ENGINEERING TRANSACTIONS • *Engng. Trans.* • **49**, 1, 3–28, 2001
Polish Academy of Sciences • Institute of Fundamental Technological Research
10.24423/engtrans.571.2001

GENERALIZED SECTION MODEL FOR ANALYSIS OF REINFORCED CONCRETE CHIMNEY WEAKENED BY OPENINGS

M. LECHMAN and P. M. LEWIŃSKI

BUILDING RESEARCH INSTITUTE

ul. Filtrowa 1, 00–611 Warsaw, Poland

In the paper, the general equilibrium equation due to bending moment and normal force is formulated for RC cross-section weakened by an arbitrary number of openings. The governing equations for the normal stresses due to bending moment and normal force are derived for the case when openings are located symmetrically as well as asymmetrically to the bending direction. The normal tensile stresses in concrete are neglected, and the reinforcing steel is continuously spaced at l layers ($l \in N$). The constitutive equations for steel are assumed to be linear elastic, while the concrete is described as an elastic material in compression and brittle in tension. Furthermore, the strains are assumed to be small and their distribution across the section to be linear. The additional reinforcement located in the zone of the flue openings is involved. Basing on the equilibrium equation about the neutral axis, the effective moment of inertia of the cracked annular cross-section with openings is derived, and its influence on the fundamental frequency of the chimney is evaluated. The effects of different parameters on stresses in concrete and steel in the considered annular cross-sections are presented and discussed.

NOTATIONS

R	external radius of ring
r	internal radius of ring
l	number of layers of reinforcement
r_s	radius of equivalent ring of steel reinforcement
r_c	centroidal radius (of concrete)
r_m	mean radius of ring
b_s	effective thickness of reinforcement
b_c	thickness of concrete
μ	the ratio of areas, steel to concrete
μ_{α_i}, μ_{β_i}	the ratios of areas, additional steel located at the angles α_i, β_i to concrete
z	distance between the neutral axis and the location of normal force
e	eccentricity of loading
α	angle determining the location of the neutral axis

α_i, β_i	angles determining the locations of openings
β	variable angle
$F_{a\alpha_i}$, $F_{a\beta_i}$	the cross-sectional areas of additional reinforcement in the surrounding of openings determined by angles α_i, β_i, respectively
E_c	modulus of elasticity of concrete
E_s	modulus of elasticity of steel
ε_c, σ_c	strain, stress in concrete, respectively
ε_c'	maximum compressive strain in concrete
$\varepsilon_s \sigma_s$	strain, stress in steel, respectively
σ_{si}	stress in additional reinforcement in the surrounding of opening determined by the angle α_i
n	ratio of moduli of elasticity of steel to concrete
m, $m1$, $m2$	number of openings
k	interval number
B	coefficient of the maximum compressive stress in concrete
C	coefficient of the maximum tensile stress in steel
N	axial force at section
A_c	concrete area
b	width of the opening
M_0	the moment of the effective cross-section about the natural axis
M	the moment of the effective cross-section about the centroidal axis
I_{0e}, S_{0e}	the effective moment of inertia and the effective static moment of compressive concrete zone and steel area about the neutral axis of the cracked annular cross-section with openings, respectively
ω_1	fundamental frequency
h_0	height of the stack
I_0	cross-sectional moment of inertia at the base
μ_0	mass per unit height
λ	coefficient taking into account the geometrical properties of the stack

1. Introduction

Over the recent years, the determination of stresses in non-circular cross-sections of reinforced concrete (RC) chimneys has been analysed as a theoretical problem as well as a practical one. In order to solve the problem, both the analytical and numerical approaches are applied. Hampe and Frentzel [1] considered a tall RC chimney treated as an axisymmetric elastic shell weakened by one opening of small size. They investigated the effect of the opening shape on the stress concentration in the surroundings of the opening. In later publications regarding this problem, the Bernoulli assumption is commonly introduced for the analysis of RC cross-sections. In the analytical approach, the equilibrium equation for loading eccentricity and the corresponding formulae for stresses in concrete and steel are derived in explicit analytical form. The governing equations for the normal stresses, derived for the case when a concrete chimney cross-section is weakened by four openings located at the same level and symmetric to the wind direction, are given by LEE and JABALI [2]. The general equations for the calculation of loading eccentricity and maximum stresses in concrete and steel reinforcement for the hollow circular section with a maximum of two openings of equal size are derived by WEN-FOO YAU [3] without the use of a thin-wall approximation. The analysis presented by ACI Committee 307 [4] concerned the problem of stresses in reinforced concrete chimneys weakened by a maximum of two openings. Another group of authors present the numerical approach to the analysed problem. Starting from equilibrium equations formulated in a general implicit form, a numerical technique is used to solve the governing equations. The results are given in form of diagrams for particular geometrical characteristics of cross-sections. The calculation of the elastic stresses in the circular cross-section of the shaft with one or two openings has been presented by PINFOLD [5]. The ultimate load analysis of a shell with the circular cross-section weakened by one opening is also included in this monograph. The method of dimensioning for the annular cross-sections of RC chimneys was given by BACHMANN [6] using linear and nonlinear material models and taking into consideration the tensile strength of concrete. Commentary on DIN 1056 [7, 8] includes diagrams for the strength analysis of annular cross-sections weakened by one and two openings located symmetrically to the bending direction. CIESIELSKI and BANAŚ [10] adopted the model given by Bachmann and performed the static analysis of multilayered ring cross-sections with weakenings and strengthenings, considering different linear and nonlinear material models by using of a computer program. The authors analysed chimney cross-sections with 1, 2, 3, 4, 5 openings symmetric to the wind direction. It should be mentioned that in the referred publications [5, 6, 7, 10], representing the numerical approach, neither the unique existence of the solution nor the convergence and the accuracy of the applied numerical technique have been discussed. Despite the generality of the discussed approaches, there are no proper analytical formulae covering the majority of important problems encountered in engineering practice. In particular, the annular cross-sections weakened by arbitrary number of openings located asymmetrically to the bending direction have not been analysed as yet. The assumption of central layout of steel reinforcement in the wall of chimney-like structures, used in the referred publications, may not be justified. Furthermore, the effect of the additional steel bars designed for reinforcing the surroundings of the openings should be examined. In the paper, the governing equations for the normal stresses due to bending moment and normal force are derived for the case when a reinforced concrete chimney cross-section is weakened by one, two, ..., m openings, located asymmetrically to the bending direction, while the reinforcing steel is spaced continuously at l layers and concentrated in the vicinity of m openings. The proposed approach covers the majority of important cases encountered in engineering practice. Some partial results have been presented by the present authors at the international conferences [11, 12].

2. General equilibrium equation

The annular cross-section, described by radii R and r, is assumed to be weakened by m openings of small size in comparison with the circumference of the section, and the Bernoulli assumption is satisfied. The locations of openings are determined by couples of the angular coordinates (α_1, α_2), (α_3, α_4), ..., $(\alpha_{2m-1}, \alpha_{2m})$, $0 \leq \alpha_1 \leq \alpha_2 \leq ... \leq \alpha_{2m-1} \leq \alpha_{2m} \leq 2\pi$, while symbols $F_{a\alpha i}$, $i = 1, ..., 2m$ denote the cross-sectional areas of additional reinforcement in the surroundings of openings (see Figs. 1 and 2). The reinforcing steel is continuously spaced at l layers, the locations of which are determined by radii r_{s1}, r_{s2}, ..., r_{sl}, respectively. The effective thickness b_{si}, $i = 1, ..., l$ is defined as the quotient of the reinforcement area and the circumference of the corresponding radius r_{si} of the layer i. The steel reinforcement spaced at l layers can be replaced by a continuous ring of equal area located on the reference circumference of radius r_s, and of effective thickness b_s. They can be found by comparing the static moments of the given reinforcements and the equivalent one about an arbitrary line perpendicular to the bending direction, according to the formula

$$
\begin{aligned}
r_s &= \sum_{i=1}^{l} b_{si} r_{si}^2 / \sum_{i=1}^{l} b_{si} r_{si}, \\
b_s &= \left(\sum_{i=1}^{l} b_{si} r_{si} \right) / r_s.
\end{aligned}
\tag{2.1}
$$

For the typical case of two layers of reinforcement, the formula (2.1) takes the form:

$$
\begin{aligned}
r_s &= \left(b_{s1} r_{s1}^2 + b_{s2} r_{s2}^2 \right) / \left(b_{s1} r_{s1} + b_{s2} r_{s2} \right) \\
b_s &= \left(b_{s1} r_{s1} + b_{s2} r_{s2} \right) / r_s.
\end{aligned}
\tag{2.2}
$$

In the similar way one defines the thickness of concrete b_c as the ratio of the concrete area A_c and the circumference of the mean radius r_m of the considered ring, $b_c = A_c / (2\pi r_m)$.

