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YIELD SURFACES AND CRITERIA OF PLASTIC YIELDING FOR A STRAIN HARDENING MATERIAL PART 2. THEORETICAL ANALYSIS

J. MIASTKOWSKI, R. SZCZEBIOT

INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH
POLISH ACADEMY OF SCIENCES
ul. Świętokrzyska 21, 00–049 Warsaw, Poland

The paper presents the experimental verification of the generalized Mróz hardening hypothesis under the conditions of complex state of stress. The analysis performed takes into account the changes in size and position in the stress space of the triaxial ellipsoid illustrating the Huber-Mises-Hencky yield condition for the plane state of stress under definite plastic strains.

1. Introduction

Real materials, most of all metals, under loading which leads to permanent strains are hardened after unloading and subsequent loading. Up to now, many hypotheses describing this phenomenon have been proposed. The most popular concepts of great practical importance are:

• The hypothesis of isotropic strain hardening. Its full analysis was given by HODGE and PRAGER [8].

• The hypothesis of kinematic strain hardening introduced and designated by PRAGER [9]. It was later developed by SHIELD and ZIEGLER [10] in the nine-dimensional space.

Both these hypotheses, in spite of their simplicity and facility, often do not give satisfactory results in the description of real material behaviour. KADASHEVITCH and NOVOZHILOV [11] combined both the hypotheses introducing a concept of microstress which occurs in the material as a result of permanent strains. It is a mathematical description similar to that by Ziegler. For a description of hardening, PRAGER [1] suggested an idea of linear segments approximation of the yield surface. BATDORF and BUDIANSKY [2] proposed the theory of hardening based on the concept of the grain slip. The theory was modified by LIN [3]. BESSELING [4] based the hardening theory on the assumption that the material is composed of sub-elements of certain elastic-plastic properties. Wells and PASLAY [5] gave a similar description of strain hardening in the case of material heterogeneity.

MRÓZ [6] proposed the interesting hypothesis of strain hardening. It may be modified according to the material properties and allows to obtain the results which comply better with the experiments. However, the hypothesis was not popular due to its complexity and difficulties at practical applications.

The hypothesis of isotropic hardening assumes that, as a result of the strain process, the initial yield surface expands uniformly in all directions while its position in the stress space as well as its geometrical form remain unchanged. To define a new surface it is sufficient to know the point lying outside the initial surface, which corresponds to the current state of stress. Through this point we may draw (without turning) the surface similar to the initial one but of different size.

The hypothesis of kinematic hardening assumes that the total yield surface, in the stress space, translates as a rigid body (surface) under plastic strains. Its form and size do not change then. Similarly to the isotropic hardening hypothesis, if the point (in the stress space) corresponding to the current stress is known, then it is possible to draw uniquely (without turning) the surface similar to the initial one, of the same size but shifted with respect to the initial one.

Many other hypotheses describing the material hardening have been developed but they did not gain popularity. Despite the passage of time, the above-mentioned hypotheses are still cited by the researchers in their considerations. They are used to verify the new concepts both in the experimental and theoretical investigation.

2. GENERALIZED MRÓZ THEORY FOR DESCRIPTION OF THE PLASTIC DEFORMATION OF THE MATERIAL

In the Mróz strain hardening theory, the initial properties of the material are approximated by a family of yield surfaces. The surfaces bound the fields of constant hardening moduli in such a way that the $\sigma(\varepsilon)$ curve may be approximated by n linear segments to define the constant hardening moduli: $K_1, K_2, ...K_n$. The hardening moduli K_i correspond to f_i surfaces.

The yield surfaces are defined by the yield functions:

$$\mathbf{f}(\mathbf{s}_{ij} - \alpha_{ij}) = \mathbf{F}(\lambda) \tag{2.1}$$

where: α_{ij} – position of the yield surface centre, α – parameter describing the growth of the yield surface.

In the hypothesis, the new law of motion of the surfaces in the stress space was moreover assumed. It is graphically presented in Fig. 1.

