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A CONTRIBUTION TO THE MODELLING OF DYNAMIC PROBLEMS FOR PERIODIC PLATES

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A certain problem of vibrations analysis of thin periodic plates is presented in this paper. The applied model describes the effect of the periodicity cell size on the overall plate behaviour. In the modelling procedure we use a concept of functions which describe oscillations inside the periodicity cell and have to be properly chosen approximations of solutions to eigenvalue problems for natural vibrations of a separated periodicity cell with periodic boundary conditions. In this paper we will show that for certain cases of that cell, an approximate form of those functions can be used.

Key words: periodic plate, length-scale effect, mode-shape function.

1. Introduction

Main objects of our considerations are thin plates with a periodic structure along one direction in planes parallel to the plate midplane. These plates are composed of many identical repeated elements. Our investigations are restricted to plates in which every element, called the periodicity cell, is treated as a thin plate with span l along one direction. An example of such plates is shown in Fig. 1.

Problems of plates of this kind were investigated by means of different methods. However, exact analysis of those plates within solid mechanics is too complicated to constitute the basis for solving most of the engineering problems. Thus, many different approximate modelling methods for periodic plates were formulated.

Effective plate rigidities were used e.g. in [3, 5, 10, 13] where periodic plates were described by governing equations of certain homogeneous plates with con-

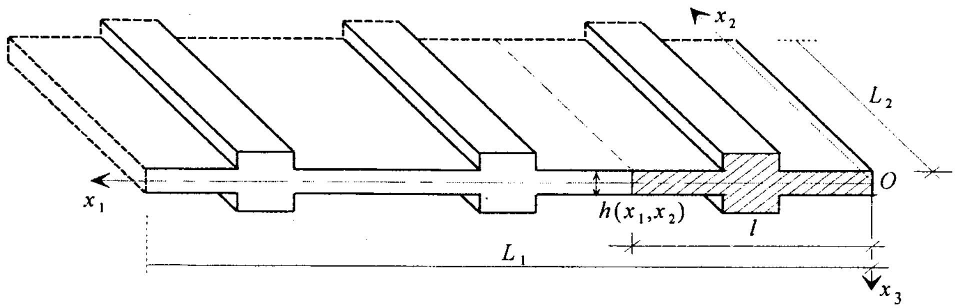

FIG. 1. Uniperiodic plate.

stant averaged rigidities and averaged mass densities. By means of the asymptotic homogenisation methods, the averaged rigidities have to be derived for every periodic structure by solving a certain boundary-value problem posed on the periodicity cell. The asymptotic procedures are restricted to the first approximation, [14], which leads to averaged models neglecting the effect of the element length called *the length-scale effect*. Moreover, the aforementioned models were restricted to the static problems.

In order to investigate dynamic problems, certain models, e.g. those based on the concept of the continuum with additional local degrees of freedom, were proposed, [15]. Short wave propagation problems were investigated in [1].

Some refined models describing long wave problems for periodic bodies were presented in [21 – 22], where it was shown that *the length-scale effect* on dynamic response of a body is a very important problem, mainly in the vibration and wave propagation investigations. The approach formulated in a general form in the aforementioned papers leads to so-called *length-scale models*. These models were applied to analyse certain dynamic problems of periodic structures, e. g. for Hencky-Reissner periodic plates [2], for composite lattice-type structures [4], for Kirchhoff periodic plates [6-9], for fibre composites [11], for periodically laminated composites [16, 20], for periodic beams [17], for periodic wavy-plates [18], for periodic visco-elastic materials [19] and others. These models are physically reasonable and simple enough to be applied in the analysis of engineering problems. In the framework of length-scale models for periodic structures we obtain the governing equations with constant coefficients. The length-scale effect on the dynamic body behaviour in the framework of the length-scale models is described by means of certain extra unknowns.

Models of this kind are different from the known modelling methods of asymptotic homogenisation. Some mathematical substantiations of those models were shown in [21 – 22]. However, for thin periodic plates certain calculations confirming correctness of the length-scale model were made in [7 – 8].

In paper [6] the modelling procedure with general assumptions and governing equations of the length-scale model for plates with periodic structure along two perpendicular directions were presented. For the considered plates with periodic structure along one direction, this procedure was changed and generalised in [23]. As a result of the generalisation, models for plates with one-directional periodic structure were obtained, taking into account the length-scale effect also in stationary processes. In order to derive the coefficients in the governing equations, we have to solve a certain eigenvalue problem for natural vibrations of a separated periodicity cell.

In this contribution, we consider the problem of solutions to this eigenvalue problem mentioned in [8] and also in [6]. The main aim of this paper is to show how the eigenvalue problem for the periodicity cell is formulated and solved, and that for many cases of periodicity cells, the eigenfunctions being solutions to this eigenvalue problem can be assumed in an approximate form which is sufficient from the computational point of view. In the presented example, our consideration will be restricted to plate bands. This makes it possible to present clearly the aforementioned problem.

Basic notations and preliminary concepts will be presented in Sec. 2. In the subsequent section, the modelling procedure and the governing equations of the presented model for linear-elastic plates with a periodic structure along one direction and constant structure along the perpendicular direction will be shown. For comparison, the governing equations of a certain homogenised model will be also presented. An analysis of free vibrations of a plate band with a periodic structure and eigenvalue problems for certain periodicity cells will be shown in Sec. 4. Subsequently, in order to evaluate the differences between applying of exact or approximate forms of solutions to those eigenvalue problems, the analysis of free vibration frequencies of a periodic plate band will be discussed. In Sec. 6 a test of correctness of the presented model will be shown. Final remarks will be formulated in the last section.

2. Preliminaries

We introduce the orthogonal Cartesian co-ordinate system $0x_1x_2x_3$ in the physical space and denote $x \equiv (x_1, x_2)$, $z \equiv x_3$. We also define t as the time coordinate. In this paper we will investigate thin linear-elastic plates with a periodic structure along one direction of the x_1-axis, and with constant properties along the x_2-axis, in planes parallel to the plate midplane. The region of an undeformed plate is denoted by $\mathbf{\Omega} \equiv \{(\mathbf{x}, z) : -h(\mathbf{x})/2 < z < h(\mathbf{x})/2, \mathbf{x} \in \Pi\}$, where Π is the rectangular plate midplane with length dimensions L_1, L_2 along the x_1- and

x_2-axis, respectively, and $h(\mathbf{x})$ is the plate thickness at the point $\mathbf{x} \in \Pi$. We shall introduce a positive value l called a period. We assume that the period l is sufficiently small compared to the minimum characteristic length dimension of the plate midplane along the direction of plate periodicity, and sufficiently large compared to the maximum plate thickness h, i. e., $h \ll l \ll L_1$. Thus, the period l will be referred to as *the mesostructure length parameter.* Denote $\Lambda \equiv [0, L_1]$ and define the interval $I(x_1) \equiv (x_1 - l/2, x_1 + l/2)$, $x_1 \in \Lambda_0$, where $\Lambda_0 = \{x_1 : x_1 \in \Lambda, I(x_1) \subset \Lambda\}$, which will be called a periodicity interval at x_1. We assume that the plates have the l-periodic heterogeneous structure and are called *uniperiodic plates.* Thus, it is assumed that plates of this kind have thickness $h(\cdot)$, which is the l-periodic function in x_1 and independent of x_2. Moreover, all the material and inertial properties of those plates, i. e. components of the elastic moduli tensor a_{ijkl}, and mass density ρ, are also l-periodic functions in x_1, independent of x_2 and even functions in z.

