary conditions on an elastic Winkler foundation. For this purpose, the effect of the translational along with rotational support, the type of beam supports and the elastic coefficient of Winkler foundation are assessed. Finally, some numerical examples are shown to present the efficiency of the sextic B-spline collocation method. To validate the analysis of the Euler-Bernoulli beam with the presented method, the results of B-spline collocation method are compared with the results of the analytical method and the integrated finite element analysis of structures (SAP2000).

Key words: Euler-Bernoulli beam, arbitrary boundary conditions, Winkler foundation, B-spline collocation method.

1. Introduction

Many geotechnical engineering problems can be studied by analyzing beams on foundations. The various foundation models such as Winkler, Pasternak, Kerr, Vlasov, Hetenyi and viscoelastic are applied in the analysis of structures on elastic foundations [1]. Among these models, the Winkler foundation model is the most common model used in such analyses. However, the modelling of soil using the Winkler approach is inadequate in the handling of the various problems [2]. The main weakness of the Winkler model lies in the fact that it neglects the shear interaction between the spring elements [3].

Analysis of statically indeterminate beams is an important problem in civil engineering. But this analysis is sometimes difficult or impossible if the degree of static indeterminacy in the beam is high. Analysis of a beam is used to determine the values of deflection, slope, shear force and bending moment. The fourth-

(or fifth-) order differential equations must be solved to obtain the displacement. The differential equations of the Euler-Bernoulli beam on the uniform elastic foundation are as follows:

- for uniformly distributed load:

$$y^{(4)}(x) + \frac{K}{EI}y(x) - \frac{q(x)}{EI} = 0, \qquad x \in [a, b], \tag{1.1}$$

- for linearly distributed load:

$$y^{(5)}(x) + \frac{K}{EI}y'(x) - \frac{dq(x)}{dx} \cdot \frac{1}{EI} = 0, \qquad x \in [ab], \tag{1.2}$$

where $y(x)$ is the transverse deflection of the mid-surface of the Euler-Bernoulli beam and $q(x)$ is the external force function on the beam. In addition, I, E and K are the second moment of area, the Young's modulus of elasticity, and the elastic coefficient of Winkler foundation, respectively. In the Euler-Bernoulli beam theory, the boundary conditions are given below:

$$\forall t \,@\, x = a : M(a) = K_{RL}\theta(a), \qquad Q(a) = -K_{TL}w(a),$$
$$\forall t \,@\, x = b : M(b) = -K_{RR}\theta(b), \qquad Q(b) = K_{TR}w(b),$$

where M and Q are the bending moment and the shear force, respectively (Fig. 1) [4]. K_{TL}, K_{TR}, K_{RL} and K_{RR} are the transverse and rotational elastic coefficients at the supports at the left and right boundary ends, respectively. For example, the boundary condition of the simple supports on both sides associated with a uniformly distributed load can be defined as

$$y(a) = 0, \quad y(b) = 0, \quad y''(a) = 0, \quad y''(b) = 0, \quad y^{(5)}(a) + \frac{K}{EI}y'(a) = 0.$$

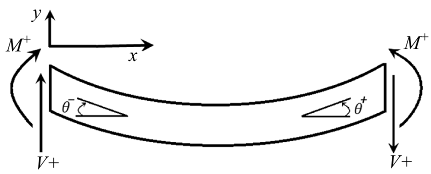

FIG. 1. Sign convention for shear forces, bending moments and slopes of the Euler-Bernoulli beam.

Also, the boundary condition of the simple supports on both sides associated with a linearly distributed load is defined as

$$y(a) = 0, \quad y(b) = 0, \quad y''(a) = 0, \quad y''(b) = 0, \quad y^{(4)}(a) + \frac{K}{EI}y(x) = q(a).$$

In this paper, the collocation method based on sextic B-spline is applied to analyze the Euler-Bernoulli beam with arbitrary boundary conditions. A spline function is the piecewise polynomial function of degree n. This function is the composite of several internal points. On the other hand, the number points must equal or be greater than $(k-1)$ degree. The differential equations with k degree are solved by B-spline functions of $(k+1)$ degree [5]. Over the years, the spline method has been used for solving the differential system of equations with different boundary conditions. For example, the sextic spline function for the solution of second-order boundary value problems associated with unilateral, obstacle and contact problems is presented by Rashidinia *et al.* [6]. Their results show that the approximate solutions obtained using the present method are better than spline and finite difference methods. A quintic non-polynomial spline method is investigated by Ramadan *et al.* for the numerical solution of the fourth-order two-point boundary value problems [7]. Based on their findings, the quintic non-polynomial spline method presents better approximations and generalizes all the existing polynomial spline methods up to fourth order. The natural frequencies of the non-uniform Euler-Bernoulli beam on elastic foundation are obtained using the spline collocation method by Hsu [8]. The Kuramoto-Sivashinsky equation is solved using septic B-spline collocation method by Zarebnia and Parvaz [9]. The solution is approximated as the linear combination of the septic B-spline functions. It is shown that this method is unconditionally stable by applying the von-Neumann stability analysis technique. Zarebnia and Parvaz presented the cubic B-spline collocation method for the numerical solution of the problem arising from chemical reactor theory [10]. Mohammadi developed a numerical method based on sextic B-spline to solve the fourth-order time-dependent partial differential equations [11]. In this paper, the convergence analysis of the sextic B-spline approximation for the Euler-Bernoulli beams with fixed and cantilever boundary conditions is discussed in detail. Reali and Gomez introduced an isogeometric analysis collocation method for the solution of the Bernoulli-Euler beam and Kirchhoff plate [12]. Akram also used the sextic spline method for solving a system of fifth-order boundary value problems [13].

In the previous studies, the Euler-Bernoulli beam on an arbitrary variable elastic Winkler foundation was not analyzed using the B-spline collocation method. On the other hand, these solutions can be generalized only to simple boundary conditions. In the present study, the solution using the sextic B-spline method is introduced to analyze the Euler-Bernoulli beam with arbitrary boundary conditions on the partial Winkler foundation. Furthermore, the analysis of the Euler-Bernoulli beam is written in a general form. Therefore, the objective of this paper is:

- To present a simple and practical numerical technique for determining the response of Euler-Bernoulli beams with elastically restrained boundary conditions, resting on a partial Winkler foundation.

- To state numerical solutions using the sextic B-spline function for an analysis of the beam with and without the partial Winkler foundation.

This paper is structured as follows. Section 2 outlines the sextic B-spline collocation method. Then, in Sec. 3, the numerical solution of the differential equation of the Euler-Bernoulli beam on uniform foundation is developed using the B-spline method. Section 4 presents some numerical examples to illustrate the efficiency of the presented method. Finally, in Sec. 5, brief conclusions are drawn.

2. Definition of B-spline curve

Let $x = (x_0, x_1, \ldots, x_N)$ be a knot vector. A B-spline function of k-degree is defined as [14]

$$B_i^0(x) = \begin{cases} 1 & \text{for } x \in [x_i, x_{i+1}), \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

$$B_i^k(x) = \frac{x - x_i}{x_{i+k} - x_i} B_i^{k-1}(x) + \frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}} B_{i+1}^{k-1}(x), \tag{2.2}$$

where $0 \le i \le N - k - 1$ and $1 \le k \le N - 1$.