Thus, in further considerations the radius r_s of equivalent reinforcement is used. In the present derivation the following assumptions are introduced:

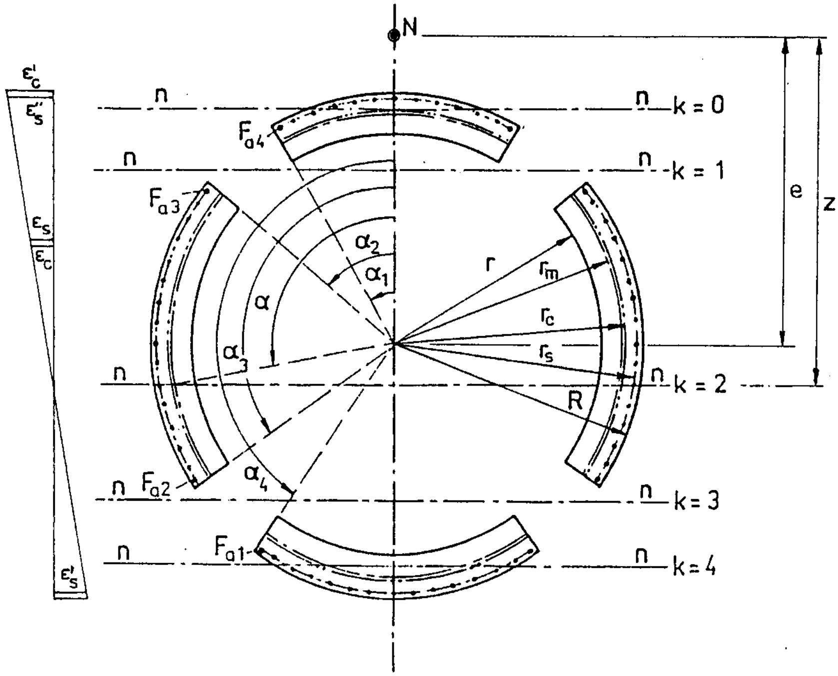

FIG. 1. Cross-section weakened by four openings.

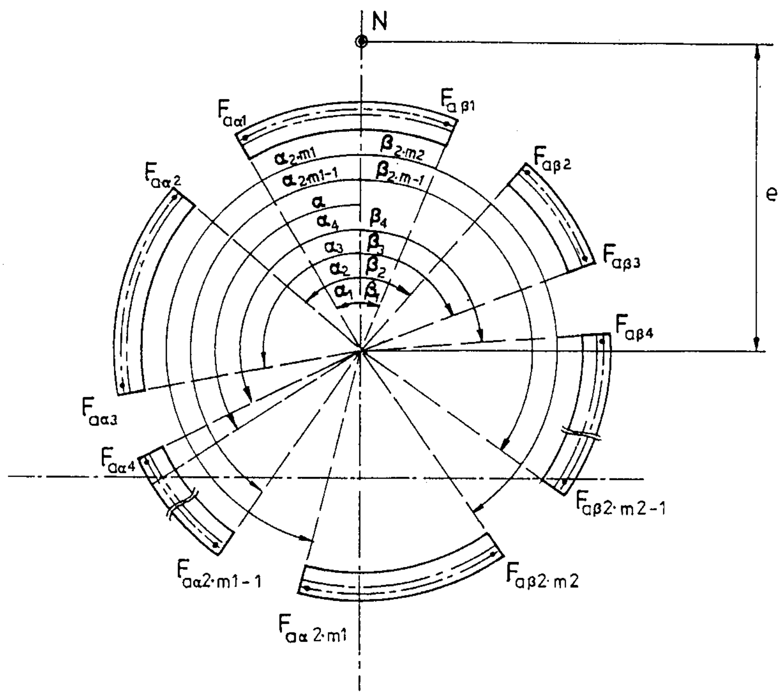

FIG. 2. Cross-section weakened by arbitrary number of openings located asymmetrically.

(i) the distribution of strain across the section is plane,

(ii) the tensile strength of concrete is ignored,

(iii) the reinforcement in both the tension and compression zone is taken into account.

Only elastic stresses are considered. As the compressive stresses in concrete of many RC chimneys in operation do not exceed the value of 50% of the uniaxial compressive strength, this assumption is justified.

The location of the neutral axis, described by an angle α, is considered outside as well as within the openings, basing on the equilibrium equation for loading eccentricity. The eccentricity of the normal force, e, is obtained as a resultant force of the weight of the chimney above the section under consideration and the wind pressure measured from the geometrical center of the chimney cross-section. The equation for loading eccentricity is obtained by considering $2m+1$ cases of the location of the neutral axis. Let us consider the case $\alpha_{2k} < \alpha < \alpha_{2k+1}$ and $\alpha_{2l} < 2\pi - \alpha < \alpha_{2l+1}$, $0 \leq k, l \leq m$. The sectional equilibrium of the bending moments about the line perpendicular to the bending axis and crossing it at the location of the normal force N, can be described in the following form (Figs. 1 and 2):

$$
\begin{aligned}
(2.3)\qquad r_m b_c \Bigg\{ & \sum_{i=0}^{k-1} \int_{\alpha_{2i}}^{\alpha_{2i+1}} \sigma_c (e - r_c \cos\beta)(1-\mu)d\beta + \int_{\alpha_{2k}}^{\alpha} \sigma_c (e - r_c \cos\beta)(1-\mu)d\beta \\
& + \int_{2\pi-\alpha}^{\alpha_{2l+1}} \sigma_c (e - r_c \cos\beta)(1-\mu)d\beta + \sum_{i=l+1}^{m} \int_{\alpha_{2i}}^{\alpha_{2i+1}} \sigma_c (e - r_c \cos\beta)(1-\mu)d\beta \\
& + \sum_{i=0}^{m} \int_{\alpha_{2i}}^{\alpha_{2i+1}} \sigma_s (e - r_s \cos\beta)\mu d\beta + \sum_{i=1}^{2m} \sigma_{si}(e - r_s \cos\alpha_i)\mu_{\alpha i} \Bigg\} = 0,
\end{aligned}
$$

where

$$
(2.4)\qquad
\begin{gathered}
\alpha_0 = 0, \qquad \alpha_{2m+1} = 2\pi, \\
\mu = \frac{2\pi r_s b_s}{2\pi r_m b_c}, \quad \mu_{\alpha i} = \frac{F_{a\alpha i}}{r_m b_c}, \quad i = 1, 2, ..., 2m, \quad r_c = \frac{2}{3}\frac{R^3 - r^3}{R^2 - r^2}
\end{gathered}
$$

and where in turn μ, $\mu_{\alpha i}$ – the ratios of areas, steel and additional steel to concrete, respectively, r_c – centroidal radius, r_m – mean radius of the ring.

The elastic stress-strain relationships for both steel and concrete in compression are assumed as:

$$
(2.5)\qquad \sigma_c = E_c \varepsilon_c, \quad \sigma_s = E_s \varepsilon_s, \quad \sigma_{si} = \sigma_s(\alpha_i).
$$

Due to the Bernoulli assumption we obtain:

$$\varepsilon_c = \varepsilon_c' \frac{r_c \cos\beta + z - e}{R + z - e}, \qquad \varepsilon_s = \varepsilon_c' \frac{r_s \cos\beta + z - e}{R + z - e}, \tag{2.6}$$

where $z - e = -r_c \cos\alpha$.

3. Derivation of equations for the section with one, two, three, four openings

In the present derivation the annular cross-section of a RC chimney weakened by four openings is considered. This case covers the majority of important cases encountered in engineering practice. The openings are symmetric to the wind direction and located at the same level. The steel reinforcement is replaced by a continuous ring of equivalent area located on the reference circumference of radius r_s (see Eq. (2.1)). The location of the neutral axis, described by an angle α, is considered outside as well as within the openings, basing on the equation for loading eccentricity.