In the analysis of periodic structures we shall also use the following introductory concepts: *the averaging operator* in a periodicity interval $I(x_1)$ and two kinds of functions which will be referred to as *slowly varying* and *highly oscillating.*

For an arbitrary integrable function φ defined on Π we define *the averaging operator*, following [10, 12 – 13, 21 – 22], given by

$$(2.1) \qquad <\varphi> = <\varphi>(x_1, x_2) \equiv l^{-1} \int\limits_{I(x_1)} \varphi(y_1, x_2) dy_1, \qquad x_1 \in \Lambda_0.$$

If the function φ is l-periodic function in x_1 and is independent of x_2, its averaged value obtained from (2.1) is constant.

The first kind of the functions used is a *slowly varying function.* A function $F(\cdot)$ defined on Λ will be called *slowly varying* if for every $y_1, y_2 \in \Lambda$ and $x_1 \in \Lambda_0$, the following condition holds: $F(y_1) \cong F(y_2)$, y_1, $y_2 \in I(x_1)$, with all its derivatives. Hence, for an arbitrary integrable function $\varphi(\cdot)$ and a slowly varying function $F(\cdot)$ we obtain $<\varphi F>(x_1) \cong <\varphi>(x_1) F(x_1)$, $x_1 \in \Lambda_0$.

Define the second kind of the functions needed. A differentiable function $\phi(\cdot)$ defined on Λ will be called *highly oscillating function* if for every $x_1 \in \Lambda_0$ it satisfies the condition $<(\phi F)_{,1}>(x_1, t) \cong <F \phi_{,1}>(x_1, t)$ for every slowly varying function F defined on Λ_0 and $l^{-2}\phi(x_1, t)$, $l^{-1}\phi_{,1}(x_1, t)$, $\phi_{,11}(x_1, t)$ are of an order $O(1)$ where l is the mesostructure length parameter, as well as there exists a periodic function ϕ_{x_1} such that $\phi \cong \phi_{x_1}$ for the interval $I(x_1)$.

Throughout the paper subscripts α, β, ...$(i, j, ...)$ run over $1, 2$ $(1, 2, 3)$ and indices A, B, ... run over $1, ..., N$. For all the aforesaid indices the summation convention holds. Displacements, strains and stresses in an arbitrary point of a plate are denoted, respectively, by u_i, e_{ij}, s_{ij}. Moreover, let p be the loads in the z-axis direction on the upper and lower plate boundaries, and b be the constant body force in the z-axis direction. By $w(\mathbf{x}, t)$ we denote the plate midplane deflection. We shall define $c_{\alpha\beta\gamma\delta} \equiv a_{\alpha\beta\gamma\delta} - a_{\alpha\beta33}a_{\gamma\delta33}(a_{3333})^{-1}$ and assume that $z = \text{const}$ are material symmetry planes; hence $c_{3\alpha\beta\gamma} = 0$, $c_{333\gamma} = 0$.

3. Modelling procedure

The modelling procedure used in this paper was presented for plates with two-directional periodic structure in [6, 8], and in the generalised form for plates with one-directional periodic structure in [23]. For the sake of selfconsistency, we remind here its key concepts.

The starting point of our considerations are the well-known Kirchhoff plate theory assumptions: *the kinematic relations, the strain-displacement equations, the stress-strain relations* (under the plane stress assumption, $s_{33} = 0$), and *the virtual work principle.* The aforementioned relations for periodic plates lead to the partial differential equation of the fourth order involving highly oscillating periodic coefficients. In order to pass to the equations with constant coefficients but retaining the length-scale effect, the additional modelling assumptions were introduced in [9, 6, 8].

Introduce the quantities which are l-periodic functions in x_1 and independent of x_2,

$$\mu := \int_{-h/2}^{h/2} \rho dz, \qquad \vartheta := \int_{-h/2}^{h/2} \rho z^2 dz, \qquad d_{\alpha\beta\gamma\delta} := \int_{-h/2}^{h/2} z^2 c_{\alpha\beta\gamma\delta} dz,$$

describing the plate properties: mass density per an unit area, rotational inertia and bending stiffnesses, respectively.

The modelling procedure of the presented *length-scale plate models* is based on three modelling assumptions formulated in [6 – 9]. Now we reduce those to only one *kinematic hypothesis.*

The fundamental kinematic hypothesis is that the averaged plate deflection, given by $W(x_1, x_2, t) \equiv < \mu >^{-1} (x_2) < \mu w > (x_1, x_2, t)$, $x_1 \in \Lambda_0$, $x_2 \in [0, L_2]$, together with its all derivatives, are *slowly varying functions* of x_1. Thus, W will be referred to as *the plate macrodeflection.* Moreover, the deflection disturbances given by $v(x_1, x_2, t) \equiv w(x_1, x_2, t) - W(x_1, x_2, t)$, $x_1 \in \Lambda_0$, $x_2 \in [0, L_2]$, are assumed to be *highly oscillating functions* of x_1.

Now the class of disturbances $v(x_1, x_2, t)$ is specified. Let us denote the plate bending stiffness and the plate mass density per unit area by $B(x_1)$ and $\mu(x_1)$, respectively, $x_1 \in \Lambda_0$. At an arbitrary $x_2 \in [0, L_2]$ the eigenvalue problem for a function $g^A(y_1)$ is given by the equation

$$(3.1)\qquad [B(y_1)g^A(y_1)_{,11}]_{,11} - \mu(y_1)(\lambda_A)^2 g^A(y_1) = 0, \qquad y_1 \in \overline{I}(x_1, \quad x_1 \in \Lambda_0,$$

and by the periodic boundary conditions on the boundary of the interval $I(x_1)$ together with the continuity conditions inside $I(x_1)$. Thus, $g^A(y_1)$, $A = 1, 2, ...$, is a sequence of eigenfunctions defined on $\overline{I}(x_1)$ and related to the sequence of eigenvalues λ_A. In the modelling procedure this sequence is restricted to the $N \geq 1$ eigenfunctions and $g^A(y_1)$, $y_1 \in I(x_1)$, $A = 1, ..., N$, will be called *mode-shape functions* g^A. We assume that the mode-shape functions $g^A(y_1)$ are linear independent, l-periodic functions, depending on the mesostructure length parameter l and such that $l^{-2}g^A(y_1)$, $l^{-1}g^A_{,1}(y_1)$, $g^A_{,11}(y_1)$ are of order $O(1)$. Moreover, it can be shown that the eigenfunctions satisfy the condition $< \mu g^A >= 0$. In the course of modelling, the disturbances v will be approximated by the finite series

$$v(y_1, x_2, t) = g^A(y_1)Q^A(x_1, x_2, t), \quad y_1 \in I(x_1), \quad x_1 \in \Lambda_0, \quad A = 1, ..., N,$$

where $g^A(y_1)$ are the known mode-shape functions, and $Q^A(x_1, x_2, t)$ are some extra unknowns being slowly varying functions in x_1. In most problems the analysis will be restricted to the simplest case $N = 1$ in which we take into account only the lowest natural vibration mode related to Eq. (3.1).

