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OBJECTIVE FUNCTIONS IN MONOCRITERIAL AND MULTICRITERIAL OPTIMIZATIONS PROBLEMS AGAINST A LOSS OF DYNAMIC STABILITY

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The paper concerns the specification and comparison of numerical examples of optimization of beams in the state of periodic parametric resonance with respect to different measures of the phenomena considered, i.e., with respect to different optimization criteria – some objective functions in monocriterion and multicriterial optimization. A formulation of monocriterion and multicriterial optimization problems for mechanical elements, subjected to a parametrically exciting force periodic in time, is given. In multicriterial optimization the scalar objective functions characterizing the parametric resonance are introduced. The paper deals with the problems of finding the control function - function of the shape (the area of cross-section of the beam) which maximizes or minimizes the objective functions under the constraint of constant volume. In some cases the optimization problems under conditions of parametric resonance reduce to the optimization problems with respect to natural frequency. The examples of parametric optimization against loss of stability are solved and analysed.

1. Introduction

Papers [1 – 6] present a brief history of parametric phenomena from Faraday and Lord Rayleigh to modern physics. In many problems, the motion of systems is described by means of the Mathieu–Hill equations. In recent years, many new applications of Mathieu equation have been found [6 – 8]. Usually the parametric resonance is a very dangerous and undesirable phenomenon in mechanical systems. Hence our aim is to avoid the resonance states or to minimize their disadvantageous effects. One of the methods leading to this is optimal structural design. In optimization procedure the resonance effect should be minimized by optimization (maximization or minimization) of some measures of the phenomenon – some objective functions.

There is a small number of papers devoted to the problems of variational optimization under dynamic stability constraints. A significant point in the monocriterion and multicriterial variational optimization procedure is determining and introducing appropriate measures of the phenomena considered, i.e.optimization criteria – some objective functions.

The problems of optimization at the loss of dynamic stability were discussed in paper [9] where, for the first time, the objective functions were introduced as some measures of dynamic instability region. Next, in paper [10] the variational optimization problems for a simply supported beam subjected to a longitudinal force periodic in time were formulated. The results presented in this paper were generalized for a beam with other boundary conditions and for multidimensional elements (e.g. the optimization of parametrically excited plate problems) in a monograph by FORYŚ [11]. In the monograph we formulated and introduced physically motivated quantities characterizing parametrically excited systems as an objective function. Some suggestions with reference to multi-criterial optimization of a system in the state of parametric resonance are presented. Next the problems of optimization discussed in the monograph are continued in papers [12, 13], devoted especially to multicriterial optimization in paper [13].

The present paper concerns the specification and comparison of numerical examples of optimization of a system in the state of periodic parametric resonance with respect to different measures of the phenomena considered, i.e. with respect to different optimization criteria – some objective functions in monocriterial and multicriterial optimization. The mechanical elements under consideration are most often made of the Kelvin–Voigt viscoelastic material. Problems of system optimization in periodic parametric resonance are reduced to static considerations.

2. Parametrically excited systems – equations of motion

The equation of motion of a non-damped "non-prismatic" parametrically excited elastic elements has the following form, cf. FORYŚ [11], FORYŚ [12]:

$$\hat{M}(\mathbf{h})\left[\frac{\partial^2 w}{\partial t^2}\right] + \hat{S}(\mathbf{h})[w] + \beta(t)\hat{P}_\beta[w] = 0, \tag{2.1}$$

where $\mathbf{h}$ is the vector of control functions or the vector of cross-sectional parameters (e.g. area of the cross-section of the rods or the thickness of the plate), $\hat{M}, \hat{S}, \hat{P}_\beta$ – are the inertia, elasticity and stability linear operators. The form of these operators depends on the kinds of mechanical elements to be considered, $w(x,t)$ is a transverse displacement of vibrating system, $\beta(t)$ is a periodic function of t. We look for an approximate solution of the above problem in the form

$$w = \sum_{k=1}^{N} f_k(t)\varphi_k^{(h)}(x,y,z), \tag{2.2}$$

where $f_k(t)$ are the unknown functions of time and $\varphi_k^{(h)}$ are the unknown eigenfunctions of the eigenvalue problem. Applying the Galerkin's method to Eq. (2.1), we obtain the system of ordinary differential equations of the second order in the matrix form, cf. [1] and [11].