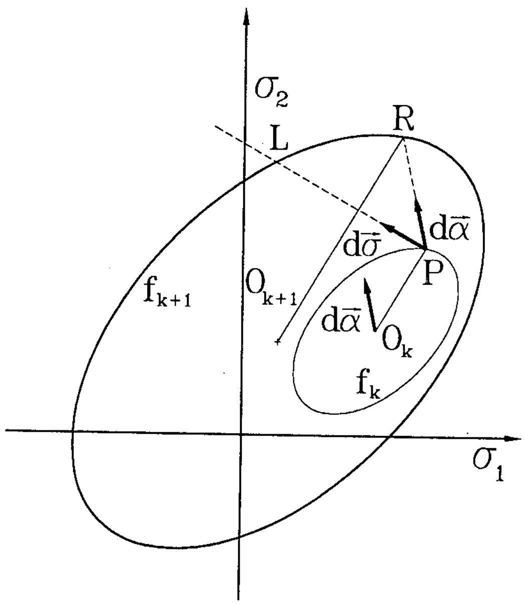

FIG. 1.

Under loading in the direction of $\mathbf{d}\sigma$ (Fig. 1), the centre O_k of the surface f_k moves towards PR defined in such a way that the normals to the surfaces f_k and f_{k+1} are parallel at the points P and R. If the stress varies by $\mathbf{d}\sigma$, then the centre of the surfaces moves by $d\alpha$:

$$d\alpha = d\mu * PR, \tag{2.2}$$

where $d\mu$ is a scalar coefficient. Now let us consider a simple case to illustrate the motion of the yield surface in the stress space.

For the initially isotropic material whose $\sigma(\varepsilon)$ diagram is presented in Fig. 2a, the surfaces f_i are similar and have a common centre at the point 0. If the stress point (Figs. 2a and 2b) moves from 0 towards C, it reaches the surface f_0 at A and the surface f_0 moves together with the stress point until it contacts the surface f_1 at B. The two tangential surfaces f_0 and f_1 move together with the stress point from B to C, where they reach the surface f_2 (Fig. 2c).

During the process of unloading, the surface f_0 starts to move when the stress point reaches E and the two tangential surfaces f_0 and f_1 translate further from F (Fig. 2d). Taking this way of the surface motion in the stress space, it is assumed that the surfaces do not intersect each other but they remain in contact and move together.

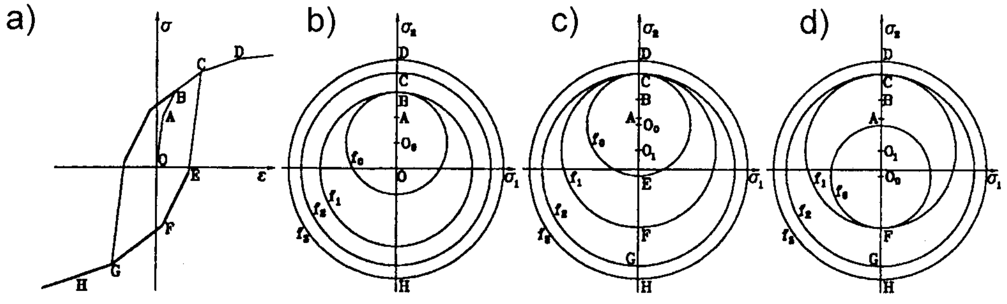

FIG. 2.

The Mróz hypothesis may be reduced to the Ziegler concept of kinematic hardening [7] by assuming that all surfaces except the first one lie at infinity. Then the vector $\mathbf{d}\alpha$ is parallel to the vector joining the stress point to the centre of the yield surface. So we get the Eq. (2.1) proposed by Ziegler and field of the hardening moduli is reduced to the linear hardening in the entire loading range,

$$d\alpha_{ij} = d\mu(\sigma_{ij} - \alpha_{ij}). \tag{2.3}$$

In the Mróz hypothesis it is also possible to change the size of the yield surface. The parameter α responsible for it is defined as equivalent to the parameter κ of the equation:

$$\kappa = W^P = \int \sigma_{ij} d\varepsilon_i^P. \tag{2.4}$$

In the case of $\varepsilon_i^p = 0$, also $\kappa = 0$.