After some manipulations, from the Kirchhoff plate theory assumptions and the kinematic hypothesis, the governing equations of the length-scale model for uniperiodic plates will be derived:

- *Equations of motion*

$$(3.2)\qquad \begin{aligned} M_{\alpha\beta,\alpha\beta} + < \mu > \ddot{W} - (< \vartheta > \ddot{W}_{,\alpha})_{,\alpha} - (\underline{< \vartheta g^B >}\ddot{Q}^B_{,2})_{,2} \\ - (\underline{< \vartheta g^B_{,\alpha} >}\ddot{Q}^B)_{,\alpha} =< p > +b < \mu > . \end{aligned}$$

- *Constitutive equations*

$$\begin{aligned} M_{\alpha\beta} &=< d_{\alpha\beta\gamma\delta} g^B_{,\alpha\beta} > Q^B + 2\underline{< d_{\alpha\beta 2\gamma} g^B_{,\gamma} >}Q^B_{,2} \\ &\quad + \underline{< d_{\alpha\beta 22} g^B >}Q^B_{,22}, \\ M^A &=< d_{\alpha\beta\gamma\delta} g^A_{,\gamma\delta} > W_{,\alpha\beta} + < d_{\alpha\beta\gamma\delta} g^A_{,\alpha\beta} g^B_{,\gamma\delta} > Q^B + 2\underline{< d_{2\alpha\beta\gamma} g^A_{,\alpha} g^B_{,\beta\gamma} >}Q^B_{,2} \\ &\quad + \underline{< d_{22\alpha\beta} g^A_{,\alpha\beta} g^B >}Q^B_{,22}, \end{aligned}$$

(3.3)

$$\begin{aligned} R^A &= \underline{< d_{22\alpha\beta} g^A >}W_{,\alpha\beta} + \underline{< d_{22\alpha\beta} g^A g^B_{,\alpha\beta} >}Q^B + 2\underline{< d_{2\alpha 22} g^A_{,\alpha} g^B >}Q^B_{,2} \\ &\quad + \underline{< d_{2222} g^A g^B >}Q^B_{,22}, \end{aligned}$$

(3.3) [cont.]
$$T^A = \underline{< d_{\alpha\beta 2\gamma} g^A_\gamma >} W_{,\alpha\beta} + \underline{< d_{2\alpha\beta\gamma} g^A_{,\alpha} g^B_{,\beta\gamma} >} Q^B + 2\underline{< d_{2\alpha 2\beta} g^A_{,\alpha} g^B_{,\beta} >} Q^B_{,2} + \underline{< d_{2\alpha 22} g^A_{,\alpha} g^B >} Q^B_{,22}.$$

- *Evolution equations*

(3.4)
$$(\underline{< \mu g^A g^B >} + \underline{< \vartheta g^A_{,\alpha} g^B_{,\alpha} >}) \ddot{Q}^B + \underline{< \vartheta g^A_{,1} >} \ddot{W}_{,1} + \underline{< \vartheta g^A g^B_{,2} >} \ddot{Q}^B_{,2} - (\underline{< \vartheta g^A >} \ddot{W}_{,2} + \underline{< \vartheta g^A g^B_{,2} >} \ddot{Q}^B + \underline{< \vartheta g^A g^B >} \ddot{Q}^B_{,2})_{,2} + M^A + R^A_{,22} - 2T^A_{,2} = \underline{< p g^A >},$$

where the underlined terms depend on the mesostructure length parameter l. This model is called *the uniperiodic plate model*, [23]. For plates with periodic structure along the x_1-axis direction and constant along the x_2-axis direction, all coefficients in brackets $< >$ in Eqs. (3.2) – (3.4) are constant (except $< p >$, $< pg^A >$ which can be slowly varying functions of x_1 and x_2), and functions g^A are dependent only on x_1.

The characteristic feature of the above averaged equations is that they describe the effect of the interval length l on the overall dynamic behaviour of uniperiodic plates.

Functions W, Q^A are the basic unknowns which have to be slowly varying functions of x_1. The function W is called the *plate macrodeflection*; functions Q^A are called *disturbance variables.* In the case of a rectangular plate with midplane $\Pi = (0, L_1) \times (0, L_2)$, two boundary conditions should be defined on the edges $x_1 = 0, L_1$ and $x_2 = 0, L_2$ for the function W; for the functions Q^A two boundary conditions should be defined only on the edges $x_2 = 0, L_2$. It is easy to see that to obtain the above equations, we must previously derive the mode-shape functions g^A, $A = 1, ..., N$, for every periodic plate under consideration as solutions to the eigenvalue problem given by (3.1). In practice, a derivation of these exact solutions is possible only for intervals with a structure which is not too complicated. In most cases we have to look for an approximate form of these solutions. We also restrict our considerations to a small number N of mode shapes. In this paper we assume that $N = 1$ and denote $g \equiv g^1$.

In the sequel it will be shown that for the interval $I(x_1)$ having not a very complicated structure, we can assume the mode-shape function in an approximate form of the solution to the eigenvalue problem.

At the end of this section let us observe that the homogenised model investigated in [6, 23] is a special case of Eqs. (3.2) – (3.4) obtained above. The governing equations of *the homogenised model* are

- *Equation of motion*

$$\tilde{M}_{\alpha\beta,\alpha\beta} + < \mu > \ddot{W} - (< \vartheta > \ddot{W}_{,\alpha})_{,\alpha} = < p > + b < \mu > . \tag{3.5}$$

- *Constitutive equations*

$$\tilde{M}_{\alpha\beta} = < d_{\alpha\beta\gamma\delta} > W_{,\gamma\delta} + < d_{\alpha\beta\gamma\delta} g^{B}_{,\gamma\delta} > Q^{B}. \tag{3.6}$$

- *Equations for the disturbance variables*

$$< d_{\alpha\beta\gamma\delta} g^{A}_{,\alpha\beta} g^{B}_{,\gamma\delta} > Q^{B} = - < d_{\alpha\beta\gamma\delta} g^{A}_{,\gamma\delta} > W_{,\alpha\beta}, \tag{3.7}$$

where the effect of mesostructure length parameter l on the overall plate behaviour is not taken into account. The above equations can be obtained from Eq. (3.2) – (3.4) by neglecting the underlined terms.

4. Analysis of a special case

4.1. Free vibrations for a periodic plate band

In order to evaluate the differences between the application of an exact or approximate form of mode-shape function g, free vibrations of a periodic plate band will be considered. Equations (3.2) – (3.4) will be used. It will be assumed that body forces b and loads p are neglected. Let us consider a plate band simply supported on the edges $x_1 = 0$, L_1 with a periodically varying piece-wise constant thickness. It is made of an isotropic, periodically varying piece-wise homogeneous material. An example of a periodicity interval is shown in Fig. 2. For the symmetric interval, a symmetric form of the mode-shape function g will be assumed. For that function it can be shown that $< \vartheta g_{,1} >= 0$. Denote $x \equiv x_1$, $L \equiv L_1$ and $Q \equiv Q^1$ as well as

$$B \equiv \frac{Eh^3}{12(1-\nu^2)}, \qquad D_{11} \equiv < d_{1111} g_{,11} >, \qquad D \equiv < d_{1111} (g_{,11})^2 >,$$

$$m \equiv < \mu >, \qquad m^{11} \equiv l^{-4} < \mu g^2 >, \qquad j \equiv < \vartheta >, \qquad j^{11} \equiv l^{-2} < \vartheta (g_{,1})^2 >,$$

where E is the plate Young's modulus, ν is the plate Poisson's ratio, h is the plate thickness. Substituting (3.3) to (3.2) and (3.4) with restriction $A = N = 1$, we arrive at

$$\begin{aligned} < B > W_{,1111} + m\ddot{W} - j\ddot{W}_{,11} + D_{11} Q_{,11} &= 0, \\ D_{11} W_{,11} + DQ + l^2 (l^2 m^{11} + j^{11}) \ddot{Q} &= 0. \end{aligned} \tag{4.1}$$