The locations of openings are determined by angles α_i, $i = 1, 2, 3, 4$, while $F_{\alpha i}$, $i = 1, ..., 4$ denote the cross-sectional areas of additional reinforcement in neighbourhood of openings (see Fig. 1). The equation for loading eccentricity is derived by considering $5 (k = 0, 1, 2, 3, 4)$ cases of the location of the neutral axis. Let us consider the case $\alpha_2 < \alpha < \alpha_3 (k = 2)$. The sectional equilibrium of the bending moments about the line perpendicular to the symmetry axis and crossing it at the location of the normal force N, can be described in the following form:

$$\begin{aligned} 2r_m b_c \Bigg\{ & \int_0^{\alpha_1} \sigma_c(e - r_c \cos\beta)(1-\mu)d\beta + \int_0^{\alpha_1} \sigma_s(e - r_s\cos\beta)\mu d\beta \\ & + \int_{\alpha_2}^{\alpha} \sigma_c(e - r_c\cos\beta)(1-\mu)d\beta + \int_{\alpha_2}^{\alpha_3} \sigma_s(e - r_s\cos\beta)\mu d\beta \\ & + \int_{\alpha_4}^{\pi} \sigma_s(e - r_s\cos\beta)\mu d\beta + \sum_{i=1}^{4} \sigma_{si}(e - r_s \cos\alpha_i)\mu_{\alpha i} \Bigg\} = 0. \end{aligned} \tag{3.1}$$

Substituting Eqs. (2.4), (2.5) and (2.6) into (3.1) and denoting $n = E_s/E_c$, $\rho = r_s/r_c$, after some rearrangements we obtain:

$$\frac{e}{r_c} = 0.5 \frac{(1-\mu)X_c(\alpha) + n\mu X_s(\alpha)}{(1-\mu)Y_c(\alpha) + n\mu Y_c(\alpha)}, \tag{3.2}$$

where the functions of α are:

$$
\begin{aligned}
X_c(\alpha) &= -2\cos\alpha(\sin\alpha_1 - \sin\alpha_2 + \sin\alpha) \\
&\qquad\qquad +0.5(\sin 2\alpha_1 - \sin 2\alpha_2 + \sin 2\alpha) + \alpha_1 - \alpha_2 + \alpha, \\
X_s(\alpha) &= -2\rho\cos\alpha(\sin\alpha_1 - \sin\alpha_2 + \sin\alpha_3 - \sin\alpha_4) \\
&+\rho^2[\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \pi) + 0.5(\sin 2\alpha_1 - \sin 2\alpha_2 + \sin 2\alpha_3 - \sin 2\alpha_4)] \\
&\qquad\qquad + \frac{\rho}{\mu}\sum_{i=1}^{4}\mu_{\alpha i}\cos\alpha_i(\rho\cos\alpha_i - \cos\alpha), \\
Y_c(\alpha) &= \sin\alpha_1 - \sin\alpha_2 + \sin\alpha - \cos\alpha(\alpha_1 - \alpha_2 + \alpha), \\
Y_s(\alpha) &= \rho(\sin\alpha_1 - \sin\alpha_2 + \sin\alpha_3 - \sin\alpha_4) \\
&\qquad - \cos\alpha(\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \pi) + \frac{1}{\mu}\sum_{i=1}^{4}\mu_{\alpha i}(\rho\cos\alpha_i - \cos\alpha).
\end{aligned}
\tag{3.3}
$$

In a general case of the arbitrary location of the neutral axis ($k = 0, 1, 2, 3, 4$, Fig. 1), the functions X_c, X_s, Y_c, Y_s in Eq. (3.2) are expressed as follows:

$$
\begin{aligned}
X_c^k(\alpha) &= -2\cos\alpha\left(\sum_{i=1}^{k}(-1)^{i-1}\sin\alpha_i + \delta_k\sin\alpha\right) \\
&\qquad +0.5\left(\sum_{i=1}^{k}(-1)^{i-1}\sin 2\alpha_i + \delta_k\sin 2\alpha\right) + \sum_{i=1}^{k}(-1)^{i-1}\alpha_i + \delta_k\alpha, \\
X_s^k(\alpha) &= -2\rho\cos\alpha\sum_{i=1}^{4}(-1)^{i-1}\sin\alpha_i + \rho^2\left[\sum_{i=1}^{4}(-1)^{i-1}\alpha_i + \pi\right. \\
&\qquad \left. +0.5\sum_{i=1}^{4}(-1)^{i-1}\sin 2\alpha_i\right] + \frac{\rho}{\mu}\sum_{i=1}^{4}\mu_{\alpha i}\cos\alpha_i(\rho\cos\alpha_i - \cos\alpha), \\
Y_c^k(\alpha) &= \sum_{i=1}^{k}(-1)^{i-1}\sin\alpha_i + \delta_k\sin\alpha - \cos\alpha\left(\sum_{i=1}^{k}(-1)^{i-1}\alpha_i + \delta_k\alpha\right), \\
Y_s^k(\alpha) &= \rho\sum_{i=1}^{4}(-1)^{i-1}\sin\alpha_i - \cos\alpha\left(\sum_{i=1}^{4}(-1)^{i-1}\alpha_i + \pi\right) \\
&\qquad + \frac{1}{\mu}\sum_{i=1}^{4}(-1)^{i-1}\mu_{\alpha_i}(\rho\cos\alpha_i - \cos\alpha), \qquad k = 0, 1, 2, 3, 4,
\end{aligned}
\tag{3.4}
$$

where:

$$\delta_k = \left((-1)^k + 1\right)/2, \quad \sum^{k=0}(\ ) = 0.$$

Considering the case $k = 2$ we can obtain Eq. (3.3) from Eq. (3.4). The maximum compressive stress in concrete can be expressed in the following form:

$$\sigma_c' = B\sigma_0, \tag{3.5}$$

where

$$\sigma_0 = N/A_c \tag{3.6}$$

is the compressive stress in concrete due to the action of the axial force N, A_c – concrete area, B – coefficient of the maximum compressive stress in concrete.

Let us consider the sectional equilibrium of the normal forces for the case $k = 2$. The equilibrium equation can be described as follows:

$$2r_m b_c \left\{ \int_0^{\alpha_1} \sigma_c(1-\mu)d\beta + \int_0^{\alpha_1} \sigma_s\mu d\beta + \int_{\alpha_2}^{\alpha} \sigma_c(1-\mu)d\beta + \int_{\alpha_2}^{\alpha_3} \sigma_s\mu d\beta + \int_{\alpha_4}^{\pi} \sigma_s\mu d\beta + \sum_{i=1}^{4} \sigma_{si}\mu_{\alpha_i} \right\} - N = 0. \tag{3.7}$$

Substituting here Eqs. (2.5) and (2.6) we obtain the equilibrium equations in the following form:

$$(1-\mu)r_c Y_c(\alpha) + n\mu r_c Y_s(\alpha) = \frac{R - r_c\cos\alpha}{2r_m b_c E_c \varepsilon_c'} N, \tag{3.8}$$

where Y_c and Y_s are described by Eqs. (3.3).

Denoting

$$A_c = 2\pi r_m b_c \quad \text{and} \quad \rho_R = R/r_c,$$

we get

$$B = \frac{\pi(\rho_R - \cos\alpha)}{(1-\mu)Y_c(\alpha) + n\mu Y_s(\alpha)}. \tag{3.9}$$

For arbitrary location of the neutral axis, the coefficient B takes the same form as above, however, the factors Y_c and Y_s are described by Eqs. (3.4). The maximum tensile stress in steel is computed from the constitutive and geometrical relations, (2.5) and (2.6):

$$\sigma_s' = C\sigma_c', \tag{3.10}$$

where C – coefficient of the maximum tensile stress in steel:

$$C = -n\frac{r_s + r_c \cos\alpha}{R - r_c \cos\alpha}, \tag{3.11}$$

or

$$C = -n\frac{\rho + \cos\alpha}{\rho_R - \cos\alpha}. \tag{3.12}$$

4. Generalization of the the obtained formulae for the section with m openings

Let us consider the annular cross-section weakened by m openings situated symmetrically with respect to the bending direction. By this assumption, the locations of the openings are determined by couples of the angular coordinates (α_1, α_2), (α_3, α_4) , ... , (α_{m-1}, α_m), $0 \le \alpha_1 \le \alpha_2 \le ... \le \alpha_{m-1} \le \alpha_m \le \pi$. The equation for loading eccentricity can be derived for this case in the form (3.2) using the principle of the mathematical induction, where the functions X_c^k, X_s^k, Y_c^k, Y_s^k are expressed as follows:

$$\begin{aligned}
X_c^k(\alpha) &= -2\cos\alpha\left(\sum_{i=1}^{k}(-1)^{i-1}\sin\alpha_i + \delta_k \sin\alpha\right) \\
&\quad + 0.5\left(\sum_{i=1}^{k}(-1)^{i-1}\sin 2\alpha_i + \delta_k \sin 2\alpha\right) + \sum_{i=1}^{k}(-1)^{i-1}\alpha_i + \delta_k\alpha, \\
X_s^k(\alpha) &= -2\rho\cos\alpha\sum_{i=1}^{m}(-1)^{i-1}\sin\alpha_i \\
&\quad + \rho^2\left[\sum_{i=1}^{m}(-1)^{i-1}\alpha_i + \pi + 0.5\sum_{i=1}^{m}(-1)^{i-1}\sin 2\alpha_i\right] \\
&\quad + \frac{\rho}{\mu}\sum_{i=1}^{m}\mu_{\alpha i}\cos\alpha_i(\rho\cos\alpha_i - \cos\alpha), \\
Y_c^k(\alpha) &= \sum_{i=1}^{k}(-1)^{i-1}\sin\alpha_i + \delta_k\sin\alpha - \cos\alpha\left(\sum_{i=1}^{k}(-1)^{i-1}\alpha_i + \delta_k\alpha\right), \\
Y_s^k(\alpha) &= \rho\sum_{i=1}^{m}(-1)^{i-1}\sin\alpha_i - \cos\alpha\left(\sum_{i=1}^{m}(-1)^{i-1}\alpha_i + \pi\right) \\
&\quad + \frac{1}{\mu}\sum_{i=1}^{m}(-1)^{i-1}\mu_{\alpha_i}(\rho\cos\alpha_i - \cos\alpha), \qquad k = 0, 1, ..., m,
\end{aligned} \tag{4.1}$$

where:

$$\delta_k = \left((-1)^k + 1\right)/2, \quad \sum^{k=0} (\,) = 0.$$

The maximum concrete stress coefficient B takes the same form as (3.9), however, the factors Y_c^k and Y_s^k are described by Eqs. (4.1). The coefficient of the maximum tensile stress in steel C is computed from Eqs. (3.11) or (3.12).