3. Theoretical analysis of behaviour of the plastic deformation of the material based on the Mróz hardening hypothesis

Assume that the $\sigma(\varepsilon)$ curve, for the material studied, is approximated by the linear segments connecting the points corresponding to the following permanent strains: 0%, 0.01%, 0.02%, 0.03%, 0.04%, 0.05%, 0.1%, 0.2%, 0.3%, 0.4%, 0.5%. It

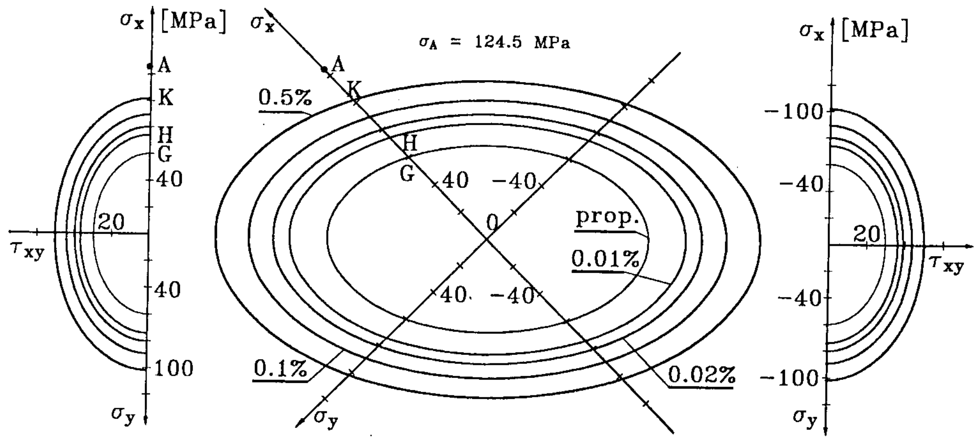

FIG. 3.

corresponds to the assumption of the family of yield surfaces bounding the fields of constant hardening moduli shown Fig. 3.

Assume that these surfaces, under prestrains of the material, do not change their size but translate in the stress space according to the rules of the Mróz hardening hypothesis.

The stresses which would create prestrains of the material according to the test programme, correspond to the points lying on the positive axis of σ_x.

Moving from the point 0 along the trajectory of initial stress we reach the point G (Fig. 3). From this point the surface of proportionality starts to move. From point H, the two tangential surfaces start to move. With further increase in stress σ_x, the corresponding surfaces start to move and all the above-defined surfaces are already tangent at K. From that place they move together. When the stress reaches the value of 124.5 MN/m^2 (it corresponds to the prestrains equal to 1.74%), then the position of the surfaces in the stress space is such as that shown in Fig. 4. In such a configuration of the yield surfaces, the point 0 lies outside the surface of proportionality.

Under unloading to the point 0 (during the test) (Fig. 4), position of the yield surface does not change till it reaches the opposite edge of the proportionality surface i.e. the point M. From this moment on, the surface of proportionality moves to touch the point 0. The surfaces configuration shown in Fig. 5 corresponds (according to the Mróz theory) to the total unloading of the material after the earlier tension producing the permanent strains of 1.74%.

This configuration of the yield surfaces should appear (theoretically) at the study of the material characterized by the above history of loading. Assuming the homogeneous field of stress and strains for a large specimen, the small specimens (cut out of the central part in the large one) exibit the same distribution of the yield surfaces at the beginning (Part 1). Certainly, the position of the surface will change under tension or compression of each small specimen but we were interested in the point of meeting the specific surface in the prescribed direction of loading, from the point 0 where it started to move. It corresponds to the position of the yield surface after preloading.