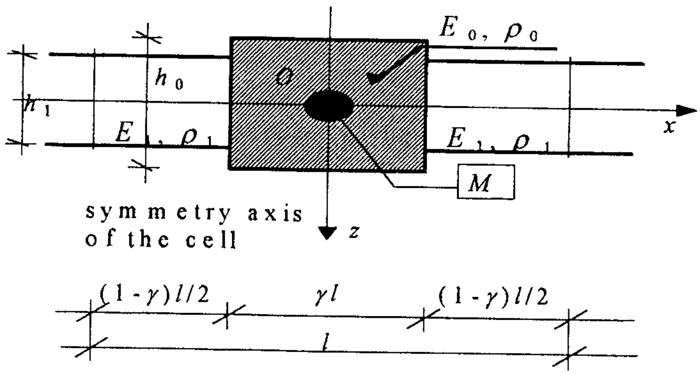

FIG. 2. The periodicity interval for the plate band with periodically distributed concentrated masses and with periodically varying ρ, E, h.

Introduce the wave number $k = 2\pi/L$. Solutions to the above equations will be assumed in the form satisfying boundary conditions for a simply supported plate band. Hence, these solutions can be assumed as

$$W(x) = A_W \sin(kx)\cos(\omega t), \qquad Q(x) = A_Q \sin(kx)\cos(\omega t), \tag{4.2}$$

where A_W, A_Q are amplitudes. Substituting these solutions to (4.1) we obtain the system of linear algebraic equations for amplitudes A_W, A_Q

$$\begin{bmatrix} <B> k^4 - \omega^2(m+jk^2) & -D_{11}k^2 \\ -D_{11}k^2 & D - \omega^2 l^2(l^2 m^{11} + j^{11}) \end{bmatrix} \begin{Bmatrix} A_W \\ A_Q \end{Bmatrix} = \{0\},$$

which has non-trivial solutions if its determinant is equal to zero. After some manipulations, in the framework of *the uniperiodic plate model* we arrive at formulae for *a lower* ω_1 *and a higher* ω_2 *free vibration frequency*

$$(4.3)\qquad \begin{aligned} (\omega_1)^2 \equiv \frac{1}{2}\left\{l^2(m+jk^2)(l^2m^{11}+j^{11})\right\}^{-1}\Big\{<B> k^4 l^2(l^2m^{11}+j^{11}) \\ +D(m+jk^2) \\ -\sqrt{[<B> k^4 l^2(l^2m^{11}+j^{11}) - D(m+jk^2)]^2 + 4(m+jk^2)l^2(l^2m^{11}+j^{11})(D_{11})^2k^4}\Big\}, \\ (\omega_2)^2 \equiv \frac{1}{2}\left\{l^2(m+jk^2)(l^2m^{11}+j^{11})\right\}^{-1}\Big\{<B> k^4 l^2(l^2m^{11}+j^{11}) \\ +D(m+jk^2) \\ +\sqrt{[<B> k^4 l^2(l^2m^{11}+j^{11}) - D(m+jk^2)]^2 + 4(m+jk^2)l^2(l^2m^{11}+j^{11})(D_{11})^2k^4}\Big\}, \end{aligned}$$

depending on the mesostructure length parameter l. In order to compare the results obtained within the presented model, *the homogenised model* will be also applied. From Eqs. (3.5) - (3.7) for the plate band under consideration, with restriction $A = N = 1$, we obtain only one equation

$$\left[< B > -(D_{11})^2 D^{-1}\right] W_{,1111} + m\ddot{W} - j\ddot{W}_{,11} = 0. \tag{4.4}$$

Assuming a solution to the above equation in the form $(4.2)_1$, after some manipulations we arrive at the following formula for only one – *a lower* ω_0 *free vibration frequency:*

$$(\omega_0)^2 \equiv \left[< B > -(D_{11})^2 D^{-1}\right] k^4 (m + jk^2)^{-1}, \tag{4.5}$$

without terms depending on the mesostructure length parameter l.

Comparing the obtained results (formulae (4.2) and (4.5)) it can be confirmed that, within the homogenised model, the length-scale effect on the dynamic plate behaviour can not be investigated. In the framework of the model we can analyse only one (lower) free vibration frequency. However, higher free vibration frequency can be investigated only within the uniperiodic plate model.

4.2. Analysis of the eigenvalue problem for different cases of periodicity interval

Let us consider a plate band with span L. The periodicity interval of the plate $I_l \equiv (-l/2, l/2)$ is shown in Fig. 2. The plate periodicity is caused by periodically distributed concentrated masses M and periodically varying plate properties. Only Poisson's ratio ν is constant. Functions describing those properties are assumed in the following form:

- the Young's modulus E as

$$E(x) = \begin{cases} E_0 & \text{if} \quad x \in (-\gamma l/2, \gamma l/2), \\ E_1 & \text{if} \quad x \in [-l/2, -\gamma l/2] \cup [\gamma l/2, l/2], \qquad \gamma \in [0,1], \end{cases} \tag{4.6a}$$

- the mass density ρ as

$$\rho(x) = \begin{cases} \rho_0 & \text{if} \quad x \in (-\gamma l/2, \gamma l/2), \\ \rho_1 & \text{if} \quad x \in [-l/2, -\gamma l/2] \cup [\gamma l/2, l/2], \qquad \gamma \in [0,1], \end{cases} \tag{4.6b}$$

- and the plate thickness h as

$$h(x) = \begin{cases} h_0 & \text{if} \quad x \in (-\gamma l/2, \gamma l/2), \\ h_1 & \text{if} \quad x \in [-l/2, -\gamma l/2] \cup [\gamma l/2, l/2], \quad \gamma \in [0,1]. \end{cases} \tag{4.6c}$$

The periodicity cell will be treated as a plate band with thickness h and span l. Denote a derivative $(\cdot)' \equiv (\cdot)_{,1}$. Eigenfunctions for the interval will be obtained by solving the eigenvalue problem (3.1) which takes the form

$$[B(x)g''(x)]'' - \mu(x)\lambda^2 g(x) = 0, \tag{4.7}$$

where g are l-periodic functions related to eigenvalues $\lambda \equiv \alpha l$ (α is the wave number); and $< \mu g >= 0$. We will look for the solution to the eigenvalue problem using the Krylov-Prager functions

$$\begin{aligned} S(\alpha x) &= \frac{1}{2}\left[\cos\mathrm{h}(\alpha x) + \cos(\alpha x)\right], \quad U(\alpha x) = \frac{1}{2}\left[\cos\mathrm{h}(\alpha x) - \cos(\alpha x)\right], \\ T(\alpha x) &= \frac{1}{2}\left[\sin\mathrm{h}(\alpha x) + \sin(\alpha x)\right], \quad V(\alpha x) = \frac{1}{2}\left[\sin\mathrm{h}(\alpha x) - \sin(\alpha x)\right]. \end{aligned} \tag{4.8}$$

Introduce the following dimensionless coefficients:

$$\zeta \equiv M(<\mu> l)^{-1}, \qquad \varepsilon \equiv E_1/E_0, \qquad \varphi \equiv \rho_1/\rho_0, \qquad \eta \equiv h_1/h_0, \qquad \eta_0 \equiv h_0/l,$$

where $\zeta \geq 0$; ε, φ, $\eta \in [0,1]$. Two cases of a plate band will be considered below.