5. The case of asymmetric location of openings

Let us consider the circular cross-section weakened by m openings situated asymmetrically to the bending direction (Fig. 2).

It is assumed that the circular cross-section is subdivided by the bending axis on two halves containing $m1$ and $m2$ openings, respectively ($m1+m2 = m$). Thus, the locations of the openings in the first half are determined by couples of the angular coordinates (α_1, α_2), (α_3, α_4), ... , $(\alpha_{2\cdot m1-1}, \alpha_{2\cdot m1})$, $0 \le \alpha_1 \le \alpha_2 \le \ldots \le \alpha_{2\cdot m1-1} \le \alpha_{2\cdot m1} \le \pi$, while the locations of the openings in the second half are described by (β_1, β_2), (β_3, β_4),..., $(\beta_{2\cdot m2-1}, \beta_{2\cdot m2})$, $0 \le \beta_1 \le \beta_2 \le \ldots \le \beta_{2\cdot m2-1} \le \beta_{2\cdot m2} \le \pi$. The symbols $F_{a\alpha i}$, $i = 1, \ldots, 2\cdot m1$ denote the cross-sectional areas of additional reinforcement in the surrounding of openings in the first half of the section, and $F_{a\beta i}$, $i = 1, \ldots, 2\cdot m2$ in the second one, respectively. The equation for loading eccentricity can be derived for the analysed case in the similiar form as (3.2) by superposition of the solutions obtained for the symmetrical problem. Ordering the values α_1, α_2, α_3, α_4, ... , $\alpha_{2\cdot m1-1}$, $\alpha_{2\cdot m1}$, β_1, β_2, β_3, β_4, ..., $\beta_{2\cdot m2-1}$, $\beta_{2\cdot m2}$ and numbering them by $j_1, j_2, \ldots, j_{2\cdot m1}$, $i_1, i_2, \ldots, i_{2\cdot m2}$, Eq. (3.2) can be written in the form:

$$\text{(5.1)} \qquad \frac{e}{r_c} = 0.5\frac{(1-\mu)\left\{X_{c1}^j(\alpha) + X_{c2}^j(\alpha)\right\} + n\mu\left\{X_{s1}^j(\alpha) + X_{s2}^j(\alpha)\right\}}{(1-\mu)\left\{Y_{c1}^j(\alpha) + Y_{c2}^j(\alpha)\right\} + n\mu\left\{Y_{s1}^j(\alpha) + Y_{s2}^j(\alpha)\right\}},$$

$$j = 0, 1, 2, \ldots, 2(m1 + m2),$$

where the functions $\{X_{c1}^j, X_{s1}^j, Y_{c1}^j, Y_{s1}^j\}$ and $\{X_{c2}^j, X_{s2}^j, Y_{c2}^j, Y_{s2}^j\}$ refer to the corresponding section halves and are expressed in the similar manner as in Eqs. (4.1):

$$\text{(5.2)} \qquad X_{c1}^j(\alpha) = -2\cos\alpha\left(\sum_{i=1}^{k}(-1)^{i-1}\sin\alpha_i + \delta_k\sin\alpha\right)$$
$$+0.5\left(\sum_{i=1}^{k}(-1)^{i-1}\sin 2\alpha_i + \delta_k\sin 2\alpha\right) + \sum_{i=1}^{k}(-1)^{i-1}\alpha_i + \delta_k\alpha,$$

$$j_k \le j \le j_{k+1}; \quad j_0 = 0,$$

(5.2) [cont.]

$$X_{s1}^{j}(\alpha) = -2\rho\cos\alpha\sum_{i=1}^{2\cdot m1}(-1)^{i-1}\sin\alpha_i + \rho^2\left[\sum_{i=1}^{2\cdot m1}(-1)^{i-1}\alpha_i + \pi\right.$$

$$\left. + 0.5\sum_{i=1}^{2\cdot m1}(-1)^{i-1}\sin 2\alpha_i\right] + \frac{\rho}{\mu}\sum_{i=1}^{2\cdot m1}\mu_{\alpha i}\cos\alpha_i(\rho\cos\alpha_i - \cos\alpha),$$

$$Y_{c1}^{j}(\alpha) = \sum_{i=1}^{k}(-1)^{i-1}\sin\alpha_i + \delta_k\sin\alpha - \cos\alpha\left(\sum_{i=1}^{k}(-1)^{i-1}\alpha_i + \delta_k\alpha\right),$$

$$j_k \le j \le j_{k+1},$$

$$Y_{s1}^{j}(\alpha) = \rho\sum_{i=1}^{2\cdot m1}(-1)^{i-1}\sin\alpha_i - \cos\alpha\left(\sum_{i=1}^{2\cdot m1}(-1)^{i-1}\alpha_i + \pi\right)$$

$$+ \frac{1}{\mu}\sum_{i=1}^{2\cdot m1}(-1)^{i-1}\mu_{\alpha i}(\rho\cos\alpha_i - \cos\alpha), \qquad k = 0, 1, ..., 2\cdot m1$$

$$\delta_k = \left((-1)^k + 1\right)/2, \qquad \sum^{k=0}(\ ) = 0,$$

$$X_{c2}^{j}(\alpha) = -2\cos\alpha\left(\sum_{i=1}^{l}(-1)^{i-1}\sin\beta_i + \delta_l\sin\alpha\right)$$

$$+ 0.5\left(\sum_{i=1}^{l}(-1)^{i-1}\sin 2\beta_i + \delta_l\sin 2\alpha\right) + \sum_{i=1}^{l}(-1)^{i-1}\beta_i + \delta_l\alpha,$$

$$i_l \le j \le i_{l+1}; \quad i_0 = 0,$$

$$X_{s2}^{j}(\alpha) = -2\rho\cos\alpha\sum_{i=1}^{2\cdot m2}(-1)^{i-1}\sin\beta_i + \rho^2\left[\sum_{i=1}^{2\cdot m2}(-1)^{i-1}\beta_i + \pi\right.$$

$$\left. + 0.5\sum_{i=1}^{2\cdot m2}(-1)^{i-1}\sin 2\beta_i\right] + \frac{\rho}{\mu}\sum_{i=1}^{2\cdot m2}\mu_{\beta_i}\cos\beta_i(\rho\cos\beta_i - \cos\alpha),$$

$$Y_{c2}^{j}(\alpha) = \sum_{i=1}^{l}(-1)^{i-1}\sin\beta_i + \delta_l\sin\alpha - \cos\alpha\left(\sum_{i=1}^{l}(-1)^{i-1}\beta_i + \delta_l\alpha\right),$$

$$i_l \le j \le i_{l+1},$$

(5.2) [cont.]

$$Y^j_{s2}(\alpha) = \rho \sum_{i=1}^{2\cdot m2} (-1)^{i-1} \sin\beta_i - \cos\alpha \left(\sum_{i=1}^{2\cdot m2} (-1)^{i-1}\beta_i + \pi \right)$$

$$+\frac{1}{\mu} \sum_{i=1}^{2\cdot m2} (-1)^{i-1} \mu_{\beta i}(\rho\cos\beta_i - \cos\alpha), \qquad l = 0, 1, ..., 2\cdot m2,$$

$$\delta_l = \left((-1)^l + 1\right)/2, \quad \sum^{l=0}(\ ) = 0, \quad j = 0, 1, ..., 2\cdot(m1 + m2).$$

For the symmetrical case $X^j_{c1} = X^j_{c2}$, $X^j_{s1} = X^j_{s2}$, $Y^j_{c1} = Y^j_{c2}$, $Y^j_{s1} = Y^j_{s2}$ and the formulae (5.1), (5.2) take the same form as (3.2), (4.1). The maximum concrete stress coefficient B is then calculated from the expression

$$B = \frac{2\pi(\rho_R - \cos\alpha)}{(1-\mu)\{Y^j_{c1}(\alpha) + Y^j_{c2}(\alpha)\} + n\mu\{Y^j_{s1}(\alpha) + Y^j_{s2}(\alpha)\}}, \tag{5.3}$$

where the factors Y^j_{c1}, Y^j_{c2}, Y^j_{s1}, Y^j_{s2} are described by the formulae (5.2). The coefficient of the maximum tensile stress in steel C is computed from Eqs. (3.11) or (3.12).