Sextic B-spline can be obtained by calculating the B-spline basis function up to sixth order using Eq. (2.2). Therefore, the sextic B-spline basis function $B_i^6(x)$ is as follows:

$$B_i^6(x) = \frac{1}{h^6} \begin{cases} (x - x_i + 3h)^6, & x \in [x_{i-3}, x_{i-2}], \\ (x - x_i + 3h)^6 - 7(x - x_i + 2h)^6, & x \in [x_{i-2}, x_{i-1}], \\ (x - x_i + 3h)^6 - 7(x - x_i + 2h)^6 \\ \qquad + 21(x - x_i + h)^6, & x \in [x_{i-1}, x_i], \\ (x - x_i + 3h)^6 - 7(x - x_i + 2h)^6 \\ \qquad + 21(x - x_i + h)^6 - 35(x - x_i)^6, & x \in [x_i, x_{i+1}], \\ (x - x_i - 4h)^6 - 7(x - x_i - 3h)^6 \\ \qquad + 21(x - x_i - 2h)^6, & x \in [x_{i+1}, x_{i+2}], \\ (x - x_i - 4h)^6 - 7(x - x_i - 3h)^6, & x \in [x_{i+2}, x_{i+3}], \\ (x - x_i - 4h)^6, & x \in [x_{i+3}, x_{i+4}], \\ 0, & \text{otherwise}. \end{cases} \tag{2.3}$$

In this paper, the solution domain $a \le x \le b$ is divided into N segments with a uniform length of $h = \frac{b-a}{N}$ at the knots x_i where $i = 0, 1, 2, \ldots, N$ and $x_{i+1} = x_i + h$ such that $a = x_0 < x_1 < \ldots < x_N = b$. In the sextic B-spline, basis function is defined as follows:

$$y(x) = \sum_{i=0}^{N+5} c_i B_i(x), \tag{2.4}$$

where $B_0(x), \ldots, B_{N+5}(x)$ are the sextic B-splines functions at the knots and are given by Eq. (2.3). $c_0, \ldots, c_{N+5}$ are unknown real coefficients that are determined by satisfying the boundary conditions at each end of the beam and the continuity conditions of displacement, slope and moment along with the shear force and the collocation form of the differential Eqs. (1.1) and (1.2). Also, first, second, third, fourth and fifth derivatives of B_i with respect to variable x are used to solve the fifth-order differential equation. Values of B_i and its derivatives at the nodal points are given in Table 1.

Table 1. Values of B_i and its derivatives at the nodal points.

	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}	x_{i+5}	x_{i+6}	x_{i+7}
B_i	0	$\frac{1}{720}$	$\frac{57}{720}$	$\frac{302}{720}$	$\frac{302}{720}$	$\frac{57}{720}$	$\frac{1}{720}$	0
B_i'	0	$\frac{6}{720h}$	$\frac{150}{720h}$	$\frac{240}{720h}$	$-\frac{240}{720h}$	$-\frac{150}{720h}$	$-\frac{6}{720h}$	0
B_i''	0	$\frac{30}{720h^2}$	$\frac{270}{720h^2}$	$-\frac{300}{720h^2}$	$-\frac{300}{720h^2}$	$\frac{270}{720h^2}$	$\frac{30}{720h^2}$	0
B_i'''	0	$\frac{120}{720h^3}$	$\frac{120}{720h^3}$	$-\frac{960}{720h^3}$	$\frac{960}{720h^3}$	$-\frac{120}{720h^3}$	$-\frac{120}{720h^3}$	0
$B_i^{(4)}$	0	$\frac{360}{720h^4}$	$-\frac{1080}{720h^4}$	$\frac{720}{720h^4}$	$\frac{720}{720h^4}$	$-\frac{1080}{720h^4}$	$\frac{360}{720h^4}$	0
$B_i^{(5)}$	0	$\frac{720}{720h^5}$	$-\frac{3600}{720h^5}$	$\frac{7200}{720h^5}$	$-\frac{7200}{720h^5}$	$\frac{3600}{720h^5}$	$-\frac{720}{720h^5}$	0

3. Construction of the proposed solution

By substituting Eq. (2.4) into Eqs. (1.1) and (1.2), equations yield as follows:

- for uniformly distributed load:

$$\sum_{i=0}^{N+5} c_i B_i^{(4)}(x_j) + \frac{K}{EI}\sum_{i=0}^{N+5} c_i B_i(x) - \frac{q(x)}{EI} = 0, \qquad \text{for } j = 0, 1, \ldots, N, \tag{3.1}$$

- for linearly distributed load:

$$\sum_{i=0}^{N+5} c_i B_i^{(5)}(x_j) + \frac{K}{EI}\sum_{i=0}^{N+5} c_i B_i'(x) - \frac{dq(x)}{dx}\cdot\frac{1}{EI} = 0, \tag{3.2}$$

$$\text{for } j = 0, 1, \ldots, N.$$

From Table 1 and Eq. (2.4), y, y_i', y_i'', y_i''', $y^{(4)}$ and $y^{(5)}$ are obtained as follows:

$$y_i = \frac{1}{720}\left(c_i + 57c_{i+1} + 302c_{i+2} + 302c_{i+3} + 57c_{i+4} + c_{i+5}\right), \tag{3.3}$$

$$y_i' = \frac{1}{720h}\left(6c_i + 150c_{i+1} + 240c_{i+2} - 240c_{i+3} - 150c_{i+4} - 6c_{i+5}\right), \tag{3.4}$$

$$y_i'' = \frac{1}{720h^2}\left(30c_i + 270c_{i+1} - 300c_{i+2} - 300c_{i+3} + 270c_{i+4} + 30c_{i+5}\right), \tag{3.5}$$

$$y_i''' = \frac{1}{720h^3}\left(120c_i + 120c_{i+1} - 960c_{i+2} + 960c_{i+3} - 120c_{i+4} - 120c_{i+5}\right), \tag{3.6}$$

$$y^{(4)} = \frac{1}{720h^4}\left(360c_i - 1080c_{i+1} + 720c_{i+2} + 720c_{i+3} - 1080c_{i+4} + 360c_{i+5}\right), \tag{3.7}$$

$$y^{(5)} = \frac{1}{720h^5}\left(720c_i - 3600c_{i+1} + 7200c_{i+2} - 7200c_{i+3} + 3600c_{i+4} - 720c_{i+5}\right). \tag{3.8}$$

y, y_i', EIy_i'', and EIy_i''' can be stated as displacement, slope, bending moment and shear force in the beam, respectively. Substituting Eq. (3.7) into Eq. (3.1), for uniformly distributed load, results in:

$$\begin{aligned}\frac{1}{720h^4}&\left(360c_j - 1080c_{j+1} + 720c_{j+2} + 720c_{j+3} - 1080c_{j+4} + 360c_{j+5}\right)\\ &+ \frac{K}{720EI}\left(c_i + 57c_{i+1} + 302c_{i+2} + 302c_{i+3} + 57c_{i+4} + c_{i+5}\right) - \frac{q(x)}{EI} = 0,\\ &i \text{ or } j = 0, \ldots, N.\end{aligned} \tag{3.9}$$

The above solution for uniformly distributed load can be written in the form of:

$$\begin{aligned}\frac{1}{720}&\left(\left(\frac{360}{h^4} + \frac{k}{EI}\right)c_j + \left(-\frac{1080}{h^4} + \frac{57k}{EI}\right)c_{j+1} + \left(\frac{720}{h^4} + \frac{302k}{EI}\right)c_{j+2}\right.\\ &+ \left(\frac{720}{h^4} + \frac{302k}{EI}\right)c_{j+3} + \left(-\frac{1080}{h^4} + \frac{57k}{EI}\right)c_{j+4}\\ &\left.+ \left(\frac{360}{h^4} + \frac{k}{EI}\right)c_{j+5}\right) - \frac{q(x)}{EI} = 0, \qquad i \text{ or } j = 0, \ldots, N.\end{aligned} \tag{3.10}$$