After some rearrangements, our problem is now described by an ordinary system of equations:

a) the eigenvalue equations with proper boundary conditions

$$\left[\hat{S}(\mathbf{h}) - \omega^2\hat{M}(\mathbf{h})\right]\varphi^{(h)} = 0 \tag{2.3}$$

and

b) the equations of the second order in the matrix form

$$\frac{d^2 f_k}{dt^2} + 2\varepsilon_k(\mathbf{h})\frac{df_k}{dt} + \omega_k^2(\mathbf{h})\left[f_k + \beta(t)\sum_{j=1}^{N} b_{kj}f_j\right] = 0, \quad k = 1,2,\dots,N, \tag{2.4}$$

where

$$\omega_k^2 = \frac{(\varphi_k, \hat{S}(\mathbf{h})[\varphi_k])}{(\varphi_k, \hat{M}(\mathbf{h})[\varphi_k])} = \frac{J_2^{(kk)}}{J_1^{(kk)}}, \tag{2.5}$$

and approximately

$$\beta_{\text{cr}} \cong -\frac{1}{b_{kk}} = -\frac{(\varphi_k, \hat{S}(\mathbf{h})[\varphi_k])}{(\varphi_k, \hat{P}_\beta[\varphi_k])} = -\frac{J_2^{(kk)}}{J_3^{(kk)}}; \tag{2.6}$$

expressions (2.5), (2.6) are the nonadditive functionals connnected with functionals

$$\begin{aligned} M_{ik} &= \int_D \varphi_i\hat{M}(\mathbf{h})[\varphi_k]d\tau = (\varphi_i, \hat{M}(\mathbf{h})[\varphi_k]) = J_1^{(ik)}, \\ S_{ik} &= \int_D \varphi_i\hat{S}(\mathbf{h})[\varphi_k]d\tau = (\varphi_i, \hat{S}(\mathbf{h})[\varphi_k]) = J_1^{(ik)}, \\ b_{ik} &= \int \varphi_i\hat{P}_\beta[\varphi_k]d\tau = (\varphi_i, \hat{P}(\mathbf{h})[\varphi_k]) = J_3^{(ik)}. \end{aligned} \tag{2.7}$$

3. Solutions of equations of motion and stability

The Equations (2.4) are a set of coupled linear equations with variable coefficients. One may distinguish two types of instability of the trivial solution of (2.4), cf. [11]: the periodic (simple) parametric resonance, the combination parametric resonance. In the present paper we confine our analysis to the first type of instability. The most popular and very effective method of determining the instability region and amplitudes is the Bolotin's method. While applying it, one assumes the solution of (2.4) at the stability limits to be a truncated Fourier series, then the harmonic balance method is applied. So the solution with the period $2T$ is assumed to be

$$f(t) = \sum_{k=1,3,5\ldots}^{\infty} \left(a_k \sin\frac{k\theta t}{2} + b_k \cos\frac{k\theta t}{2} \right), \tag{3.1}$$

and the solution with the period T is

$$f(t) = b_0 + \sum_{k=2,4,6\ldots}^{\infty} \left(a_k \sin\frac{k\theta t}{2} + b_k \cos\frac{k\theta t}{2} \right). \tag{3.2}$$

The non-zero solution of the set of Mathieu–Hill type of linear Eq. (2.4) exists if the proper determinants equal zero ([1]).

4. Parametric systems – one degree of freedom

Now we confine ourselves to one degree of freedom – one Mathieu–Hill equation is taken into account. Taking $\beta(t) = \beta_0 + \beta_t \cos\theta t$, after some transformations we have

$$\frac{d^2 f}{dt^2} + 2\varepsilon(\mathbf{h})\frac{df}{dt} + \omega^2(\mathbf{h})[1+\beta_0 b(\mathbf{h})]\left[\left(1 + \frac{b(\mathbf{h})\beta_t}{1+b(\mathbf{h})\beta_0}\cos\theta t\right)\right] f = 0. \tag{4.1}$$