The solution algorithm consists of three steps:

• calculation of roots (α) of the equation for loading eccentricity (3.2), (4.1) or (5.1), (5.2),

• determination of the maximum compressive stress in concrete basing on the formulae (3.5) (3.6), (3.9) or (5.3),

• determination of the maximum tensile stress in steel basing on the formulae (3.10), (3.11) or (3.12).

6. Dynamic analysis of the model

Dynamic analysis of the model requires the determination of the dominant (smallest) eigenvalue of the problem. In this approach, the evaluation of the effective cross-sectional characteristics of RC ring is necessary. In the problem under consideration, the following constitutive model for determination of stiffness of RC chimney is proposed:

(i) in the state before cracking, the effective cross-sectional characteristics of RC ring is adopted,

(ii) after cracking, the tensile strength of the concrete is neglected so that only the compressive zone of concrete in the effective cross-sectional characteristics is considered.

The case (i) leads to a purely geometrical task. For the case (ii) the general theory of reinforced concrete is used in order to determine the stiffness after cracking. Due to Eurocode 2, the cross-sectional stiffness after cracking should be calculated about the neutral axis of the effective cross-section. This moment, denoted by M_0, can be obtained from the equation similar to Eq. (2.3) by substituting the expression $(e - r_c \cos\beta)$ by $(r_c \cos\beta + z - e)$ and $(e - r_s \cos\beta)$ by $(r_s \cos\beta + z - e)$. Similarly, the moment about the centroidal axis of the effective cross-section, denoted by M, also can be calculated from the equation similar to Eq. (2.3) by replacing $(e - r_c \cos\beta)$ by $(r_c \cos\beta)$ and $(e - r_s \cos\beta)$ by $(r_s \cos\beta)$, and using Eqs. (2.5) and (2.6). Comparing the obtained equations for M_0, M and N (which is described by Eq. (3.7)), we can get the obvious equilibrium equation:

$$M_0 = M + N(z - e), \tag{6.1}$$

where

$$\begin{aligned} M_0 &= \sigma_c' \cdot I_{0e}/x, \\ N &= \sigma_c' \cdot S_{0e}/x, \end{aligned} \tag{6.2}$$

where in turn the depth of compressive zone is described by $x = R + z - e = R - r_c \cos\alpha$, as the function of location of the neutral axis (described by an angle α), I_{0e} – effective moment of inertia of compressive concrete zone and steel area about the neutral axis of the cracked annular cross-section with openings, S_{0e} – effective static moment of compressive concrete zone and steel area about the neutral axis of the cracked cross-section. The searched value of I_{0e} can be derived directly, but it is possible to evaluate it easily using the known coefficient B. Substituting Eqs. (6.2) to (6.1) we can get the equivalent equilibrium equation:

$$I_{0e} = S_{0e}(e - r_c \cos\alpha), \tag{6.3}$$

which enables us to establish the location of the neutral axis, described by an angle α, as by Eq. (3.2). The effective static moment can be derived by comparing Eqs. (6.2) and (3.5):

$$S_{0e} = A_c \cdot x/B. \tag{6.4}$$

Substituting Eqs. (6.4) to (6.3) we get:

$$I_{0e} = A_c \cdot (R - r_c \cos\alpha) \cdot (e - r_c \cos\alpha)/B. \tag{6.5}$$

For the determination of the eigenvalue of the problem it is possible to evaluate the fundamental (natural) frequency by using the subsequent formula:

$$\omega_1 = \frac{\lambda}{h_0^2}\sqrt{\frac{E_c I_0}{\mu_0}}, \tag{6.6}$$

where h_0 – height of the stack, I_0 – cross-sectional moment of inertia at the base, μ_0 – mass per unit height and λ – coefficient taking into account the geometrical properties of the stack. In case of the cylindrical structure of the chimney $\lambda = 3.5151$ [14]. In particular, it applies to chimney structures in the form of truncated conical shells. Exact analysis allows for point-to-point changes in stiffness and mass of the structure, but in present consideration it is possible to deduce some qualitative conclusions. Taking the effect of openings and cracking of concrete into account, we can observe the decrease of the fundamental frequency (increase of the natural period of vibrations). More general eigenvalue calculations could also be performed. The forced vibrations can be analysed by explicit integration of the equations of motion. Such kind of integration for structural system can be simply treated for such cases where a lumped mass is used. In the equations of motion the effective moment of inertia I_{0e} should be used for the generation of the stiffness matrix, dependent on the state of deformation, what means that the problem is nonlinear. An explicit consistent algorithm can be given using the central difference method. If the internal force vector is computed using the new displacement and velocity, the new acceleration can be computed for a lumped matrix by merely dividing the force by the approximate mass term. The fundamental frequency of the stack with lumped masses can be determined by using the Kayser-Trochy method, where the solution depends directly on the displacements along the chimney height, which should be calculated according to the stiffness of the shaft at particular cross-sections.

7. Numerical examples

The numerical iterative technique is applied for the solution of the equations for loading eccentricity. The presented approach enables the evaluation of stresses in the considered cross-section of the R/C chimneys by the interactive analysis. For presentation of the proposed Eqs. (3.2), (3.4), (3.9), (3.11) or (3.12), (5.2), (5.3), three particular designs with one, two and three openings are chosen. Figure 3 shows the curves plotted according to the equation for loading eccentricity (3.2), (3.4) for a section weakened by one opening. Values of the coefficients B and C for this section are plotted directly as a function of the external loading and the width of the opening (i.e. B, C versus e/R and b/R, Figs. 4, 5). Similarly, the values of α, B, C for the section with two openings are shown in Figs. 6, 7, 8, respectively. As the next example, the cross-section weakened by three openings is considered. Values of α, B, C are expressed as a function of the external eccentricity (Figs. 9, 10, 11). It is apparent that the obtained curves are discontinuous because of the skip of the neutral axis from the location

$\alpha \in (\alpha_4, \pi)$ to the location $\alpha' \in (\alpha_3, \alpha_4)$. As it is shown in Figs. 10 and 11, the additional reinforcement in the surrounding of openings involved in the above mentioned equations results in lower stresses in concrete and steel. The obtained results were compared with those calculated for the circular cross-section (dashed line). The presented examples indicate that the dependence of the coefficient B on the arguments e/R and b/R is approximately linear (Figs. 4, 7, 10) while the changes of the coefficient C and the angle α are nonlinear (Figs. 3, 5, 6, 8).

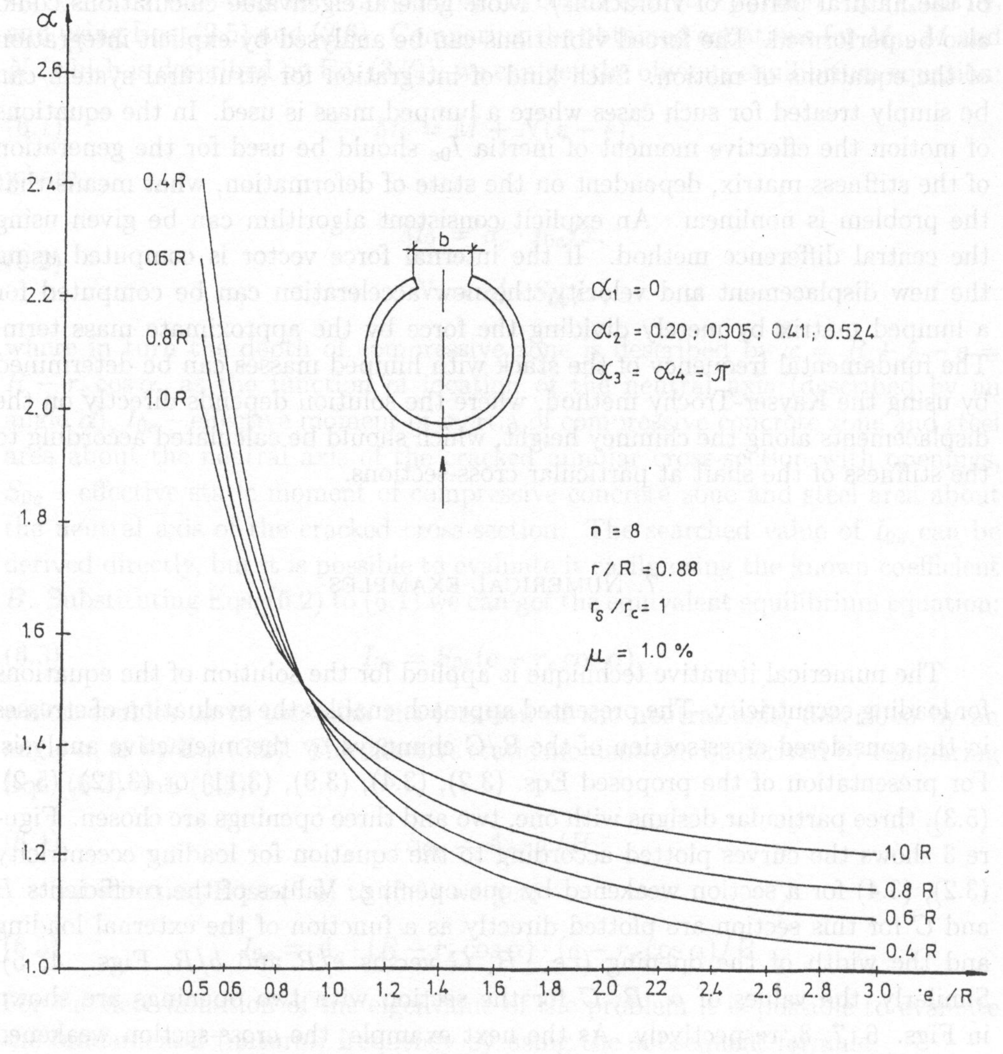

FIG. 3. Values of α for a section weakened by one opening.