4.2.1. A plate band with periodically distributed concentrated masses. Let us consider a simply supported plate band of span L. The plate periodicity is caused only by periodically distributed concentrated masses M. In this case we have $\zeta \neq 0$, $\varepsilon = \varphi = \eta = 1$. Using (4.8), the solution to the eigenvalue problem (4.7) is looked for in the form

$$g(x)l^{-2} = \begin{cases} A(\lambda) S\left(\lambda \frac{x}{l}\right) + U\left(\lambda \frac{x}{l}\right), & \text{if} \quad x \in \left[-\frac{1}{2}l, 0\right], \\ A(\lambda)\left\{S\left(\lambda\frac{x}{l}\right)\right. \\ \left. +\lambda\zeta V\left[\lambda\left(\frac{x}{l} - \frac{1}{2}\right)\right] S\left(\frac{1}{2}\lambda\right)\right\} \\ +U\left(\lambda\frac{x}{l}\right) \\ +\lambda\zeta V\left[\lambda\left(\frac{x}{l} - \frac{1}{2}\right)\right] U\left(\frac{1}{2}\lambda\right), & \text{if} \quad x \in \left(0, \frac{1}{2}l\right], \end{cases} \tag{4.9}$$

where $A(\lambda) \equiv -C\left[V(\lambda) + \lambda\zeta S\left(\frac{1}{2}\lambda\right) U\left(\frac{1}{2}\lambda\right)\right]\left\{T(\lambda) + \lambda\zeta\left[S\left(\frac{1}{2}\lambda\right)\right]^2\right\}^{-1}$,

C is constant. Restricting our considerations to symmetric vibrations, the equation for the eigenvalue λ takes the form

$$\left\{T(\lambda)+\zeta\lambda\left[S\left(\frac{1}{2}\lambda\right)\right]^{2}\right\}\left\{T(\lambda)+\zeta\lambda\left[U\left(\frac{1}{2}\lambda\right)\right]^{2}\right\}$$
$$-\left[V(\lambda)+\zeta\lambda S\left(\frac{1}{2}\lambda\right)U\left(\frac{1}{2}\lambda\right)\right]^{2}=0.$$

From the above equation we can derive eigenvalues λ_A, $A = 1,2,.....$, but we restrict our analysis to $A = N = 1$. Hence, we obtain the smallest eigenvalue λ dependent on the quotient ζ, and the exact form of the mode-shape function g related to this eigenvalue is given by (4.9).

4.2.2. A plate band with periodic structure. Let us consider a simply supported plate band of span L and periodically varying piece-wise constant properties: Young's modulus E, mass density ρ and thickness h, without concentrated masses. The periodicity interval can be treated as a plate band with span l. In this case we have $\zeta = 0$, $\varepsilon \leq 1$, $\varphi \leq 1$, $\eta \leq 1$. Using (4.8) and denoting

$$\Gamma_1(\lambda) \equiv T\left(\frac{1}{2}\gamma\lambda\right)+\left[\varepsilon(\varphi\eta^2)^3\right]^{\frac{1}{4}}\left\{S\left(\frac{1}{2}\gamma\lambda\right)T\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]\right.$$
$$+\left(\frac{\varepsilon\eta^2}{\varphi}\right)^{\frac{1}{4}}V\left(\frac{1}{2}\gamma\lambda\right)U\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]$$
$$+\left(\varepsilon\varphi\eta^4\right)^{-\frac{1}{2}}U\left(\frac{1}{2}\gamma\lambda\right)V\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]$$
$$\left.+\left[\varepsilon\left(\varphi\eta^2\right)^3\right]^{-\frac{1}{4}}T\left(\frac{1}{2}\gamma\lambda\right)\left[S\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]-1\right]\right\},$$

$$\Xi_1(\lambda) \equiv V\left(\frac{1}{2}\gamma\lambda\right)+\left[\varepsilon(\varphi\eta^2)^3\right]^{\frac{1}{4}}\left\{U\left(\frac{1}{2}\gamma\lambda\right)T\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]\right.$$

[cont.]

$$+ \left(\frac{\varepsilon\eta^2}{\varphi}\right)^{\frac{1}{4}} T\left(\frac{1}{2}\gamma\lambda\right) U\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]$$

$$+ \left(\varepsilon\varphi\eta^4\right)^{-\frac{1}{2}} S\left(\frac{1}{2}\gamma\lambda\right) V\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]$$

$$+ \left[\varepsilon(\varphi\eta^2)^3\right]^{-\frac{1}{4}} V\left(\frac{1}{2}\gamma\lambda\right)\left[S\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]-1\right]\Bigg\},$$

$$\Gamma_2(\lambda) \equiv S\left(\frac{1}{2}\gamma\lambda\right) V\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]$$

$$+ \left(\frac{\varepsilon\eta^2}{\varphi}\right)^{\frac{1}{4}} V\left(\frac{1}{2}\gamma\lambda\right) S\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]$$

$$\left(\varepsilon\varphi\eta^4\right)^{-\frac{1}{2}} U\left(\frac{1}{2}\gamma\lambda\right) T\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]$$

$$+ \left[\varepsilon(\varphi\eta^2)^3\right]^{-\frac{1}{4}} T\left(\frac{1}{2}\gamma\lambda\right) U\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right],$$

$$\Xi_2(\lambda) \equiv U\left(\frac{1}{2}\gamma\lambda\right) V\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]$$

$$+ \left(\frac{\varepsilon\eta^2}{\varphi}\right)^{\frac{1}{4}} T\left(\frac{1}{2}\gamma\lambda\right) S\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]$$

$$+ \left(\varepsilon\varphi\eta^4\right)^{-\frac{1}{2}} S\left(\frac{1}{2}\gamma\lambda\right) T\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]$$

$$+ \left[\varepsilon(\varphi\eta^2)^3\right]^{-\frac{1}{4}} V\left(\frac{1}{2}\gamma\lambda\right) U\left[\frac{1}{2}(1-\gamma)\lambda\left(\frac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right],$$