Next, we transform Eq. (4.1) into (cf. Foryś [11]),

$$\ddot{f} + 2\varepsilon(\mathbf{h})\dot{f} + \Omega^2(\mathbf{h})(1 - 2\mu(\mathbf{h})\cos(\theta t))f = 0, \tag{4.2}$$

where we define

$$\Omega(\mathbf{h}) = \omega(\mathbf{h})\sqrt{1+\beta_0 b(\mathbf{h})} = \omega(\mathbf{h})\sqrt{1 - \frac{\beta_0}{\beta_{\mathrm{cr}}(\mathbf{h})}}, \qquad b = -1/\beta_{\mathrm{cr}}, \tag{4.3}$$

and exciting parameters μ

$$\mu(\mathbf{h}) = -\frac{\beta_t b(\mathbf{h})}{2(1+\beta_0 b(\mathbf{h}))} = \frac{\beta_t}{2(\beta_{cr}(\mathbf{h}) - \beta_0)}, \tag{4.4}$$

connected with constant part β_0 and amplitude β_t of the oscillating part of external parametric excitation $\beta(t)$, cf. [11 – 13]. After rearrangement, our problem — of one degree parametric system is described by the ordinary system of equations:

a) the eigenvalue Eq. (2.3) with proper boundary conditions

$$\left[\hat{S}(\mathbf{h}) - \omega^2 \hat{M}(\mathbf{h})\right] \varphi^{(h)} = 0, \tag*{$(4.5)_1$}$$

b) Eq. (4.2) of the second order (cf. [1, 11])

$$\ddot{f} + 2\varepsilon(\mathbf{h})\dot{f} + \Omega^2(\mathbf{h})(1 - 2\mu(\mathbf{h})\cos(\theta t))f = 0. \tag*{$(4.5)_2$}$$

For parametrically excited beams, the inertia, elasticity and stability operators take the form

$$\begin{aligned}
\hat{M}(\mathbf{h}) &= m(x) = \rho(x)h(x), \\
\hat{S}(\mathbf{h}) &= \frac{\partial^2}{\partial x^2}\left[K_\alpha h^\alpha(x)\frac{\partial^2}{\partial x^2}\right], \\
\hat{P}_\beta &= \frac{\partial^2}{\partial x^2}.
\end{aligned}$$

Now the function of state φ satisfies the equation of state (2.3) written below

$$[K_\alpha h^\alpha(x)\varphi''(x)]'' - \rho h(x)\omega^2\varphi(x) = 0. \tag{4.6}$$

The Mathieu–Hill Eq. $(4.5)_2$ plays an important role in the optimization procedure of structural, parametrically excited elements against a loss of dynamic stability. On the basis of Eq. $(4.5)_2$, we introduce in Sec. 4.1 the objective function in the form of nonadditive functionals. In optimization, procedure, the equation of state (2.3) or (4.6) and proper boundary conditions are some of the constraints.

The amplitude – frequency characteristics are important in the resonance phenomena. Linear theory is capable of determining the region in which the trivial solution becomes dynamically unstable and it predicts that the unstable motion grows without limits. However, as the amplitude of motion grows, the nonlinear effects limit the growth. The nonlinear equation gives finite amplitude of motion in the region of instability. This amplitude may be calculated on the basis of nonlinear equation, (cf. [11]):

$$\ddot{f} + 2\varepsilon(\mathbf{h})\dot{f} + \Omega^2(\mathbf{h})(1 - 2\mu(\mathbf{h})\cos(\theta t))f + \psi(f, \dot{f}, \ddot{f}) = 0, \tag{4.7}$$

where the functions $\psi(f, \dot{f}, \ddot{f})$ include nonlinear effects, e.g. such geometrical nonlinearities as nonlinear damping, nonlinear elasticity etc. In some types of nonlinearities there are several solutions, stable and nonstable. The stability of solutions is determined on the basis of linearized variational equation of motion.