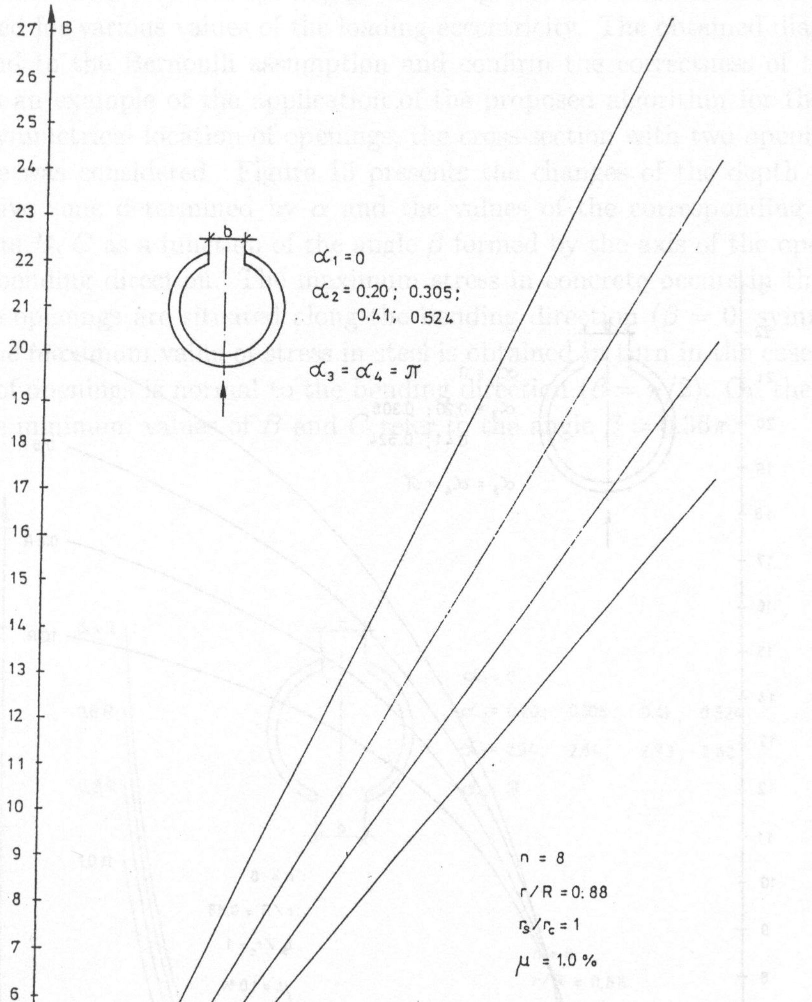

FIG. 4. Values of the maximum concrete stress coefficient B for a section weakened by one opening.

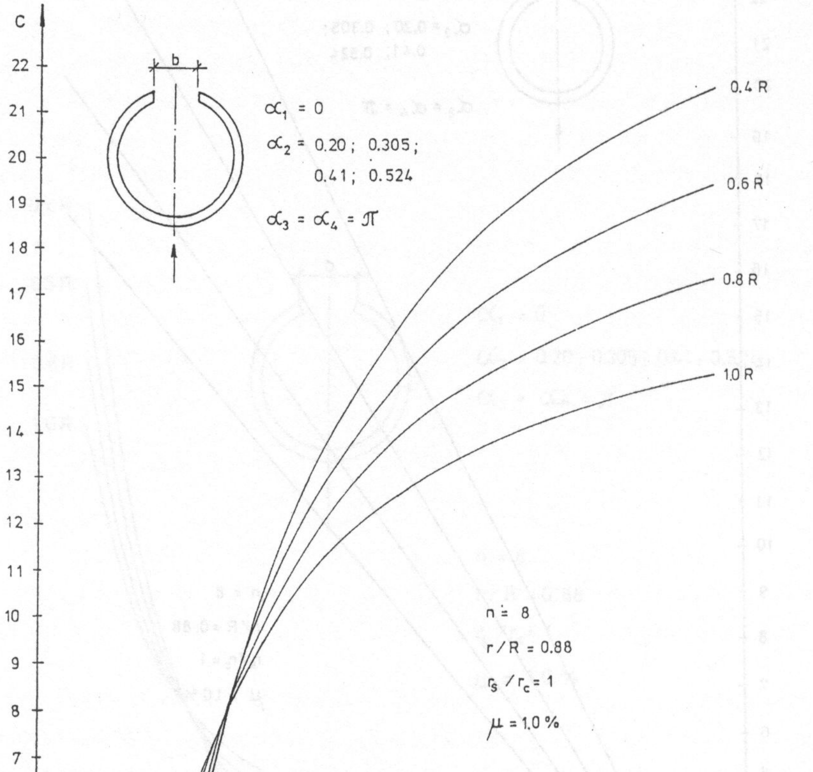

FIG. 5. Values of the maximum steel stress coefficient C for a section weakened by one opening.

For the section with one opening given in Fig. 12, the stress distribution was determined for various values of the loading eccentricity. The obtained diagrams correspond to the Bernoulli assumption and confirm the correctness of the results. As an example of the application of the proposed algorithm for the case of the asymmetrical location of openings, the cross-section with two openings of equal size was considered. Figure 13 presents the changes of the depth of the compressive zone determined by α and the values of the corresponding stress coefficients B, C as a function of the angle β formed by the axis of the openings and the bending direction. The maximum stress in concrete occurs in the case when the openings are situated along the bending direction ($\beta = 0$, symmetric case). The maximum value of stress in steel is obtained in turn in the case when the axis of openings is normal to the bending direction ($\beta = \pi/2$). On the other hand, the minimum values of B and C refer to the angle $\beta = 0.36\pi$.

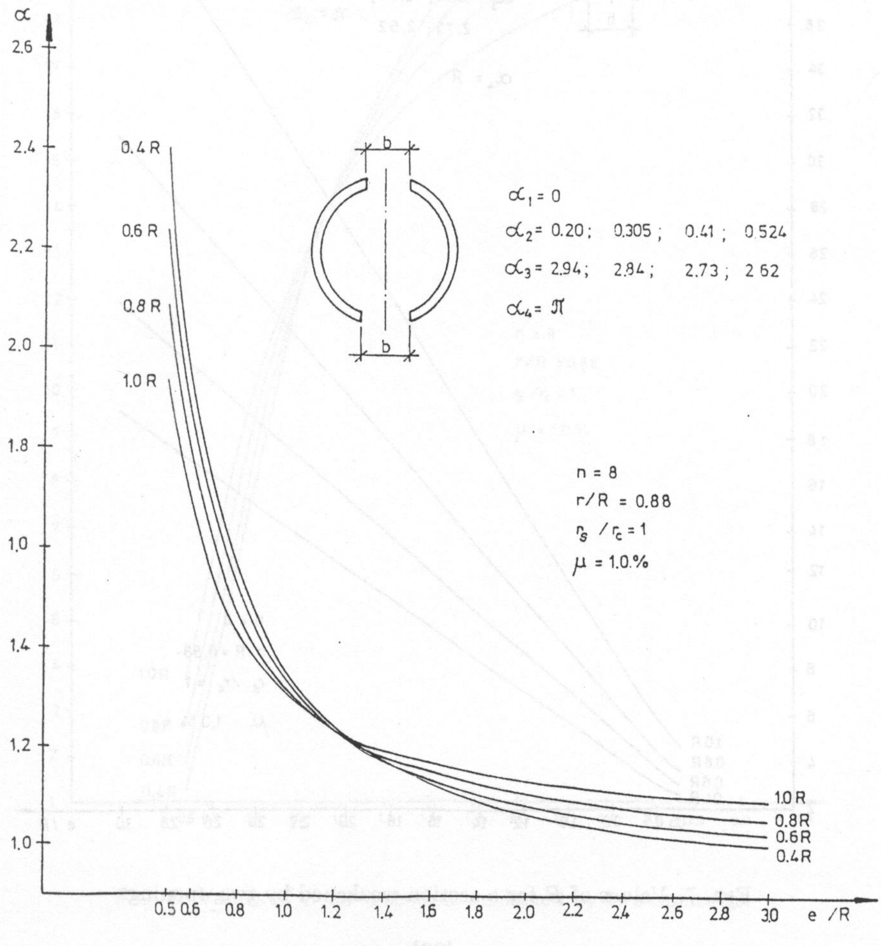

FIG. 6. Values of α for a section weakened by two openings.