the solution to the eigenvalue problem (4.7) is sought in the form

$$
(4.10)\qquad g(x)l^{-2} = \begin{cases} A(\lambda)S\left(\lambda\dfrac{|x|}{l}\right) + U\left(\lambda\dfrac{|x|}{l}\right), & \\ \hfill \text{if} \quad |x| \le \dfrac{1}{2}\gamma l, & \\ A(\lambda)\left\{S\left(\dfrac{1}{2}\gamma\lambda\right)S\left[\lambda\left(\dfrac{|x|}{l}-\dfrac{1}{2}\gamma\right)\left(\dfrac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]\right. & \\ \quad + \left(\dfrac{\varphi\eta^2}{\varepsilon}\right)^{-\frac{1}{4}} V\left(\dfrac{1}{4}\gamma\lambda\right) T\left[\lambda\left(\dfrac{|x|}{l}-\dfrac{1}{2}\gamma\right)\left(\dfrac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right] & \\ \quad + (\varphi\varepsilon\eta^4)^{-\frac{1}{2}} U\left(\dfrac{1}{2}\gamma\lambda\right) U\left[\lambda\left(\dfrac{|x|}{l}-\dfrac{1}{2}\gamma\right)\left(\dfrac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right] & \\ \quad \left. + [\varepsilon(\varphi\eta^2)^3]^{-\frac{1}{4}} T\left(\dfrac{1}{2}\gamma\lambda\right) V\left[\lambda\left(\dfrac{|x|}{l}-\dfrac{1}{2}\gamma\right)\left(\dfrac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right]\right\} & \\ + U\left(\dfrac{1}{2}\gamma\lambda\right) S\left[\lambda\left(\dfrac{|x|}{l}-\dfrac{1}{2}\gamma\right)\left(\dfrac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right] & \\ + \left(\dfrac{\varphi}{\varepsilon\eta^2}\right)^{-\frac{1}{4}} T\left(\dfrac{1}{2}\gamma\lambda\right) T\left[\lambda\left(\dfrac{|x|}{l}-\dfrac{1}{2}\gamma\right)\left(\dfrac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right] & \\ + (\varphi\varepsilon\eta^4)^{-\frac{1}{2}} S\left(\dfrac{1}{2}\gamma\lambda\right) U\left[\lambda\left(\dfrac{|x|}{l}-\dfrac{1}{2}\gamma\right)\left(\dfrac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right] & \\ + \left[(\varphi\eta^2)^3\right]^{-\frac{1}{4}} V\left(\dfrac{1}{2}\gamma\lambda\right) V\left[\lambda\left(\dfrac{|x|}{l}-\dfrac{1}{2}\gamma\right)\left(\dfrac{\varphi}{\varepsilon\eta^2}\right)^{\frac{1}{4}}\right], & \\ \hfill \text{if} \quad |x| \in \left(\dfrac{1}{2}\gamma l, \dfrac{1}{2}l\right], & \end{cases}
$$

where $A(\lambda) \equiv -C\Xi_1(\lambda)\Gamma_1(\lambda)^{-1}$, C is constant. Restricting our considerations to symmetric vibrations, the equation for the eigenvalue λ takes the form

$$
\Gamma_1(\lambda)\Xi_2(\lambda) - \Gamma_2(\lambda)\Xi_1(\lambda) = 0.
$$

From the above equation we can derive eigenvalues λ_A, $A = 1, 2, \ldots$. Our analysis is restricted to $A = 1$. Hence, we obtain the smallest eigenvalue λ dependent on the parameters φ, ε, η, and the exact form of the mode-shape function g related to this eigenvalue is defined by (4.10).

5. Numerical results

In this section it will be shown that the mode-shape function g for the interval in Fig. 2 related to the eigenvalue λ (for $A = N = 1$) can be assumed in the approximate form

$$g(x) = l^2[\cos(2\pi x/l) + c], \qquad x \in (-l/2, l/2), \tag{5.1}$$

where the constant c is derived from the condition $< \mu g >= 0$ is $c =< \mu >^{-1}$ $< \mu \cos(2\pi x/l) >$.

5.1. The application of exact and approximate form of mode-shape functions to analyse free vibrations of a periodic plate band

In order to evaluate the differences between the results obtained by using exact (4.9) or (4.10) and approximate form (5.1) of mode-shape function, free vibration frequencies of the periodic plate bands from Sec. 4 will be analysed. Taking into account (4.3) we denote the frequencies obtained for the approximate form (5.1) of mode-shape function by $\tilde{\omega}$, and those for the exact form (4.9) or (4.10) by ω. Introduce dimensionless coefficients:

$$q \equiv l/L, \qquad \Omega^- \equiv \tilde{\omega}_1/\omega_1, \qquad \Omega^+ \equiv \tilde{\omega}_2/\omega_2,$$

where q is called the dimensionless mesostructure parameter, Ω^- and Ω^+ are the ratios of lower and higher free vibration frequencies, respectively. Some numerical results are presented below.

- *A plate band with periodically distributed concentrated masses*

For the plate band with constant properties and with periodically distributed concentrated masses M we have: $\zeta \neq 0$, $\varepsilon = \varphi = \eta = 1$. The diagram of Ω^+ versus the parameter $\zeta \in [0, 50]$ is presented in Fig. 3a. This diagram is made for $\eta_0 = 0.1$, $q = 0.01$.

- *A plate band with periodically varying mass density ρ*

For the plate band without masses and with periodically varying mass density ρ given by (4.6b) we have $\zeta = 0$, $\varepsilon = \eta = 1$, $\varphi \leq 1$. Diagrams of ratio Ω^+ versus the parameter $\varphi \in [0.2, 1]$ are shown in Fig. 3b. It is made for parameters $\eta_0 = 0.1$, $q = 0.01$, $\gamma = 0.25, 0.5, 0.75$.

- *A plate band with periodically varying Young's modulus E*

For the plate band with periodically varying Young's modulus E given by (4.6a), i. e. $\zeta = 0$, $\varphi = \eta = 1$, $\varepsilon \leq 1$, diagrams of ratios Ω^-, Ω^+ versus the parameter $\varepsilon \in [0.1, 1]$ are shown in Fig. 4. It is made for parameters $\eta_0 = 0.1$, $q = 0.01$, $\gamma = 0.25, 0.5, 0.75$.

- *A plate band with periodically varying thickness h*

For the plate band with the plate thickness h periodically varying according to (4.6c), i. e. $\zeta = 0$, $\varepsilon = \varphi = 1$, $\eta \leq 1$, diagrams of ratios Ω^-, Ω^+ versus the parameter $\eta \in [0.7, 1]$ are shown in Fig. 5. It is made for parameters $\eta_0 = 0.1$, $q = 0.01$, $\gamma = 0.25, 0.5, 0.75$.

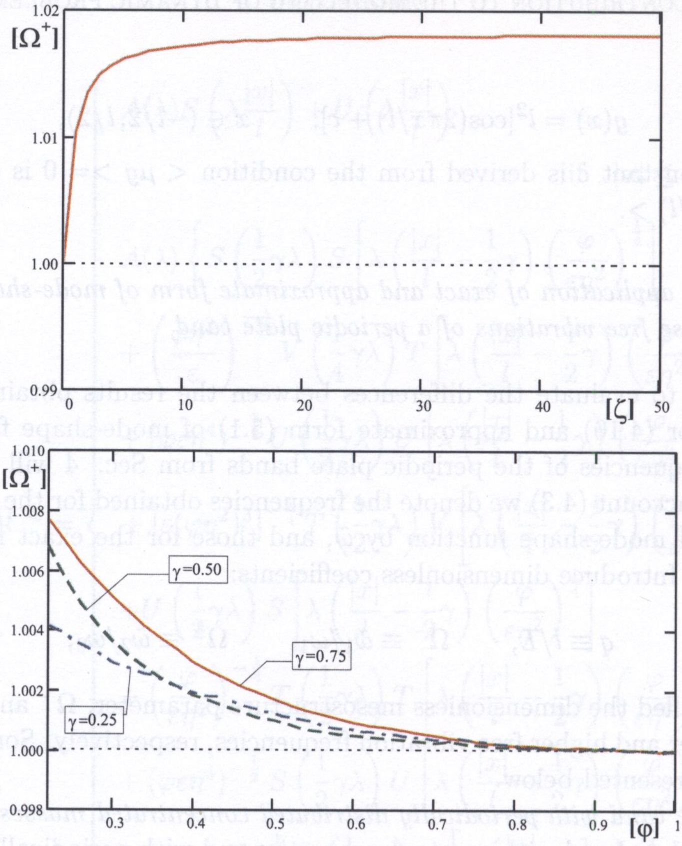

FIG. 3. Diagrams of ratio Ω^+ for: a) the plate band with periodically distributed concentrated masses M, b) for the plate band with periodically varying mass density ρ.