4.1. Objective function

We analyse the periodic parametric resonance (the first type of instability) and restrict ourselves to one degree of freedom and the first, most important instability region. On the basis of general theory of differential equations with variable coefficients and papers [1] and [11], the boundary of the first instability region is determined by the relation

$$\theta \cong 2\Omega(\mathbf{h})\sqrt{1 \pm \sqrt{\mu^2(\mathbf{h}) - \left(\frac{\Delta(\mathbf{h})}{\pi}\right)^2}}, \tag{4.8}$$

where

$$\Delta(\mathbf{h}) = \frac{2\pi\varepsilon(C(\mathbf{h}))}{\omega(\mathbf{h})\sqrt{1 - \dfrac{\beta_0}{\beta_{cr}(\mathbf{h})}}} = \frac{2\pi\varepsilon(C(\mathbf{h}))}{\Omega(\mathbf{h})}. \tag{4.9}$$

The first, main purpose of the paper is to determine and define the proper measures of periodic parametric resonance. These measures are the objective functions in optimization procedure. In papers [11 – 13] the author proposed four physically motivated quantities characterizing the parametrically excited systems. The periodic parametric resonance occurs if in a parametrically excited system the proper relations between the frequency of external excitation θ and natural frequencies take place. The most dangerous, main parametric periodic resonance occurs in the neighborhood of the doubled value of the first natural frequency $\theta \cong 2\omega$.

The first natural frequency (2.5) is the proper objective function in optimization procedure when we maximize the nonresonance region $0 \leq \theta \leq 2\omega$. So the parametrical system will be stable if the value of natural frequency is maximal in optimization with the constraints: the equation of state and proper boundary conditions (4.4) and constant volume constraint.

The formula (4.8) gives the boundary of instability region in the $(\mu, \theta/2\omega)$ plane, Fig. 1. If the expression under the inner square root is positive, the formula (4.8) gives two real values of critical frequency. The critical value of the exciting parameter denoted by μ^* is

$$\mu^* = \frac{\Delta(\mathbf{h})}{\pi} = \frac{2\varepsilon(\mathbf{h})}{\omega(\mathbf{h})\sqrt{1+\beta_0 b(\mathbf{h})}}, \tag{4.10}$$

where $\Delta(\mathbf{h})$ was defined in (4.9). If additionally $\beta_0 = 0$, one has

$$\mu^* = \frac{\Delta(\mathbf{h})}{\pi} = \frac{2\varepsilon(\mathbf{h})}{\omega(\mathbf{h})}. \tag{4.11}$$

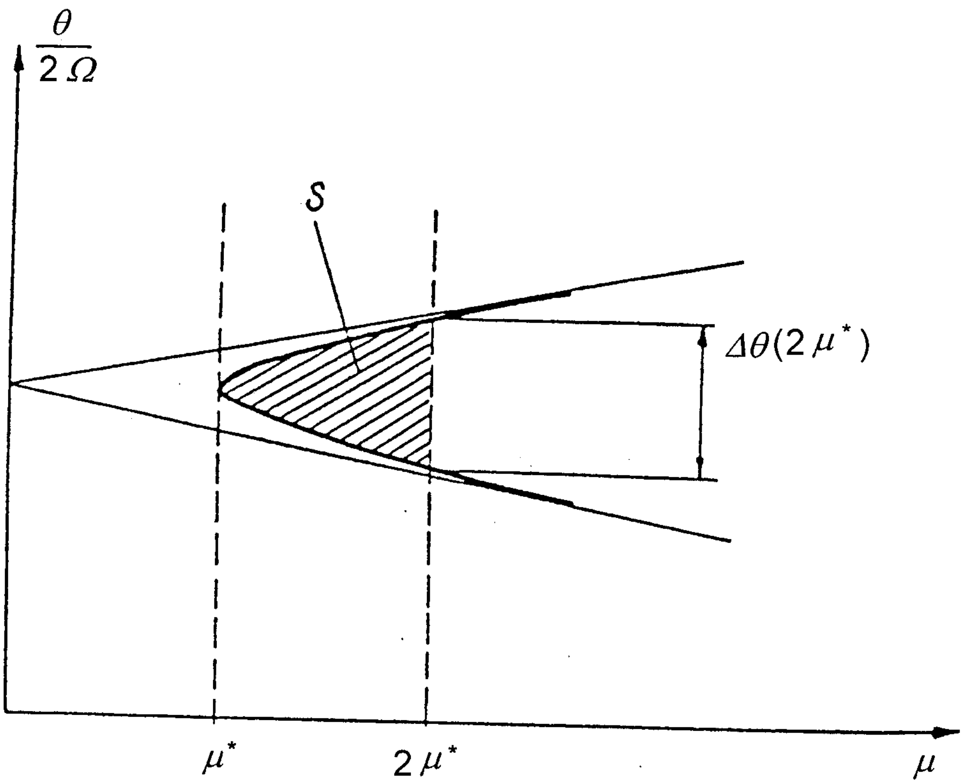

FIG. 1. Instability regions in $(\mu, \theta/2\omega)$ plane,-μ^* the critical value of exciting parameter, S-the area of a part of the instability region between μ^* and $2\mu^*$.