Similarly, it is possible to develop the solution for the linearly distributed load by substituting Eq. (3.8) into Eq. (3.2):

$$\frac{1}{720h^5}\left(720c_j - 3600c_{j+1} + 7200c_{j+2} - 7200c_{j+3} + 3600c_{j+4} - 720c_{j+5}\right) + \frac{K}{720hEI}(6c_i + 150c_{i+1} + 240c_{i+2} - 240c_{i+3} - 150c_{i+4} - 6c_{i+5}) - \frac{dq(x)}{dx}\cdot\frac{1}{EI} = 0, \qquad i \quad \text{or} \quad j = 0, 1, \ldots, N.$$

By simplifying the above solution, the solution for linearly distributed load can be rewritten as follows:

$$\frac{1}{720}\left(\left(\frac{720}{h^5} + \frac{6k}{hEI}\right)c_j + \left(-\frac{3600}{h^5} + \frac{150k}{hEI}\right)c_{j+1} + \left(\frac{7200}{h^5} + \frac{240k}{hEI}\right)c_{j+2} - \left(\frac{7200}{h^5} + \frac{240k}{hEI}\right)c_{j+3} + \left(\frac{3600}{h^5} - \frac{150k}{hEI}\right)c_{j+4} - \left(\frac{720}{h^5} + \frac{6k}{hEI}\right)c_{j+5}\right) - \frac{dq(x)}{dx}\cdot\frac{1}{EI} = 0, \qquad i \quad \text{or} \quad j = 0, 1, \ldots, N. \tag{3.11}$$

The systems (3.10) and (3.11) consist of $N + 1$ equations in the $N + 6$ unknowns $\{c_0, c_j, \ldots, c_{N+5}\}$. Thus, the five equations are needed at this stage. Therefore, the boundary conditions are used to obtain theses extra equations. Four extra equations are explicitly obtained using two boundary conditions at each end of the beam depending on the type of end support and one extra equation for uniformly distributed load is given below:

$$\frac{1}{720h^5}\left(720c_0 - 3600c_1 + 7200c_2 - 7200c_3 + 3600c_4 - 720c_5\right) + \frac{K}{720hEI}(6c_0 + 150c_1 + 240c_2 - 240c_3 - 150c_4 - 6c_5) = 0.$$

The above solution for uniformly distributed load can be rewritten as

$$\frac{1}{720}\left(\left(\frac{720}{h^5} + \frac{6k}{hEI}\right)c_0 + \left(-\frac{3600}{h^5} + \frac{150k}{hEI}\right)c_1 + \left(\frac{7200}{h^5} + \frac{240k}{hEI}\right)c_2 - \left(\frac{7200}{h^5} + \frac{240k}{hEI}\right)c_3 + \left(\frac{3600}{h^5} - \frac{150k}{hEI}\right)c_4 - \left(\frac{720}{h^5} + \frac{6k}{hEI}\right)c_5\right) = 0. \tag{3.12}$$

Also, one extra equation for linearly distributed load is obtained as

$$\frac{1}{720h^4}\left(360c_0 - 1080c_1 + 720c_2 + 720c_3 - 1080c_4 + 360c_5\right) + \frac{K}{720EI}y(x)\left(c_i + 57c_{i+1} + 302c_{i+2} + 302c_{i+3} + 57c_{i+4} + c_{i+5}\right) = q(a).$$

By simplifying the above solution, the solution for linearly distributed load can be given as follows:

$$\frac{1}{720}\left(\left(\frac{360}{h^4}+\frac{k}{EI}\right)c_0+\left(-\frac{1080}{h^4}+\frac{57k}{EI}\right)c_1+\left(\frac{720}{h^4}+\frac{302k}{EI}\right)c_2 + \left(\frac{720}{h^4}+\frac{302k}{EI}\right)c_3+\left(-\frac{1080}{h^4}+\frac{57k}{EI}\right)c_4+\left(\frac{360}{h^4}+\frac{k}{EI}\right)c_5\right)-\frac{q(x)}{EI}=0. \tag{3.13}$$

In addition, the continuity conditions of displacement, slope and moment along with the shear force in the vicinities of the different segment connections are defined as [4]

$$\begin{aligned} Y(a) &= y(a),\\ \theta(a) &= y'(a),\\ M(a) &= EIy''(a),\\ V(a) &= EIy'''(a). \end{aligned} \tag{3.14}$$

By applying the relationships between the individual physical quantities and the B-spline function, the continuity conditions at the first and last knot (the end knots) can be rewritten as follows:

$$\begin{aligned} Y(a) &= \frac{1}{720}\left(c_0+57c_1+302c_2+302c_3+57c_4+c_5\right),\\ \theta(a) &= \frac{1}{720h}\left(6c_0+150c_1+240c_2-240c_3-150c_4-6c_5\right),\\ M(a) &= \frac{EI}{720h^2}\left(30c_0+270c_1-300c_2-300c_3+270c_4+30c_5\right),\\ V(a) &= \frac{EI}{720h^3}\left(120c_0+120c_1-960c_2+960c_3-120c_4-120c_5\right), \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} Y(b) &= \frac{1}{720}\left(c_N+57c_{N+1}+302c_{N+2}+302c_{N+3}+57c_{N+4}+c_{N+5}\right),\\ \theta(b) &= \frac{1}{720h}(6c_N+150c_{N+1}+240c_{N+2}-240c_{N+3}\\ &\qquad -150c_{N+4}-6c_{N+5}),\\ M(b) &= \frac{EI}{720h^2}(30c_N+270c_{N+1}-300c_{N+2}\\ &\qquad -300c_{N+3}+270c_{N+4}+30c_{N+5}),\\ V(b) &= \frac{EI}{720h^3}(120c_N+120c_{N+1}-960c_{N+2}\\ &\qquad +960c_{N+3}-120c_{N+4}-120c_{N+5}), \end{aligned} \tag{3.16}$$

where Y, θ, M, and V are displacement, slope, bending moment, and shear force of the Euler-Bernoulli beam, respectively. Finally, the matrix equation is given as

$$[A] \times [C] = [F], \tag{3.17}$$

where the coefficient matrix $[A]$, matrix $[C]$ and the load matrix $[F]$ are cited in the appendix.

4. Numerical examples

To validate the sextic B-spline method, the results of different examples are presented. First, the high computational efficiency of the method is shown and then it is examined for the feedback with arbitrary boundary conditions. In all the examples, E and I are assumed as

$$E = 2038901.91 \ \frac{\text{kg}}{\text{cm}^2}, \qquad I = 6572.4175 \ \text{cm}^4.$$

4.1. Euler-Bernoulli beam under uniformly distributed load

For the purpose of verification of the presented method, the Euler-Bernoulli beam with translational restraint supported under a uniformly distributed load is considered (Fig. 2). The beam is assumed to have the following characteristics:

$$q = 15 \ \frac{\text{kg}}{\text{cm}}, \qquad k = 2500 \ \frac{\text{kg}}{\text{cm}}, \qquad L = 500 \ \text{cm}.$$

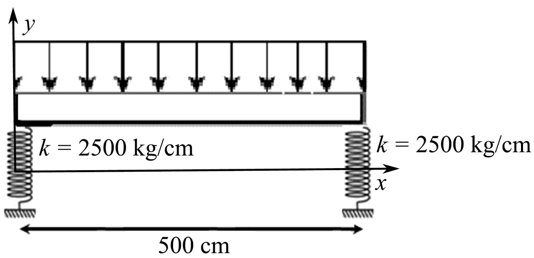

Fig. 2. Euler-Bernoulli beam under with translational restraint supported under uniformly distributed load.