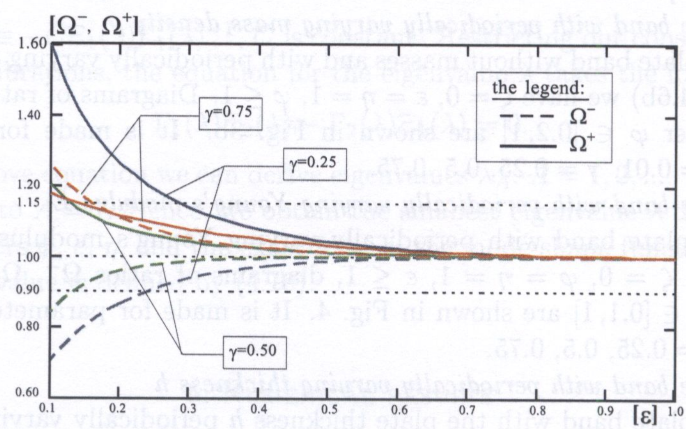

FIG. 4. Diagrams of ratios Ω^-, Ω^+ for the plate band with periodically varying Young's modulus E.

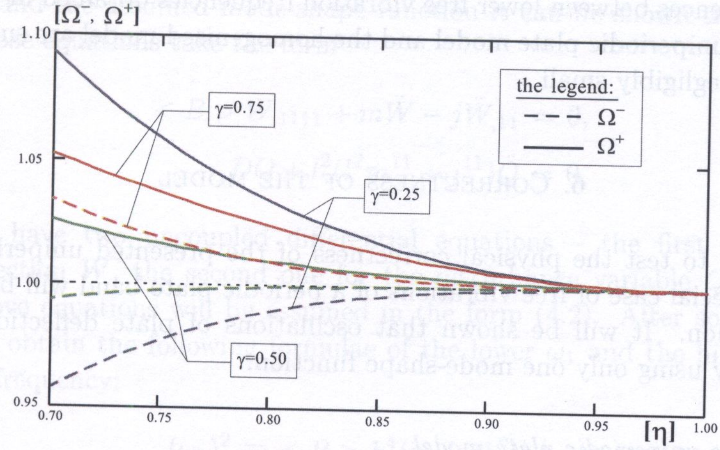

FIG. 5. Diagrams of ratios Ω^-, Ω^+ for the plate band with periodically varying thickness h.

5.2. *Discussion of the obtained results*

1° We can observe that in the framework of both (the uniperiodic and homogenised plate) models, the lower free vibration frequencies obtained by using the approximate form of mode-shape function (5.1) are identical with those obtained by the exact form of that function for the following cases:

– plates having constant thickness and all material properties, with periodically distributed concentrated masses,

– plates having constant thickness and all material properties except the mass density ρ being periodically variable (given by (4.6b)).

2° Differences between higher free vibration frequencies obtained for the two above cases of plates by using the exact or approximate forms of mode-shape function are very small (for large value of mass M, $\zeta = 50$, the difference is less than 2%).

3° For plates having periodically varying Young's modulus E or thickness h (given by (4.6a) or (4.6c)) and all the remaining properties constant, differences between the free vibration frequencies obtained by using the exact or approximate form of mode-shape function are visible not only for higher but also for lower free vibration frequencies:

– differences between lower free vibration frequencies are greater than 10% for $\varepsilon \leq 0.25$ or than 5% for $\eta \leq 0.70$,

– differences between higher free vibration frequencies are greater than 15% for $\varepsilon \leq 0.25$ or than 10% for $\eta \leq 0.70$.

4° Differences between lower free vibration frequencies obtained in the framework of the uniperiodic plate model and the homogenised model are smaller than 10^{-4}, i. e. negligibly small.

6. Correctness of the model

In order to test the physical correctness of the presented uniperiodic plate model, a special case of free vibrations of a periodic plate band will be analysed in this section. It will be shown that oscillations of plate deflections can be described by using only one mode-shape function.

6.1. The uniperiodic plate model

Let us consider a simply supported plate band with periodically varying thickness h and with span L along the x-axis, and made of an isotropic homogeneous material. The periodicity interval is shown in Fig. 6. The thickness h is given by

$$h(x) = \begin{cases} h_0 & \text{if } x \in (-(1+\gamma)l/4, -(1-\gamma)l/4) \cup ((1-\gamma)l/4, \\ & \quad (1+\gamma)l/4), \\ h_1 & \text{if } x \in [-l/2, -(1+\gamma)l/4] \cup [-(1-\gamma)l/4, \\ & \quad (1-\gamma)/l/4] \cup [(1+\gamma)l/4, l/2], \quad \gamma \in [0,1]. \end{cases} \tag{6.1}$$

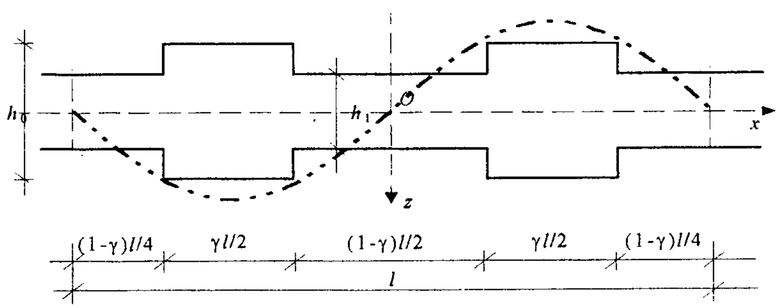

FIG. 6. The periodicity interval with the mode-shape function g.

Moreover, we would like to analyse oscillations of plate deflections described by one mode-shape function assumed as an approximate solution to eigenvalue problem (4.7):

$$g(x) = l^2 \sin(2\pi x/l). \tag{6.2}$$

It can be shown that $<\mu g>= 0$, $<\vartheta g_{,1}>= 0$. To investigate free vibrations of the plate band within the uniperiodic plate model we will apply Eqs. (4.1). For the plate and the assumed mode-shape function it can be shown that $D_{11} = 0$. Hence, those equations take the form

$$\begin{aligned} < B > W_{,1111} + m\ddot{W} - j\ddot{W}_{,11} &= 0, \\ DQ + l^2(l^2 m^{11} + j^{11})\ddot{Q} &= 0, \end{aligned} \tag{6.3}$$

where we have two uncoupled differential equations – the first one for the macrodeflection W, the second one for the disturbance variable Q. Solutions to the above equations will be assumed in the form (4.2). After some manipulations we obtain the following formulae of the lower ω_1 and the higher ω_2 free vibration frequency:

$$\begin{aligned} (\omega_1)^2 &\equiv < B > k^4(m + jk^2)^{-1}, \\ (\omega_2)^2 &\equiv Dl^{-2}(l^2 m^{11} + j^{11})^{-1}, \end{aligned} \tag{6.4}$$

where $k = 2\pi/L$. It can be observed that for the plate under consideration and the assumed mode-shape function, only the higher frequency ω_2 depends on the mesostructure length parameter l. In the sequel, our considerations will be restricted only to the analysis of the higher free vibration frequency ω_2. Using notations of D, B, m^{11} and j^{11} from Sec. 4, we have

$$\begin{aligned} D &= < B(g_{,11})^2 >= \frac{2E\pi^3}{3(1-\nu^2)}\left\{(h_0^3 - h_1^3)[\pi\gamma + \sin(\pi\gamma)] + \pi h_1^3\right\}, \\ m^{11} &= l^{-4} < \mu g^2 >= \frac{1}{2\pi}\rho\{(h_0 - h_1)[\sin(\pi\gamma) + \pi\gamma] + \pi h_1\}, \\ j^{11} &= l^{-2} < \vartheta(g_{,1})^2 >= \frac{1}{6}\rho\pi\left\{(h_0^3 - h_1^3)[\pi\gamma - \sin(\pi\gamma)] + \pi h_1^3\right\}. \end{aligned}$$