So μ^* is the special value of μ connected with the energy dissipation in parametric systems. The form of critical value of the exciting parameter depends on the model of damping. For the Kelvin–Voigt viscoelastic material and for parametric excitation we usualy assume that the matrix of damping $\mathbf{D}$ is proportional to elasticity matrix $\mathbf{S}$. On the basis of paper [12] one has

$$\mu^*(\mathbf{h}) = \frac{\Delta(\mathbf{h})}{\pi} = \frac{\tau J_2}{(J_1(J_2+\beta_0 J_3))^{\frac{1}{2}}} = J, \tag{4.12}$$

$J = F(J_1, J_2, J_3)$ being the nonadditive function of functional, $J_1 = J_1^{(11)}$, $J_2 = J_2^{(11)}$, $J_3 = J_3^{(11)}$ (cf. (7)). If we consider the special case $\beta_0 = 0$ then

$$\mu^{*2}(\mathbf{h}) = \tau^2\omega^2. \tag{4.13}$$

Now we will analyze parametrically excited system in the $(\beta_t, \theta/2\omega)$ plane. On the basis of (4.4) we have the amplitude of the oscillating part of parametric excitation $\beta_t(\mathbf{h}) = 2\mu(\beta_{\mathrm{cr}} - \beta_0)$. So we look for such a value of amplitude of the oscillating part of excitation for which the unstable solution occurs. If $\mu = \mu^*$, $\beta_t(\mathbf{h})$ equals the critical value of amplitude of oscillating part of excitation

$$\beta_t^*(\mathbf{h}) = 2\mu^*(\mathbf{h})(\beta_{\mathrm{cr}}(\mathbf{h}) - \beta_0) = . - 2\mu^*(\mathbf{h})(1+\beta_0 b)/b. \tag{4.14}$$

If additionally $\beta_0 = 0$, one has

$$\beta_t^*(\mathbf{h}) = 2\mu^*(\mathbf{h})\beta_{\mathrm{cr}}(\mathbf{h}) = -2\mu^*(\mathbf{h})/b. \tag{4.15}$$

For the Kelvin–Voigt viscoelastic material μ^* is determined by (4.12). If additionally $\beta_0 = 0$, one has

$$\beta_t^*(\mathbf{h}) = 2\tau\omega\beta_{\mathrm{cr}}(\mathbf{h}) = -2\tau\omega/b. \tag{4.16}$$

For the $\mu > \mu^*$ or for $\beta_t > \beta_t^*$ the region of instability appears.

Our optimization problem against the loss of dynamic stability of the parametrically excited beam consists in determining the values of parameter h, which extremizes the critical value of exciting parameter μ^* or the critical value of amplitude of oscillating part of excitation β^*. The vibrating parametrically excited system will be most stable if the μ^* attains maximum in the $(\mu, \theta/2\omega)$ plane or β_t^* attains maximum in the $(\beta_t, \theta/2\omega)$ plane. One can see that critical parameters: critical value of exciting parameter or critical value of amplitude of the oscillating part of excitation, separate stable and unstable solutions (cf. Fig. 1). Maximization of the values of the critical parameters μ^*, β^* also allows us to move away from unstable solution regions. For such objective function we not only control the geometrical and physical parameters of the system or its sourranding but also influence the instability region through the change of the coefficient of damping and through the change of the parameter β_0.