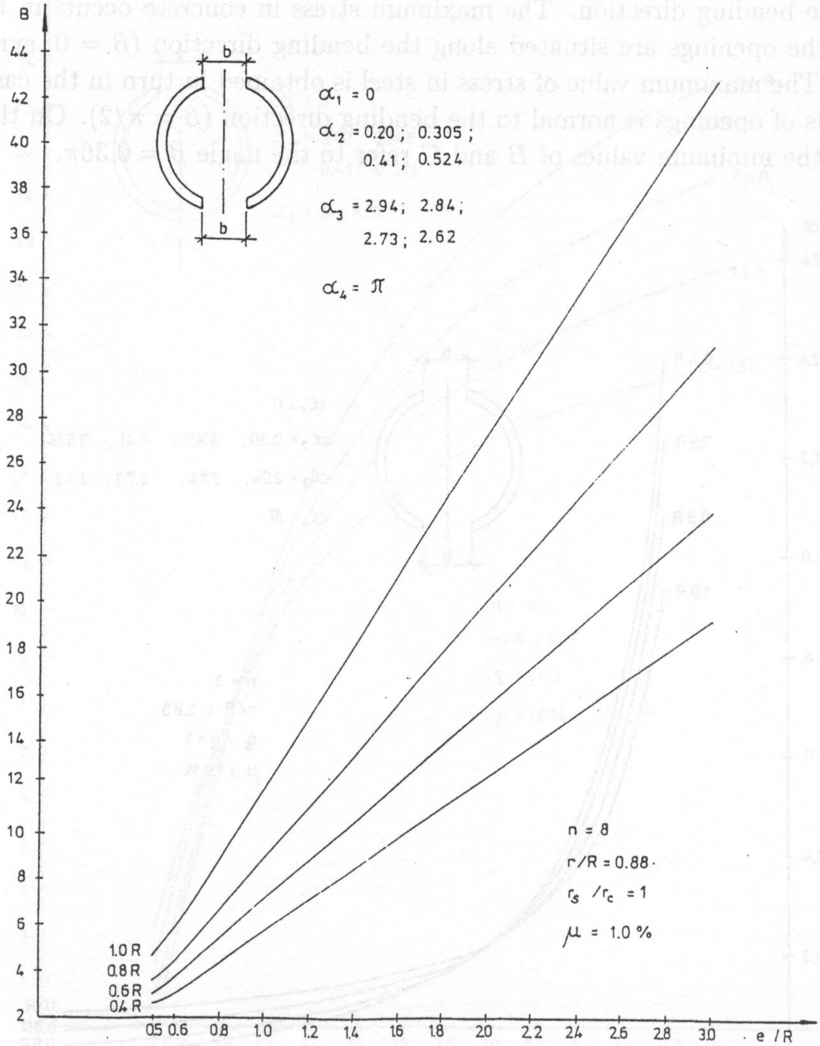

FIG. 7. Values of B for a section weakened by two openings.

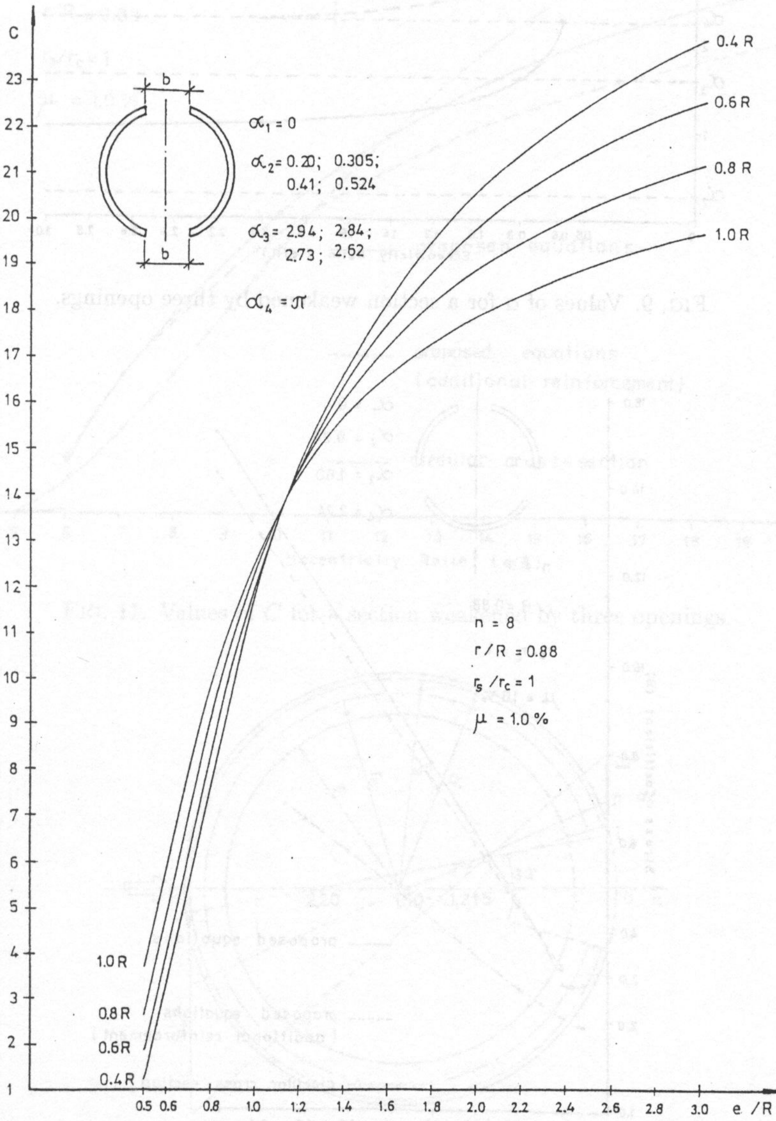

FIG. 8. Values of C for a section weakened by two openings.

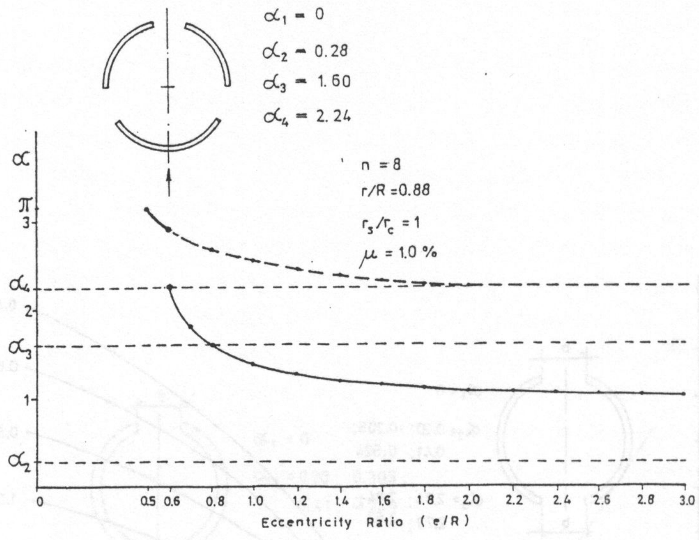

FIG. 9. Values of α for a section weakened by three openings.

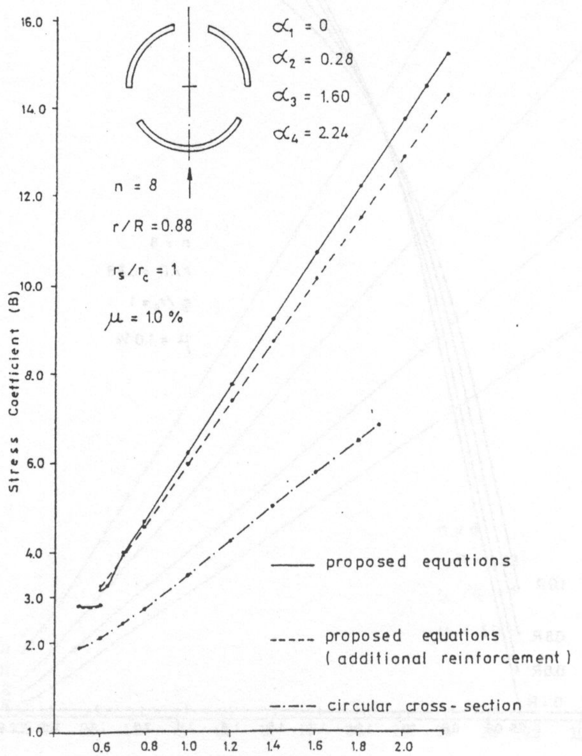

FIG. 10. Values of B for a section weakened by three openings.