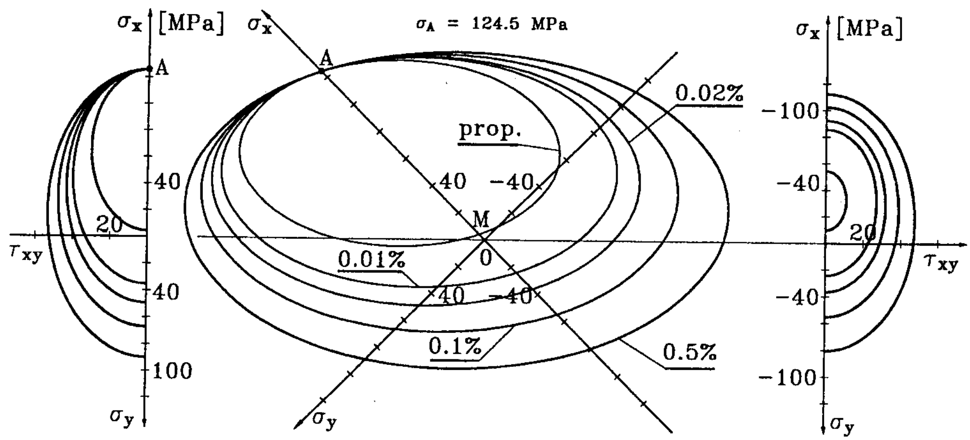

FIG. 4.

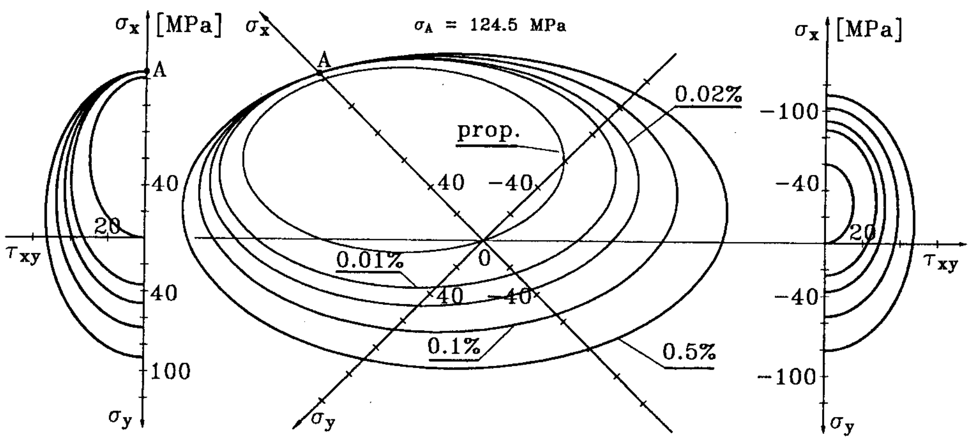

FIG. 5.

Similarly to the first level of the initial stress (124.5 MN/m^2), the positions for the others is obtained: 154.6 MN/m^2 and 185.5 MN/m^2. But when the stress reaches 154.6 MN/m^2, the following three surfaces pass the point 0: the proportionality surface, 0.01% and 0.02%. When the stress is 185.5 MN/m^2, then the nine of the surfaces under consideration pass the point 0. These are the following surfaces: the proportionality surface, 0.01%, 0.02%, 0.03%, 0.04%, 0.05%, 0.1%, 0.2%, 0.3%.

The observed discrepancies between the obtained theoretical yield surfaces and the experimental results show that changes of the yield sueface dimensions due to plastic strains should be taken into account in theoretical.

As it was just mentioned, the Mróz hypothesis enables us to take into account the change in size of the yield surface. Assume that the dimensions of the yield surface will be experimentally determined by the parameters calculated by the method of least squares, and these surfaces will move in accordance with the Mróz theory rules.

Figure 6 presents the surfaces $\sigma_{0.02}$ obtained experimentally (solid lines) and the theoretical ones (dashed lines) for the material in initial state, and for the three different values of prestrains. Similar comparisons are illustrated in Fig. 7 for the surfaces $\sigma_{0.1}$, and in Fig. 8 for the surfaces $\sigma_{0.5}$.