Substituting the above coefficients to $(6.4)_2$, the higher frequency $\overline{\omega}_2$ is given by

$$(\overline{\omega}_2)^2 = \frac{E}{(1-\nu^2)\rho} \frac{4\pi^4\{(h_0^3 - h_1^3)[\pi\gamma + \sin(\pi\gamma)] + \pi h_1^3\}}{\{3l^2[(h_0 - h_1)(\sin(\pi\gamma) + \pi\gamma) + \pi h_1] + \pi^2[(h_0^3 - h_1^3)(\pi\gamma - \sin(\pi\gamma)) + \pi h_1^3]\}\, l^2}. \tag{6.5}$$

6.2. The Ritz method

In order to evaluate the results obtained above let us consider a simply supported plate band with span l and with varying thickness h, Eq. (6.1). The plate is shown in Fig. 6. The Ritz method will be used to investigate free vibrations of the plate under assumptions of the Kirchhoff plate theory. For the plate we can write the following formulae of the potential $\mathcal{E}$ and the kinetic $\mathcal{K}$ energy:

$$\mathcal{E} = \frac{1}{2}\int_{\Pi}\int_{-h/2}^{h/2} c_{\alpha\beta\gamma\delta}e_{\alpha\beta}e_{\gamma\delta}dzda, \qquad \mathcal{K} = \frac{1}{2}\int_{\Pi}\int_{-h/2}^{h/2} \rho(\dot{u}_i)^2 dzda.$$

Using the well known Kirchhoff plate theory relations and assuming the plate band deflection in the form

$$w = A\sin(2\pi x/l)\cos(\breve{\omega}t),$$

the maximal potential and kinetic energy can be written as

$$\mathcal{E}_{\max} = A^2\frac{E}{3(1-\nu^2)}\frac{\pi^3}{l^3}[(h_0^3 - h_1^3)(\pi\gamma + \sin(\pi\gamma)) + \pi h_1^3],$$

$$\mathcal{K}_{\max} = A^2\breve{\omega}^2\rho\left\{\frac{1}{4\pi}l[(h_0 - h_1)(\pi\gamma + \sin(\pi\gamma)) + \pi h_1] + \frac{1}{12}\frac{\pi}{l}[(h_0^3 - h_1^3)(\pi\gamma - \sin(\pi\gamma)) + \pi h_1^3]\right\}.$$

The Ritz method demands that

$$\frac{d(\mathcal{E}_{\max} - \mathcal{K}_{\max})}{dA} = 0.$$

Hence, we arrive at the formula for the free vibration frequency $\breve{\omega}$

$$\breve{\omega}^2 = \frac{E}{(1-\nu^2)\rho} \frac{4\pi^4\{(h_0^3 - h_1^3)[\pi\gamma + \sin(\pi\gamma)] + \pi h_1^3\}}{\{3l^2[(k_0 - h_1)(\sin(\pi\gamma) + \pi\gamma) + \pi h_1] + \pi^2[((h_0^3 - h_1^3)(\pi\gamma - \sin(\pi\gamma)) + \pi h_1^3]\}l^2}, \tag{6.6}$$

which is identical with the higher frequency (6.5). Summing up, it can be observed that for special cases, the results obtained in the framework of the presented uniperiodic plate model can be verified by the known method, such as the Ritz method.

7. Final remarks

The length-scale model derived in [23] has been applied in the paper to analyse vibrations of plates with periodic structure along one direction and constant structure along the perpendicular direction. The refined models, formulated in the framework of the averaged length-scale theory of periodic bodies [21 – 22], were applied to many dynamic problems of periodic structures in several papers [2, 4, 6 – 9, 11, 16 – 20] aforementioned in Sec. 1. The models are physically reasonable and simple enough to be applied in the analysis of engineering problems.

The presented modelling procedure for periodic plates leads to a system of differential equations with constant coefficients for the macrodeflection W and the disturbance variables Q^A, $A = 1, ..., N$. The governing equations (3.2) – (3.4) take into account the length-scale effect on the plate behaviour by certain coefficients (underlined in these equations) dependent on the mesostructure length parameter l. The length-scale effect is described by the extra unknowns Q^A, $A = 1, ..., N$, called disturbance variables, and by additional mode-shape functions g^A. Mode-shape functions g^A describe oscillations of deflections inside the periodicity intervals. These functions should be obtained as properly chosen approximations of solutions to the eigenvalue problems for natural vibrations of separated periodicity intervals. It was shown in Sec. 3 and 4. For uniperiodic plates, the mode-shape functions g^A are l-periodic functions.

Mode-shape functions obtained in Sec. 4 for special cases of a plate band were used to analyse free vibrations for those plates in Sec. 5. The results presented there make it possible to evaluate the differences between the free vibration frequencies obtained by using the approximate and exact forms of the mode-shape function. Analysing these results, we can formulate the following general conclusions:

1° For many special problems the mode-shape functions g^A, $A = 1, ..., N$, can be assumed as approximate solutions to the eigenvalue problem for the periodicity interval. It is sufficient from the computational point of view.

– Plates with all properties constant and with periodically distributed concentrated masses or with periodically varying mass density ρ, can be analysed by using the approximate form (5.1) of mode-shape function.

– Plates with all properties constant excluding the Young's modulus E or thickness h which are periodically variable, can be investigated by using the approximate form of mode-shape function if differences between the values of the Young's modulus or the thickness inside the periodicity interval are relatively small, i. e. $\varepsilon \geq 0.25$ or $\eta \geq 0.70$; for $\varepsilon \leq 0.25$ or $\eta \leq 0.70$ the exact form of mode-shape function should be used.

2° Comparing the results obtained in the framework of both the models – the uniperiodic plate and the homogenised models, it can be observed that

– higher free vibration frequencies (and also higher order vibrations) caused by a periodic plate structure can be analysed only in the framework of the presented uniperiodic plate model,

– lower free vibration frequencies (and also lower vibrations) can be investigated within the homogenised model.

3° Comparison between the results obtained for the special case of the plate band by using the uniperiodic plate model and the Ritz method makes it possible to confirm that the presented model is physically correct.

Summarising, the analytical exact solutions to the eigenvalue problem (3.1) can be obtained only for plates having intervals with rather not very complicated structure. In most cases, instead of the exact solutions to eigenvalue problems, we have to look for an approximate form of mode-shape functions.

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