In the resonance state other objective functions may be introduced. They are some measures of instability region. One of them is associated with the area $S(\mathbf{h})$ of a part of the instability region, e.g. that enclosed between μ^* and $2\mu^*$. If we make the assumptions that μ changes in the interval $(\mu^*, 2\mu^*)$, the expression $\varepsilon = \sqrt{\mu^2 - \mu^{*2}} \ll 1$ is a small quantity in the region of μ changes and the area of the part of instability region is (cf. (17))

$$S(\mu^*, 2\mu^*) = \int_{\mu^*}^{2\mu^*} \sqrt{1 + \sqrt{\mu^2 - \mu^{*2}}}d\mu - \int_{\mu^*}^{2\mu^*} \sqrt{1 - \sqrt{\mu^2 - \mu^{*2}}}d\mu \tag{4.17}$$

$$\cong \int_{\mu^*}^{2\mu^*} \sqrt{\mu^2 - \mu^{*2}}d\mu = 1.074\mu^{*2}.$$

For the Kelvin–Voigt viscoelastic material μ^* is determined by (4.12). If additionally $\beta_0 = 0$, one has

$$S(\mu^*, 2\mu^*) = 1.074\tau^2\omega^2. \tag{4.18}$$

Another measure of the instability region is determined as the width of dynamic instability region for $\mu = 2\mu^*$

$$Z = \frac{\Delta\theta(2\mu^*)}{2\Omega} = \sqrt{1 + \sqrt{\mu^2 - \mu^{*2}}} - \sqrt{1 - \sqrt{\mu^2 - \mu^{*2}}} \cong \sqrt{3}\mu^*. \tag{4.19}$$

For the Kelvin–Voigt viscoelastic material μ^* is determined by (4.12). It enables us to define the interval of frequency $\Delta\theta(\mu^*, 2\mu^*)$ of external excitation where the solution is unstable and determined by

$$\Delta\theta(2\mu^*) = 2\Omega Z(2\mu^*) = 2\Omega\sqrt{1 + \sqrt{\mu^2 - \mu^{*2}}} - 2\Omega\sqrt{1 - \sqrt{\mu^2 - \mu^{*2}}} \tag{4.20}$$

$$\cong \sqrt{3}\mu^* = 2\Omega\sqrt{3}\mu^*.$$

For the Kelvin–Voigt viscoelastic material μ^* is determined by (4.12). If additionally $\beta_0 = 0$, so

$$\mu^*(\mathbf{h}) = \tau\omega \quad \text{(cf. (4.13))} \tag{4.21}$$

and

$$\Delta\theta(2\mu^*) = 2\sqrt{3}\tau\omega^2. \tag{4.22}$$

If it is not possible to move away from the unstable solution region by means of optimization, the phenomenon of parametric resonance occurs (there exists a non-stable solution of equation of motion), the resonance amplitude grows to infinity. The nonlinearities limit the growth, so amplitudes of parametric resonance are finite in the region of instability. In such examples the proper objective function is the amplitude of steady state of parametric resonance. The resonance amplitude can be obtained on the basis of nonlinear equation of motion. For one mode the amplitude equals

$$A\left(\mu(\mathbf{h}), \theta/2\Omega(\mathbf{h}), \mu^*(\mathbf{h})\right) = \sqrt{a^2+b^2}, \tag{4.23}$$

where a and b are coefficient of sine and of cosine in the solution of Eq. (4.1). For example, for non-linear elasticity $\psi(f, \dot{f}, \ddot{f}) = \gamma f^3$, where γ is the coefficient of nonlinear elasticity, the amplitude in main parametric periodic resonance has the form

$$A = \frac{2\Omega(\mathbf{h})}{\sqrt{3\gamma}}\sqrt{n^2(\mathbf{h}) - 1 \pm \sqrt{\left(\mu^2(\mathbf{h}) - \left(\frac{n\Delta(\mathbf{h})}{\pi}\right)^2\right)}}, \tag{4.24}$$

where $n = \theta/2\Omega$, $\Delta = 2\pi\varepsilon/\Omega$.

Now in the optimization procedure we look for the minimum value of the resonance amplitude determined for one degree of freedom by (4.23) or e.g. by (4.24).