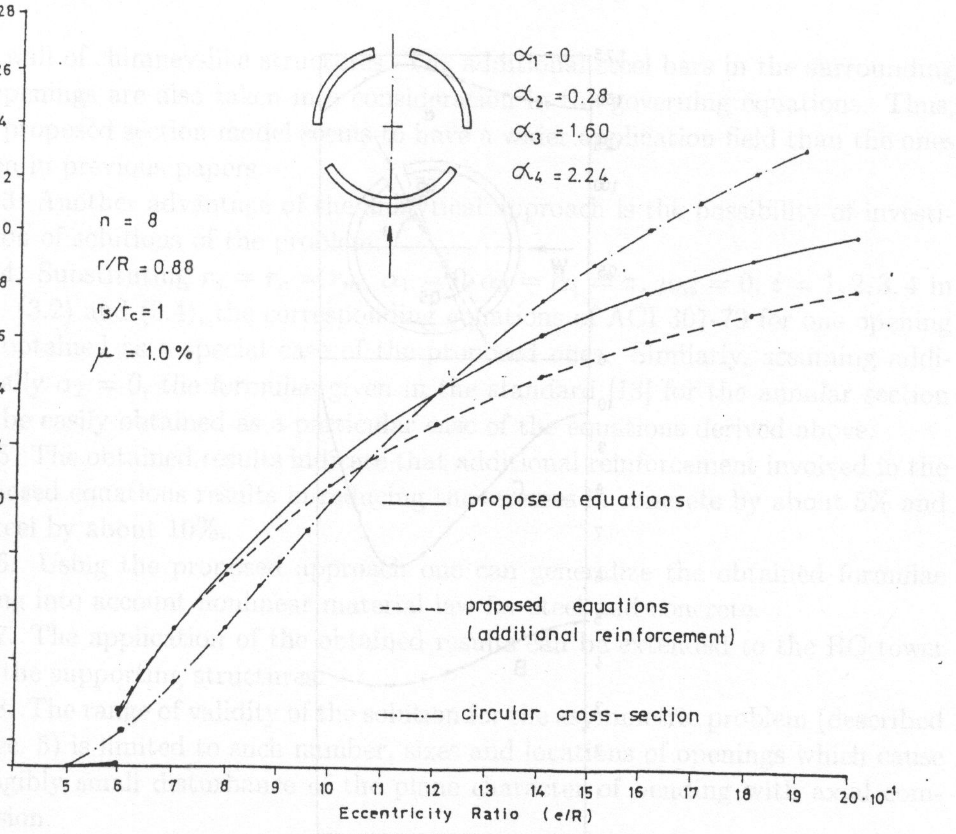

FIG. 11. Values of C for a section weakened by three openings.

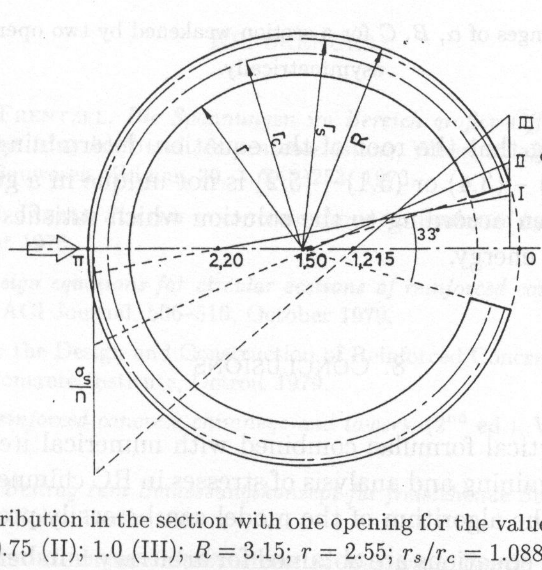

FIG. 12. Stress distribution in the section with one opening for the values of loading eccentricity $e/R = 0.5$ (I); 0.75 (II); 1.0 (III); $R = 3.15$; $r = 2.55$; $r_s/r_c = 1.088$; $\mu = 0.5\%$; $n = 6.56$.

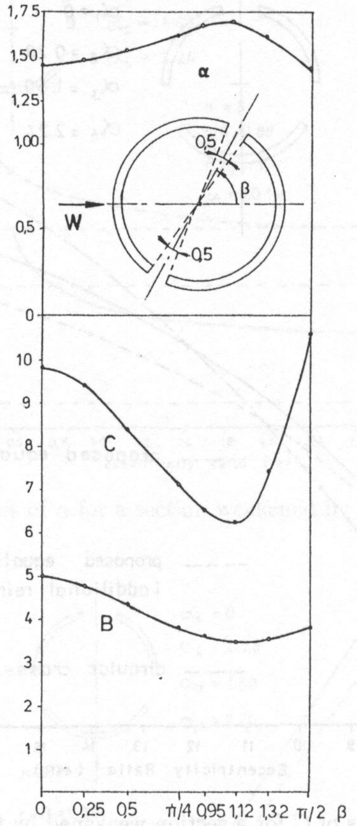

FIG. 13. Value changes of α, B, C for a section weakened by two openings located asymmetrically.

It is worth noting that the root of the equation determining the location of the neutral axis (3.2) – (3.4) or (5.1) – (5.2) is not unique in a general case. The correct value is chosen according to the solution which satisfies the principle of the minimum elastic energy.

8. CONCLUSIONS

1. General analytical formulae combined with numerical iterative technique is proposed for determining and analysis of stresses in RC chimney sections weakened by openings. The algorithm of the model can be easily processed.

2. The governing equations are obtained for arbitrary number of openings and arbitrary bending direction, as well as for steel reinforcement spaced at l layers in

the wall of chimney-like structures. The additional steel bars in the surrounding of openings are also taken into consideration in the governing equations. Thus, the proposed section model seems to have a wider application field than the ones given in previous papers.

3. Another advantage of the analytical approach is the possibility of investigation of solutions of the problem.

4. Substituting $r_s = r_c = r_m$, $\alpha_1 = 0, \alpha_3 = \alpha_4 = \pi$, $\mu_{\alpha i} = 0$, $i = 1, 2, 3, 4$ in Eqs. (3.2) and (3.4), the corresponding equations of ACI 307-79 for one opening are obtained as a special case of the proposed ones. Similarly, assuming additionally $\alpha_2 = 0$, the formulae given in the standard [13] for the annular section can be easily obtained as a particular case of the equations derived above.

5. The obtained results indicate that additional reinforcement involved in the proposed equations results in reducing the stresses in concrete by about 5% and in steel by about 10%.

6. Using the proposed approach one can generalize the obtained formulae taking into account nonlinear material law for steel and concrete.

7. The application of the obtained results can be extended to the RC tower and the supporting structures.

8. The range of validity of the solution for the asymmetric problem (described in Sec. 5) is limited to such number, sizes and locations of openings which cause negligibly small disturbance of the plane character of bending with axial compression.

References

1. E. Hampe, R. Frentzel, *Die Spannungen im Bereich großer Öffnungen von Stahlbetonschornsteinen für Längskraft und Momentenbelastung*, Wissenschaftliche Zeitschrift für Architektur und Bauwesen Weimar, **20**, 3, 245–253, 1973.

2. C. H. Lee, H. H. Jabali, *Stresses in concrete chimneys weakened by openings*, ACI Journal, 465–468, August 1976.

3. W. F. Yau, *Design equations for circular sections of reinforced concrete without thin wall approximation*, ACI Journal, 506–510, October 1979.

4. Specification for the Design and Construction of Reinforced Concrete Chimneys, ACI 307-79, American Concrete Institute, Detroit 1979.

5. M. Pinfold, *Reinforced concrete chimneys and towers*, (2nd ed.), Viewpoint Publications, London 1984.

6. K. Bachmann, *Beitrag zum Bemessungskonzept fur freistehende Schornsteine aus Stahlbeton*, W. Ernst u. Sohn, Berlin 1981.

7. H. Nieser, V. Engel, *Industrieschornsteine in Massivbau*, Beuth, Berlin 1986.

8. DIN-1056, *Freistehende Schornsteine in Massivbauart.*

9. DIN-1045, *Beton und Stahlbeton.*

10. R. Ciesielski, K. Banaś, *Analysis of multilayered ring cross-sections for bending with axial force considering material nonlinearities* [in Polish], Ossolineum Publishing House, Constructions and Building Materials, 1990.

11. M. Lechman, P. M. Lewiński, *Determination of normal stresses in non-circular cross-sections of reinforced concrete chimneys*, CICIND Conference, Florence, 9/11, September 1993.

12. M. Lechman, P. M. Lewiński, *The constitutive equations for nonlinear analysis of RC chimneys with non-circular cross-sections*, Proc. of EURO-C 1994 International Conference "Computational Modelling of Concrete Structures", H. Mang, N. Bičanič, R. de Borst [Eds.], 22nd-25th March, 745–754, Innsbruck 1994, Pineridge Press Ltd.

13. PN-88/B-03004, *Polish code for brickwork and concrete chimneys* [in Polish].

14. W. Nowacki, *Mechanics of building structures* [in Polish], PWN, Warszawa 1960.

Received November 10, 1999.