Analytical solution of the displacement, slope, shear force, and bending moment of the Euler-Bernoulli beam with the translational restraint supported under a uniformly distributed load can be determined as

$$Y(x) = \frac{1}{EI}\left(-0.625X^4 + 625X^3 - 78125000x - 1.5EI\right),$$
$$\theta(x) = \frac{1}{EI}\left(-2.5X^3 + 1875X^2 - 78125000\right),$$
$$V(x) = 3750 - 15x, \qquad M(x) = -7.5x^2 + 3750x.$$

Table 2 compares the values of the displacement, slope, bending moment, shear force of the Euler-Bernoulli beam with translational restraint supported under a uniformly distributed load. It can be seen that the results are fairly close. The maximum difference of the obtained results is approximately 0.0004%.

Table 2. The values of the displacement, slope, bending moment, shear force of the Euler-Bernoulli beam under uniformly distributed load – translational restraint supported case.

Location [cm]	Displacement [cm]		Slope [rad]		Shear force [kg]		Bending moment [kg/cm]	
	Analytical solution	*B*-spline function	Analytical solution	*B*-spline function	Analytical solution	*B*-spline function	Analytical solution	*B*-spline function
0	−1.50000	−1.50000	−0.00583	−0.00583	3750.0	3750.0	0.0	0.0
50	−1.78596	−1.78596	−0.00550	−0.00550	3000.0	3000.0	168 750.0	168 750.0
100	−2.04102	−2.04102	−0.00462	−0.00462	2250.0	2250.0	300 000.0	300 000.0
150	−2.24070	−2.24070	−0.00331	−0.00331	1500.0	1500.0	393 750.0	393 750.0
200	−2.36750	−2.36750	−0.00173	−0.00173	750.0	750.0	450 000.0	450 000.0
250	−2.41093	−2.41094	0.0	0.0	0.0	0.0	468 750.0	468 750.0
300	−2.36750	−2.36750	0.00173	0.00173	−750.0	−750.0	450 000.0	450 000.0
350	−2.24070	−2.24070	0.00331	0.00331	−1500.0	−1500.0	393 750.0	393 750.0
400	−2.04102	−2.04102	0.00462	0.00462	−2250.0	−2250.0	300 000.0	300 000.0
450	−1.78596	−1.78596	0.00550	0.00550	−3000.0	−3000.0	168 750.0	168 750.0
500	−1.50000	−1.50000	0.00583	0.00583	−3750.0	−3750.0	0.0	0.0

4.2. Euler-Bernoulli beam under linearly and uniformly distributed load

In order to illustrate the accuracy of the presented method, the Euler-Bernoulli beam with arbitrary boundary conditions under uniformly and linearly distributed load is considered (Fig. 3). Analytical solution of the displace-

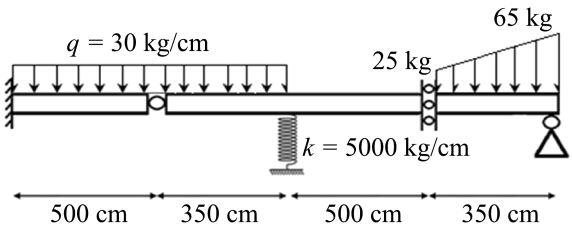

FIG. 3. Euler-Bernoulli beam with arbitrary boundary conditions under uniformly and linearly distributed load.

ment, slope, shear force, and bending moment of the Euler-Bernoulli beam with translational restraint supported under a uniformly distributed load can be determined as

$$y(x)=\frac{1}{EI}\begin{cases} -1.25x^4+\dfrac{80875}{18}x^3-4864583.334x^2 & 0\le x\le 500\text{ cm},\\ -1.25x^4+\dfrac{80875}{18}x^3-4864583.334x^2 \\ \quad +0.30100763EIx-\ 150.503816EI & 500\le x\le 850\text{ cm},\\ 1173958.334x^2+0.03255167EI\cdot x \\ \quad +90.67052EI & 850\le x\le 1350\text{ cm},\\ -\dfrac{1}{1050}x^5+\dfrac{905}{168}x^4-\dfrac{82125}{7}x^3+13215476.19x^2 \\ \quad -0.382741EI\cdot x-246.48704EI & 1350\le x\le 1700\text{ cm},\end{cases}$$

$$\theta(x)=\frac{1}{EI}\begin{cases} -5x^3+\dfrac{80875}{6}x^2-9729166.667x & 0\le x\le 500\text{ cm},\\ -5x^3+\dfrac{80875}{6}x^2-9729166.667x \\ \quad +0.30100763EI & 500\le x\le 850\text{ cm},\\ 2347916.667x+0.03255167EI & 850\le x\le 1350\text{ cm},\\ -\dfrac{1}{210}x^4+\dfrac{905}{42}x^3-\dfrac{492750}{14}x^2 \\ \quad +26430952.381x-0.382741EI & 1350\le x\le 1700\text{ cm},\end{cases}$$

$$M(x)=\begin{cases} -15x^2+\dfrac{80875}{3}x-9729166.667 & 0\le x\le 850\text{ cm},\\ 2347916.667 & 850\le x\le 1350\text{ cm},\\ -\dfrac{2}{105}x^3+\dfrac{905}{14}x^2-\dfrac{492750}{7}x+26430952.381 & 1350\le x\le 1700\text{ cm},\end{cases}$$

$$V(x)=\begin{cases} -30x+\dfrac{80875}{3} & 0\le x\le 850\text{ cm},\\ 0 & 850\le x\le 1350\text{ cm},\\ -\dfrac{2}{35}x^2+\dfrac{905}{7}x-\dfrac{492750}{7} & 1350\le x\le 1700\text{ cm}.\end{cases}$$

Table 3 presents the values of the displacement, slope, bending moment, shear force of the Euler-Bernoulli beam with arbitrary boundary conditions under uniformly and linearly distributed load. It can be seen that the results are fairly close. The maximum difference of the obtained results is approximately 0.00005%.

Table 3. The values of the displacement, slope, bending moment, shear force of the Euler-Bernoulli beam under uniformly and linear distributed load – arbitrary boundary conditions case.