4.2. One degree of freedom-variational optimization

The optimization problem against the loss of dynamic stability of the beam consists in determining the control function $h(x)$ (e.g. the area of the cross-section of the beam) which extremizes the functional J (e.g. functionals ((4.10) – (4.24)) in the form

$$J = F(J_1, J_2, J_3), \qquad J_i = \int_D f_i(x, h, \varphi, \varphi'')dx, \qquad i = 1, 2, 3. \tag{4.25}$$

The functions f_i are the known functions of space variables, of control function h and of function of state $\varphi(h)$.

The constraints (e.g. constant volume constraint) are in the form

$$F_i(J_1, J_2, J_3) = \text{const}. \tag{4.26}$$

The function of state $\varphi(h)$ satisfies the natural transverse vibrations of the nonprismatic rods without damping

$$\hat{L}[h(x)]\varphi(h) = [\hat{S}(h) - \omega^2\hat{M}(h)]\varphi(h) \qquad \text{in } \mathbf{D}$$

and boundary conditions

$$\{N[h(x)]\varphi(h)\}_\Gamma = O \qquad \text{on } \Gamma. \tag{4.27}$$

The length of the beam and its material parameters are fixed. In calculations the geometrical constraints are adopted

$$h_1 \leq h(x) \leq h_2. \tag{4.28}$$

Variational calculus is used to find the solution. The necessary conditions for extreme values can be derived by setting to zero the first variation of non-additive functionals with constraints. BANICHUK *et al.* [14] derived the necessary conditions of the optimality for non-additive functionals.

4.3. One degree of freedom-examples

The papers [11, 12, 13] were devoted to the problems of monocriterion variational optimization of the parametrically excited Kelvin–Voigt viscoelastic simply supported beam with respect to the critical values of exciting parameter μ^*, but make the assumption that the constant part of parametrical excitation equals zero: $\beta_0 = 0$, so $\beta(t) = \beta_t \cos\theta t$ and the objective function was in the form (4.13). In this case, the optimization with respect to the maximum of μ^* resolves itself into optimization problems with respect to the maximum of natural frequency, cf. paper [12].

We considered in [12] also the examples with additional geometrical constraints. The optimization of the beams with respect to the square of natural frequency for the first mode is a well-known problem (cf. [15, 16]) and enables us to obtain numerical results of our optimization problem with respect to the maximum of objective function μ^*. For example in Figs. 2, 3, the results for simply supported-fixed and cantilever beams with circular cross-section are presented (cf. [16]).

Examples in the present paper concern the problems of monocriterion optimization of the parametrically excited Kelvin–Voigt viscoelastic beam with respect to the maximum of critical values of exciting parameter μ^*, and also with respect to the maximum of critical value of amplitude of excitation β_t^* , but now $\beta_0 \neq 0$, $(\beta(t) = \beta_0 + \beta_t \cos\theta t)$. The objective function μ^* takes the form (4.12)

$$\mu^*(h) = \frac{\Delta(h)}{\pi} = \frac{\tau J_2}{(J_1(J_2 + \beta_0 J_3))^{\frac{1}{2}}} = J. \tag{4.29}$$

The system is most stable if μ^* attains the maximum. In numerical calculations we minimize the objective function $f = 1/\mu^*$. The calculations of optimal shape of the rod were made by the method described in paper [13]. The function h which describes the area of cross-section is approximated on the basis of the values of the area in eleven equidistant points of the beam. The ten quantities $A_0, .., A_{10}$ are the parameters of optimization. In calculations, the geometrical constraints are adopted $h_1 \le h(x) \le h_2$. These values h_1, h_2 may be connected with strength constraints. The length and volume of the beam and its material parameters are fixed. The beams are assumed to be of circular cross-section.

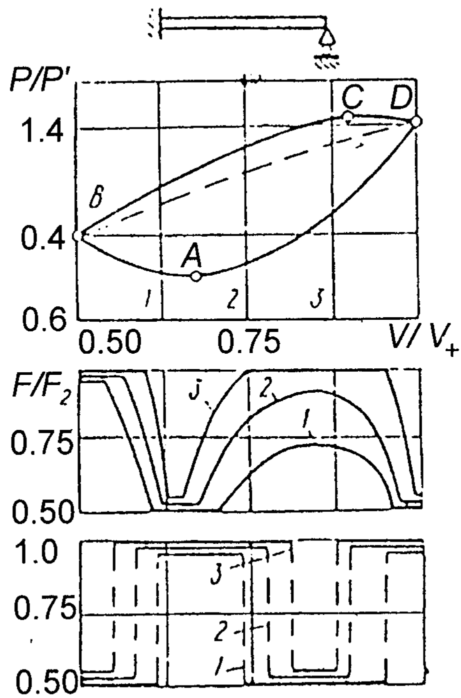

FIG. 2. Results of optimization of simply supported-fixed beam with respect to the maximum and minimum of the first natural frequency.