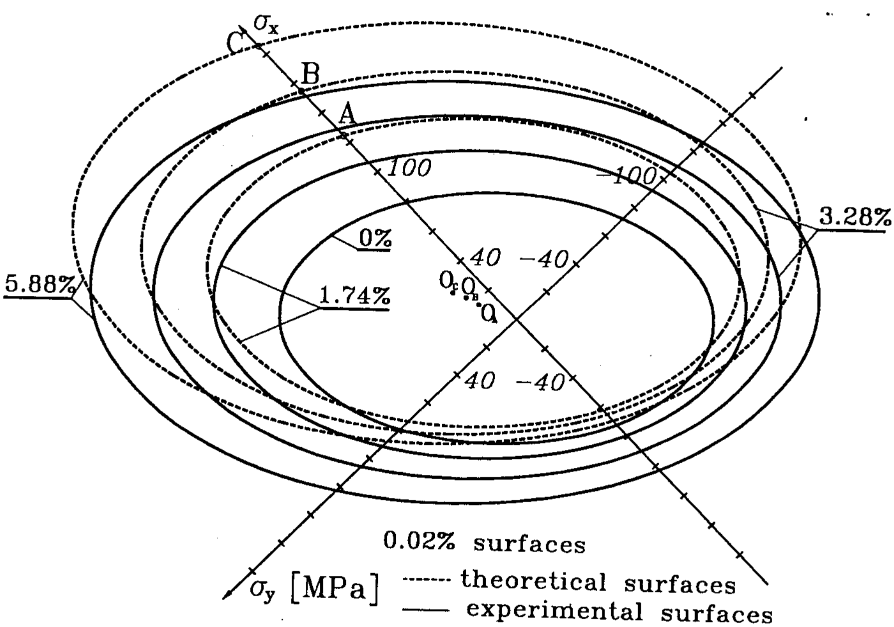

FIG. 6.

Figures 6, 7, 8 show a considerably better conformity of the experimental surface with the Mróz theoretical ones than the Mróz surfaces which did not change their size but were translated only. It is clearly seen in Figs. 7 and 8 that the surfaces mentioned almost overlap. Slightly larger discrepancies remain for the surfaces defined by the small plastic strains, what is illustrated in Fig. 6 for the surfaces $\sigma_{0.2}$. After all, it is a significant improvement in comparison to the considerations disregarding the isotropy.

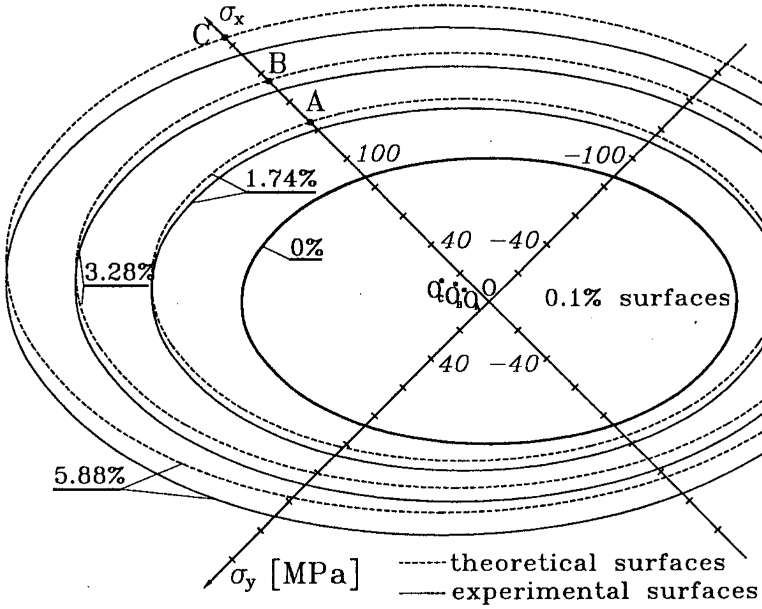

FIG. 7.