Location [cm]	Displacement [cm]		Angle [rad]		Shear [kg]		Bending moment [kg/cm]	
	Analytical solution	*B*-spline function	Analytical solution	*B*-spline function	Analytical solution	*B*-spline function	Analytical solution	*B*-spline function
0	0.00	5.92E-15	0.00000	−1.4E-13	26958.33	26958.33	−9729166.67	−9729165.94
100	−3.30418	−3.30418	−0.06292	−0.06292	23958.33	23958.34	−7183333.33	−7183332.78
200	−11.98752	−11.98750	−0.10796	−0.10796	20958.33	20958.27	−4937500.00	−4937499.75
300	−24.37406	−24.37410	−0.13735	−0.13735	17958.33	17958.45	−2991666.67	−2991666.39
400	−39.01177	−39.01180	−0.15335	−0.15335	14958.33	14958.55	−1345833.33	−1345833.36
500	−54.67244	−54.67240	−0.15819	−0.15819	11958.33	11958.75	0.00	−1.05
500	−54.67245	−54.67240	0.14282	0.14282	11958.33	11958.54	0.00	0.58
600	−40.25101	−40.25100	0.14691	0.14691	8958.33	8958.15	1045 833.33	1045 833.12
700	−25.06780	−25.06780	0.15768	0.15768	5958.33	5958.15	1791666.67	1791666.79
800	−8.56622	−8.56622	0.17290	0.17290	2958.33	2958.33	2237500.00	2237499.39
850	0.29166	0.29167	0.18148	0.18148	1458.33	1458.29	2347916.67	2347916.15
850	0.29166	0.29167	0.18148	0.18148	0.0	0.01	2347916.67	2347916.12
900	9.586411	9.58473	0.19024	0.19024	0.00	−0.04	2347916.67	2347916.12
1000	29.48662	29.48493	0.20776	0.20776	0.00	0.02	2347916.67	2347916.45
1100	51.13893	51.13725	0.22528	0.22528	0.00	0.55	2347916.67	2347917.36
1200	74.54336	74.54167	0.24280	0.24281	0.00	0.15	2347916.67	2347917.84
1300	99.69989	99.69820	0.26033	0.26033	0.00	0.19	2347916.67	2347915.09
1350	112.93520	112.93350	0.26909	0.26909	0.00	0.78	2347916.67	2347916.78
1350	−103.3721	−103.3720	0.26909	0.26909	0.00	−0.03	2347916.67	2347915.83
1400	−89.69930	−89.69930	0.27781	0.27781	−1392.86	−1392.86	2314285.71	2314285.04
1500	−61.08280	−61.08280	0.29414	0.29414	−5035.71	−5035.70	2002380.95	2002380.48
1600	−30.99825	−30.99830	0.30664	0.30664	−9821.43	−9821.43	1269047.62	1269047.35
1700	0.00	2.06E-16	0.31175	0.31175	−15750.00	−15750.00	0.00	0.00

4.3. Indeterminate beam under uniformly and linearly distributed load

The indeterminate beam under uniformly and linearly distributed load with spring supports is evaluated. The beam characteristics are shown in Fig. 4. Analytical solution of the displacement, slope, shear force, and bending moment

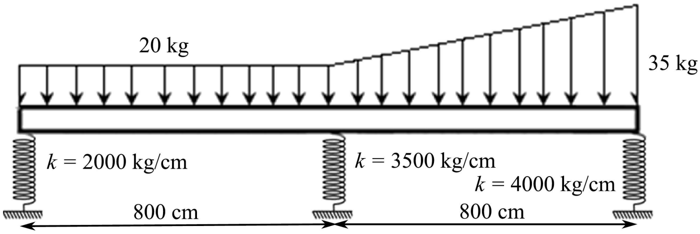

FIG. 4. Indeterminate beam under uniformly and linearly distributed load.

of the indeterminate beam under uniformly and linearly distributed load with springs supports can be determined as

$$Y(x)=\frac{1}{EI}\begin{cases} -\frac{5}{6}x^4+988.77103x^3 \\ \qquad -262393786.2x-2.96631309EI & 0\le x\le 800 \text{ cm}, \\ -\frac{1}{6400}x^5-\frac{5}{24}x^4+3677.895637x^3 \\ \qquad -8053899.056x^2+0.4851101EI \\ \qquad -140.09774885EI & 800\le x\le 1600 \text{ cm}, \end{cases}$$

$$\theta(x)=\frac{1}{EI}\begin{cases} -\frac{10}{3}x^3+2966.31309x^2-262393786.2 & 0\le x\le 800 \text{ cm}, \\ -\frac{1}{1280}x^4-\frac{5}{6}x^3+11033.68691x^2 \\ \qquad -16107798.112x+0.4851101EI & 800\le x\le 1600 \text{ cm}, \end{cases}$$

$$M(x)=\begin{cases} 5932.62618x-10x^2 & 0\le x\le 800 \text{ cm}, \\ -0.003125x^3-2.5x^2+22067.37382x \\ \qquad -16107798.112 & 800\le x\le 1600 \text{ cm}, \end{cases}$$

$$V(x)=\begin{cases} 5932.62618-20x & 0\le x\le 800 \text{ cm}, \\ -0.009375x^2-5x+22067.37382 & 800\le x\le 1600 \text{ cm}. \end{cases}$$

Table 4 presents the values of the displacement, slope, bending moment, shear force of the beam under uniformly and linearly distributed load.

Table 4. The values of the displacement, slope, bending moment, shear force of the indeterminate beam under uniformly and linearly distributed load.

Location [cm]	Displacement [cm]		Angle [rad]		Shear [kg]		Bending moment [kg/cm]	
	Analytical solution	B-spline function	Analytical solution	B-spline function	Analytical solution	B-spline function	Analytical solution	B-spline function
0	−2.96631	−2.96631	−0.0195809	−0.0195809	5932.626	5932.625	0.000	0.005
100	−4.85683	−4.85683	−0.0176160	−0.0176160	3932.626	3932.626	493262.618	493262.620
200	−6.39170	−6.39170	−0.0127165	−0.0127165	1932.626	1932.626	786525.236	786525.232
300	−7.35206	−7.35206	−0.0063748	−0.0063748	−67.374	−67.373	879787.854	879787.849
400	−7.66833	−7.66833	−0.0000834	−0.0000834	−2067.374	−2067.374	773050.472	773050.473
500	−7.42016	−7.42016	0.0046653	0.0046653	−4067.374	−4067.373	466313.090	466313.099
600	−6.83644	−6.83644	0.0063788	0.0063788	−6067.374	−6067.374	−40424.292	−40424.294
700	−6.29533	−6.29533	0.0035645	0.0035645	−8067.374	−8067.371	−747161.674	−747161.671
800	−6.32421	−6.32421	−0.0052700	−0.0052700	−10067.374	−10067.374	−1653899.056	−1653899.059
800	−6.32421	−6.32421	−0.0052700	−0.0052700	12067.374	12067.374	−1653899.056	−1653899.051
900	−7.32456	−7.32456	−0.0133640	−0.0133640	9973.624	9973.627	−550286.674	−550286.677
1000	−8.74916	−8.74916	−0.0140270	−0.0140270	7692.374	7692.373	334575.708	334575.693
1100	−9.93885	−9.93885	−0.0089613	−0.0089613	5223.624	5223.625	981938.090	981938.088
1200	−10.41172	−10.41172	−0.0000092	−0.0000092	2567.374	2567.374	1373050.472	1373050.470
1300	−9.87706	−9.87706	0.0108471	0.0108471	−276.376	−276.376	1489162.854	1489162.854
1400	−8.24939	−8.24939	0.0214856	0.0214856	−3307.626	−3307.626	1311525.236	1311525.235
1500	−5.66245	−5.66245	0.0296441	0.0296441	−6526.376	−6526.376	821387.618	821387.617
1600	−2.48316	−2.48316	0.0329206	0.0329206	−9932.626	−9932.626	0.000	0.000

4.4. Euler-Bernoulli beam on Winkler foundation under uniformly distributed load

Now the Euler-Bernoulli beam on the uniform Winkler foundation under a uniformly distributed load with spring supports is considered. The beam characteristics are shown in Fig. 5. Figure 6 presents the displacement, slope, bending moment, shear force of the beam on the Winkler foundation under uniformly distributed load.

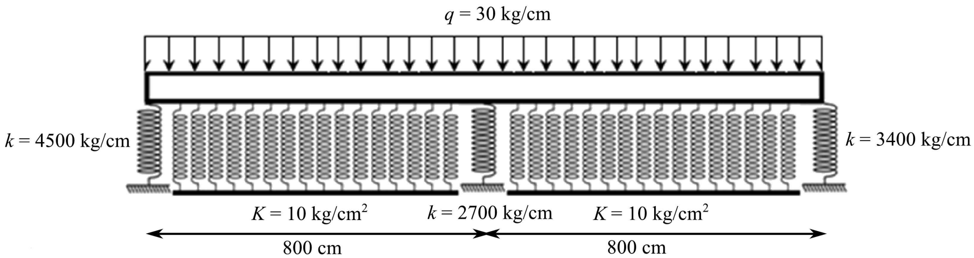

FIG. 5. Euler-Bernoulli beam on Winkler foundation under uniformly distributed load.