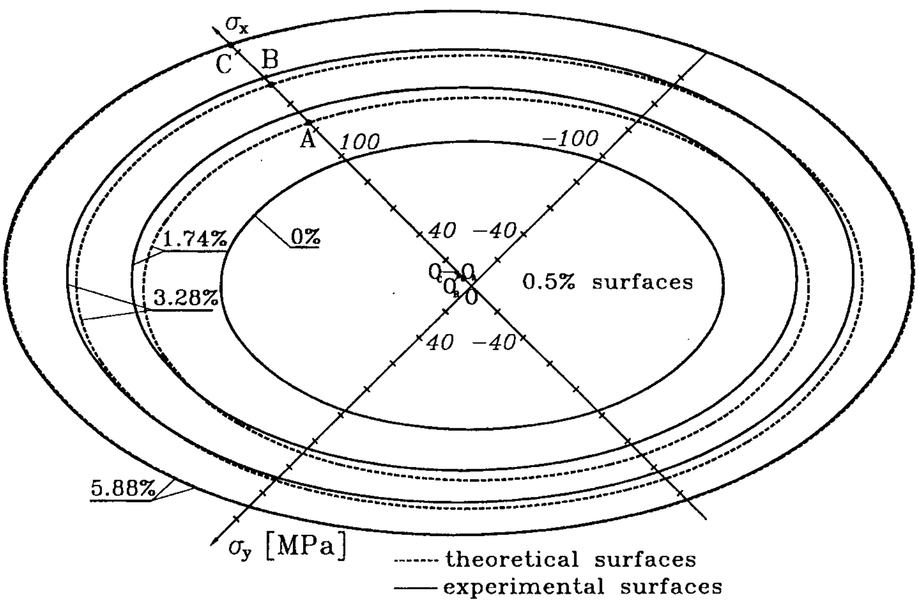

FIG. 8.

4. Conclusions

Basing on the investigation results, for the plastically strained material, an analysis was performed of the quantitative participation of the isotropic and kinematic hardening hypotheses in the behaviour of the yield surfaces.

The analysis enabled us to determine the change in size of the surface and the position of its centre as functions of the initial plastic strain and, additionally, to find a dependence of these parameters on various definitions of the surfaces.

The following conclusions result from the investigation performed and the subsequent analysis: The parameters of the yield surfaces, for a plastically strained material, depend on the size of the permanent deformations.

With increasing initial strains of the material, the size of the surface and its displacement in the direction of the initial stress also grow. The intensity of change in the surface size and the position of its centre gradually decreases as the initial stress increases.

The parameters defining the size of the surface and the position of its centre considerably depend on the accepted definition according to which these surfaces were determined.

The largest displacement of the centre occurs for the surface of proportionality. For the surfaces defined by larger permanent sets, these displacements gradually decrease and then the surfaces (0.5 exhibit only a slight displacement of the centre.

In the case of the strain diagrams considered in the article, the Mróz hypothesis gives a good description of the behaviour of the material under plastic strains but only when isotropy is taken into account.

References

1. W. Prager, *On the use of singular yield conditions and associated flow rules*, J. Appl. Mech., **20**, 317–320, 1953.

2. S. B. Batdorf, B. Budianski, *A mathematical theory of plasticity based on the concept of slip*, NACA Technical Note 1871, 1949.

3. T. N. Lin, *A proposed theory of plasticity based on slips*, Proc. of the 2^{nd} U.S. National Congress of Appl. Mech., 461–468, 1954.

4. J. F. Besseling, *A theory of plastic flow for anisotropic hardening in plastic deformation of an initially isotropic material*, Nat. Aero. Inst., Amsterdam Rep. S-410, 1953.

5. C. H. Welis, P. R. Paslay, *A small-strain plasticity theory for planar slip materials*, J. Appl. Mech., 1969.

6. Z. Mróz, *On the description of anisotropic workhardening*, Mech. Phys. Solids, **15**, 163–175, 1967.

7. H. Ziegler, *A modification of Prager's hardening rule*, Quart. Appl. Math., **17**, 55–65, 1959.

8. P. G. Hodge W. Prager, *A variational principle for plastic materials with strain hardening*, J. Math. And Phys., **27**, 1–10, 1948.

9. W. Prager, *The theory of plasticity – a survery of recent archievements*, James Clayton Lecture, Proc. Inst. Mech. Engng., **169**, 41–57, 1955.

10. R. T. Shield, H. Ziegler, *On Prager's hardening rule*, Angew. Math. Phys. (ZAMP), Vol. IXa, 260–276, 1958.

11. Q. I. Kadaszewicz, W. W. Nowożiłow, *Theory of plasticity taking into account residual microstresses*, Prikl. Mat. Mech., **22**, 78–89, 1958.

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