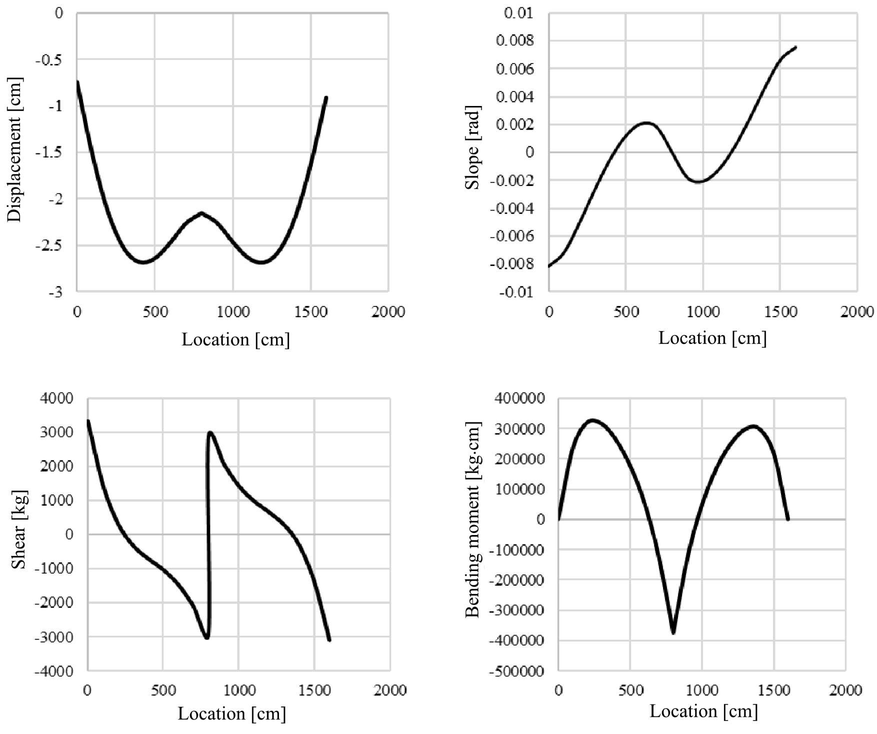

FIG. 6. Displacement, slope, bending moment, shear force of Euler-Bernoulli beam on Winkler foundation under uniformly distributed load.

4.5. Beam with the translational and rotational support on Winkler foundation under uniformly distributed load

The Euler-Bernoulli beam with the translational and rotational support on the uniform Winkler foundation under uniformly distributed load is considered in this section. The beam characteristics are shown in Fig. 7. Table 5 compares the values of the displacement, slope, bending moment, shear force of the Euler-Bernoulli beam using the B-spline collocation method along with the integrated finite element analysis of structures (SAP2000) [14]. It can be seen that the

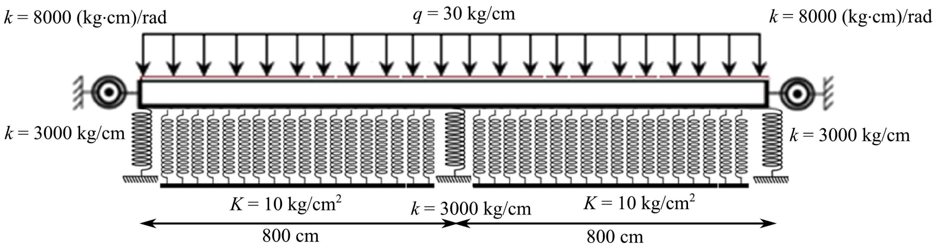

Fig. 7. Euler-Bernoulli beam with the translational and rotational support on Winkler foundation under uniformly distributed load.

Table 5. The values of the displacement, slope, bending moment, shear force of the beam with the translational and rotational support on Winkler foundation under uniformly distributed load.

Location [cm]	Displacement [cm]		Slope [rad]		Shear force [kg]		Bending moment [kg/cm]	
	SAP2000	B-spline function	SAP2000	B-spline function	SAP2000	B-spline function	SAP2000	B-spline function
0	−1.00753	−1.00054	−0.007267	−0.007207	3026.986	3001.625	−57.6438	−57.65583
100	−1.69972	−1.68964	−0.006325	−0.006313	1354.721	1354.339	211985	212091.6
200	−2.23826	−2.22866	−0.004356	−0.004369	330.1139	329.7931	291704.9	291798.2
300	−2.56225	−2.55476	−0.002129	−0.002152	−259.742	−259.963	292492.5	292572.2
400	−2.67034	−2.66535	−0.0000919	−0.0001016	−632.781	−632.901	246944.8	247012.1
500	−2.59651	−2.59373	0.00147	0.00145	−990.25	−990.288	166401.2	166457.8
600	−2.40161	−2.40044	0.00228	0.00227	−1486.36	−1486.35	44201.65	44249.44
700	−2.17809	−2.17786	0.00197	0.00196	−2199.71	−2199.67	−138219	−138178
800	−2.06343	−2.06349	1.7E-17	0.000000086	−3146.82	−3095.23	−401981	−401943
800	−2.06343	−2.06349	1.7E-17	0.000000086	3146.819	3095.25	−401981	−401943
900	−2.17809	−2.17785	−0.001966	−0.001959	2199.706	2199.685	−138219	−138176
1000	−2.40161	−2.4004	−0.002279	−0.002265	1486.363	1486.332	44201.65	44250.98
1100	−2.59651	−2.59367	−0.001469	−0.001448	990.2504	990.2263	166401.2	166455.7
1200	−2.67034	−2.66528	0.000092	0.00012	632.7814	632.7757	246944.8	247000.7
1300	−2.56225	−2.55468	0.00213	0.00215	259.742	259.7593	292492.5	292544.3
1400	−2.23826	−2.22859	0.00436	0.00437	−330.114	−330.075	291704.9	291746
1500	−1.69972	−1.68963	0.00632	0.00631	−1354.72	−1354.67	211 985	212 008.3
1600	−1.00753	−1.00064	0.00727	0.00721	−3026.99	−3001.92	−57.6438	−57.6425

results are close. Table 5 shows that the maximum difference of obtained results is approximately 9.57%.

4.6. *Euler-Bernoulli beam with general boundary conditions partially supported on Winkler foundation under uniformly and linearly distributed load*

In this section, the Euler-Bernoulli beam partially supported on the Winkler foundation under uniformly and linearly distributed load is assumed with general boundary conditions. The beam characteristics are shown in Fig. 8. Figure 9 presents the displacement, slope, bending moment, shear force of the Euler-

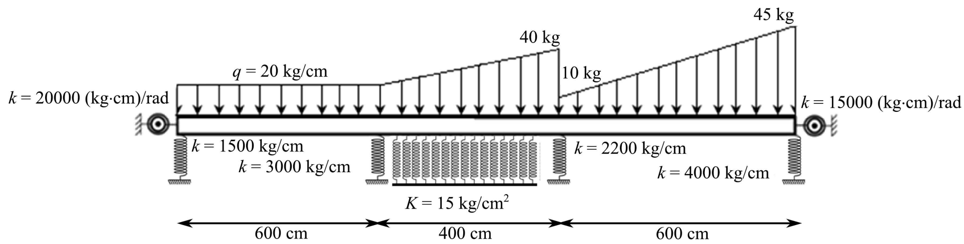

FIG. 8. Euler-Bernoulli beam with general boundary conditions partially supported on Winkler foundation under uniformly and linearly distributed load.

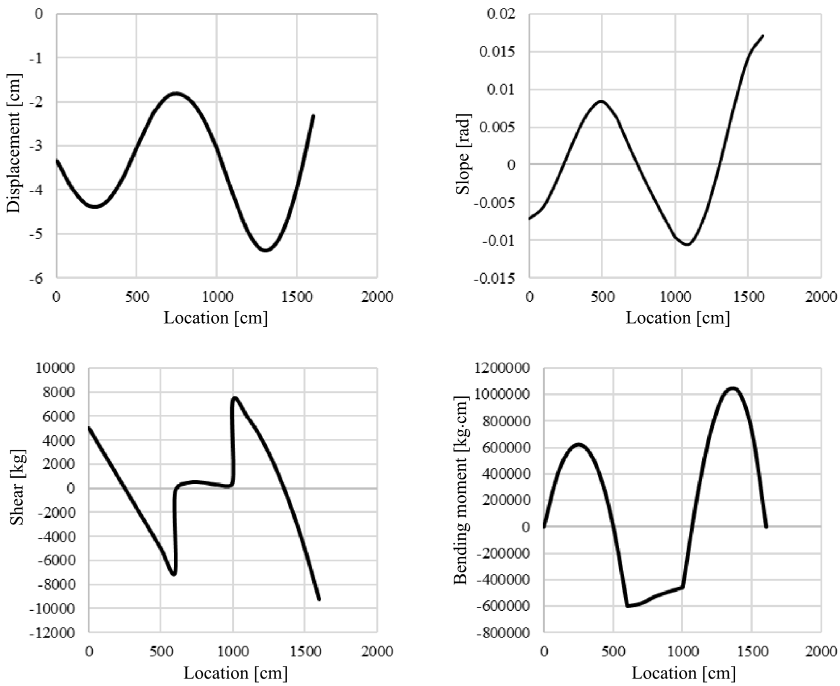

FIG. 9. Displacement, slope, bending moment, shear force of Euler-Bernoulli beam with general boundary conditions partially supported on Winkler foundation under uniformly and linear distributed load.

Bernoulli beam with general boundary conditions partially supported on the Winkler foundation under uniformly and linearly distributed load.

4.7. Euler-Bernoulli beam with arbitrary boundary conditions supported on partial Winkler foundation under uniformly and linearly distributed load

The Euler-Bernoulli beam arbitrary boundary conditions supported on the partial Winkler foundation under uniformly and linearly distributed load is considered. The beam characteristics are shown in Fig. 10. Figure 11 presents the

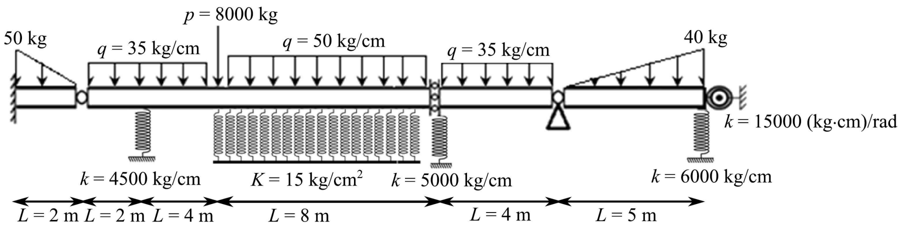

FIG. 10. Euler-Bernoulli beam with general boundary conditions partially supported on Winkler foundation under uniformly, linearly distributed and point loads.

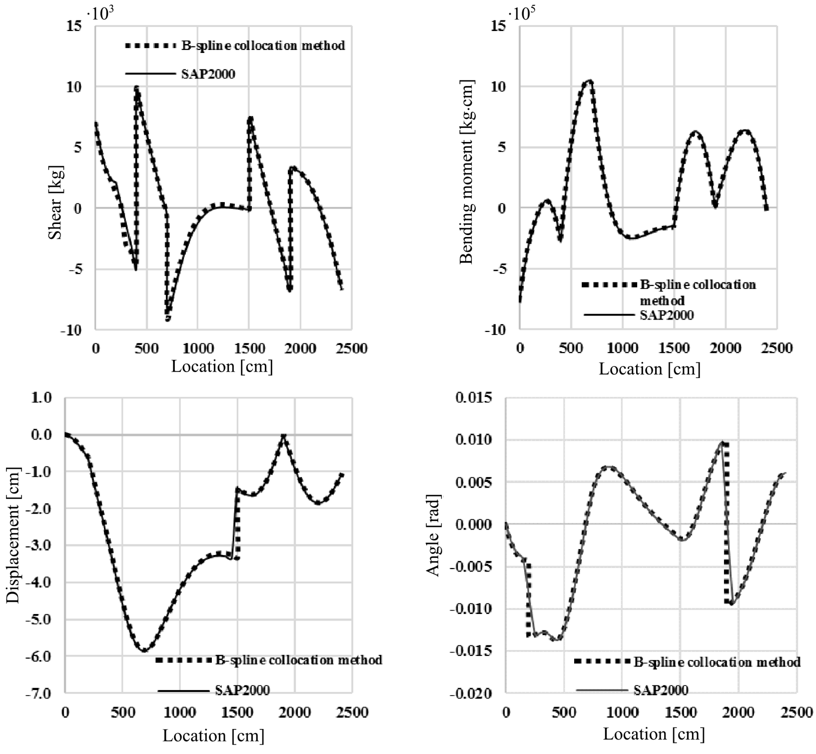

FIG. 11. Displacement, slope, bending moment, shear force of Euler-Bernoulli beam with general boundary conditions partially supported on Winkler foundation under uniformly, linearly distributed and point loads.

values of the displacement, slope, bending moment, shear force of the Euler-Bernoulli beam using the B-spline collocation method along with the integrated finite element analysis of structures (SAP2000) [14]. It can be seen that the results are close. Figure 10 shows the maximum difference of the obtained results of approximately 4.32%.

5. Conclusion

This paper presents the analysis of the Euler-Bernoulli beam with arbitrary boundary conditions partially supported on a Winkler foundation using the sextic B-spline collocation method. A direct modeling technique is introduced for modeling the beam with arbitrary boundary conditions. Thus, the effect of translational along with rotational support flexibilities, the type of beam support, and the elastic coefficient of foundation are assessed. Finally, some numerical examples are shown to present the efficiency of the sextic B-spline collocation method. To validate the analysis of the Euler-Bernoulli beam with the presented method, the results of the B-spline collocation method are compared with the results of the analytical method and the integrated finite element analysis of structures (SAP2000).

Appendix

The coefficient matrix $[A]$, matrix $[C]$ and the load matrix $[F]$ in Eq. (3.17) are given for all boundary condition by:
1) for uniformly distributed load:

$$\mathbf{F} = \begin{bmatrix} q/EI \\ q/EI \\ q/EI \\ \vdots \\ q/EI \\ q/EI \\ qEI \\ 0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_N \\ c_{N+1} \\ c_{N+2} \\ c_{N+3} \\ c_{N+4} \\ c_{N+5} \end{bmatrix},$$

where value for m_i is dependent on boundary conditions

$$A_1 = \frac{360}{h^4} + \frac{k}{EI}, \qquad A_2 = -\frac{1080}{h^4} + \frac{57k}{EI}, \qquad A_3 = \frac{720}{h^4} + \frac{302k}{EI},$$

$$A_4 = \frac{720}{h^5} + \frac{6k}{hEI}, \qquad A_5 = -\frac{3600}{h^5} + \frac{150k}{hEI}, \qquad A_6 = \frac{7200}{h^5} + \frac{240k}{hEI},$$

$$\mathbf{A} = \frac{1}{720}\begin{bmatrix} A_1 & A_2 & A_3 & A_3 & A_2 & A_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_1 & A_2 & A_3 & A_3 & A_2 & A_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_1 & A_2 & A_3 & A_3 & A_2 & A_1 & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & A_1 & A_2 & A_3 & A_3 & A_2 & A_1 \\ A_4 & A_5 & A_6 & -A_6 & -A_5 & -A_4 & 0 & \cdots & 0 & 0 & 0 \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & 0 & \cdots & 0 & 0 & 0 \\ u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & u_{13} & u_{14} & u_{15} & u_{16} & u_{17} & u_{18} \\ 0 & 0 & 0 & \cdots & 0 & u_{19} & u_{20} & u_{21} & u_{21} & u_{22} & u_{23} \end{bmatrix},$$

where u_i and m_i depend on the boundary conditions that are determined by the kind of support and toggle.

2) for linearly distributed load:

$$\mathbf{F} = \begin{bmatrix} \frac{dq(x)}{dx} \cdot \frac{1}{EI} \\ \frac{dq(x)}{dx} \cdot \frac{1}{EI} \\ \frac{dq(x)}{dx} \cdot \frac{1}{EI} \\ \vdots \\ \frac{dq(x)}{dx} \cdot \frac{1}{EI} \\ \frac{dq(x)}{dx} \cdot \frac{1}{EI} \\ \frac{dq(x)}{dx} \cdot \frac{1}{EI} \\ 0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_N \\ c_{N+1} \\ c_{N+2} \\ c_{N+3} \\ c_{N+4} \\ c_{N+5} \end{bmatrix},$$

where value for m_i is dependent on boundary conditions

$$\mathbf{A} = \frac{1}{720}\begin{bmatrix} A_4 & A_5 & A_6 & -A_6 & -A_5 & -A_4 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_4 & A_5 & A_6 & -A_6 & -A_5 & -A_4 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_4 & A_5 & A_6 & -A_6 & -A_5 & -A_4 & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & A_4 & A_5 & A_6 & -A_6 & -A_5 & -A_4 \\ A_1 & A_2 & A_3 & A_3 & A_2 & A_1 & 0 & \cdots & 0 & 0 & 0 \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & 0 & \cdots & 0 & 0 & 0 \\ u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & u_{13} & u_{14} & u_{15} & u_{16} & u_{17} & u_{18} \\ 0 & 0 & 0 & \cdots & 0 & u_{19} & u_{20} & u_{21} & u_{21} & u_{22} & u_{23} \end{bmatrix}.$$

Simple spring at the first and last member of a system

Simple spring at the first and last member:

$$Y \cdot k_1 = V, \qquad M = 0,$$

$$[C_3] = [C_1] =$$
$$\left[k_1 - \frac{120}{h^3}EI \quad 57k_1 - \frac{120}{h^3}EI \quad 302k_1 + \frac{960}{h^3}EI \quad 302k_1 - \frac{960}{h^3}EI \quad 57k_1 + \frac{120}{h^3}EI \quad k_1 + \frac{120}{h^3}EI\right],$$

$$[C_4] = [C_2] = \left[\begin{array}{cccccc} \frac{30}{h^2} & \frac{270}{h^2} & \frac{-300}{h^2} & \frac{-300}{h^2} & \frac{270}{h^2} & \frac{30}{h^2} \end{array}\right],$$

$$m_4 = m_3 = m_2 = m_1 = 0,$$

where k_1 is a stiffness coefficient of a simple spring.

Simple spring in the middle of a system

Simple spring at the first member:

$$Y \cdot k_1 = \left(V_{(\text{right})} - V_{(\text{left})}\right), \qquad M_{(\text{right})} = M_{(\text{left})},$$

$$[C_1] =$$
$$\left[k_1 - \frac{120}{h^3}EI \quad 57k_1 - \frac{120}{h^3}EI \quad 302k_1 + \frac{960}{h^3}EI \quad 302k_1 - \frac{960}{h^3}EI \quad 57k_1 + \frac{120}{h^3}EI \quad k_1 + \frac{120}{h^3}EI\right],$$

$$[C_2] = \left[\begin{array}{cccccc} \frac{30}{h^2} & \frac{270}{h^2} & \frac{-300}{h^2} & \frac{-300}{h^2} & \frac{270}{h^2} & \frac{30}{h^2} \end{array}\right],$$

$$m_1 = -V_{(\text{left spring})}, \qquad m_2 = M_{(\text{left support})}.$$

Simple spring at the last member:

$$Y_{(\text{right spring})} = Y_{(\text{left spring})}, \qquad \theta_{(\text{right spring})} = \theta_{(\text{left spring})},$$

$$[C_3] = \begin{bmatrix} 1 & 57 & 302 & 302 & 57 & 1 \end{bmatrix},$$

$$[C_4] = \begin{bmatrix} \frac{30}{h^2} & \frac{270}{h^2} & \frac{-300}{h^2} & \frac{-300}{h^2} & \frac{270}{h^2} & \frac{30}{h^2} \end{bmatrix},$$

$$m_3 = Y_{(\text{right spring})}, \qquad m_4 = \theta_{(\text{right spring})}.$$

Toggle in the middle of a system

Toggle at the first member:

$$M_{(\text{right toggle})} = 0, \qquad V_{(\text{right toggle})} = V_{(\text{left spring})},$$

$$[C_1] = \begin{bmatrix} \frac{30}{h^2} & \frac{270}{h^2} & \frac{-300}{h^2} & \frac{-300}{h^2} & \frac{270}{h^2} & \frac{30}{h^2} \end{bmatrix},$$

$$[C_2] = \begin{bmatrix} \frac{120}{h^3} & \frac{120}{h^3} & \frac{-960}{h^3} & \frac{960}{h^3} & \frac{-120}{h^3} & \frac{-120}{h^3} \end{bmatrix},$$

$$m_1 = 0, \qquad m_1 = V_{(\text{left spring})}.$$

Toggle at the last member:

$$M_{(\text{leftt toggle})} = 0, \qquad Y_{(\text{right toggle})} = Y_{(\text{left toggle})},$$

$$[C_3] = \begin{bmatrix} 1 & 57 & 302 & 302 & 57 & 1 \end{bmatrix},$$

$$[C_4] = \begin{bmatrix} \frac{30}{h^2} & \frac{270}{h^2} & \frac{-300}{h^2} & \frac{-300}{h^2} & \frac{270}{h^2} & \frac{30}{h^2} \end{bmatrix},$$

$$m_3 = Y_{(\text{right spring})}, \qquad m_4 = 0,$$

where matrices $[C_1]$, $[C_2]$, $[C_3]$ and $[C_4]$ are:

$$[C_1] = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{bmatrix},$$

$$[C_2] = \begin{bmatrix} u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} \end{bmatrix},$$

$$[C_3] = \begin{bmatrix} u_{13} & u_{14} & u_{15} & u_{16} & u_{17} & u_{18} \end{bmatrix},$$

$$[C_4] = \begin{bmatrix} u_{19} & u_{20} & u_{21} & u_{22} & u_{23} & u_{24} \end{bmatrix}.$$

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Received September 23, 2016; February 14, 2017.