Numerical calculations were performed for a cantilever beam (Figs. 4, 5, 6) for different values of constant part of excitation $\overline{\beta}_0 = 0$; $\overline{\beta}_0 = 0.2$; $\overline{\beta}_0 = 0.6$, respectively. We start the calculations from the prismatic beam. In the examples, the nondimentional geometrical constraints are: $h_{\min} = 0.5$; $h_{\max} = 1.5$. On the graphs in Figs. 4 – 5, the following quantities are shown:

i) The nondimensional areas A of cross-section of the beam versus the nondimensional coordinate x.

ii) The shape of the beam.

On the basis of simple calculation we have

$$\frac{\Delta\mu^*}{\mu^*_{\mathrm{pr}}} = \frac{f_{\mathrm{pr}} - f}{f}. \tag{4.30}$$

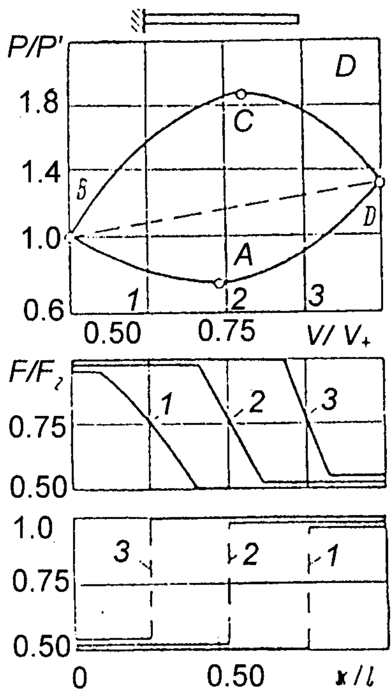

FIG. 3. Results of optimization of the cantilever beam with respect to the maximum and minimum of the first natural frequency.

In Table 1 we present the results of numerical calculation for diffrent values of constant parts of parametric excitation. We introduce the notations:

Table 1.

$\overline{\beta}_0$	f_{pr}	f	$\overline{\beta}_{\mathrm{cr}}$	$\Delta\mu^*/\mu^*_{\mathrm{pr}}100\%$	$\Delta\beta^*_t/\beta^*_{t\mathrm{pr}}100\%$
0.0–Fig. 4	0.284	0.15710	1.021	80.8%	84.8%
0.2–Fig. 5	0.282	0.157155	1.114	79.4%	101.7%
0.4	0.280	0.155713	1.108	79.8%	103.0%
0.6–Fig. 6	0.278	0.152552	1.048	82.2% (optimum)	94.2%
0.8	0.276	0.155017	1.158	78.0%	120.1%
1.0	0.274	0.153558	1.156	78.4%	125.7%

$$f = 1/\overline{\mu}^*, \qquad \beta_0 = \overline{\beta}_0 \frac{4}{\pi^2}(\beta_{cr})_{pr}, \qquad \beta_{cr} = (\overline{\beta}_{cr})(\beta_{cr})_{pr},$$

$$(\beta_{cr})_{pr} = \frac{\pi^2 E J_0}{4l^2}, \tag{4.31}$$

$$(\mu^*)^2 = \frac{EJ_0}{\rho A_0 l^4}(\overline{\mu}^*)^2, \quad A_0, J_0, l \text{ are the parameters of prismatic beam.}$$

The proper shaping of the beam stabilizes the object moving it away from unstable solution region, cf. Fig. 1. One can see that for situation illustrated in Fig. 6, the value of μ^* attains maximum and the system is most stable. We can compare the results for $\overline{\beta}_0 = 0$, $(\beta(t) = \beta_t \cos\theta t)$ Fig. 4 with the results of Griniev and Filipov calculations [16], Fig. 3. One can see the good compatibility of results.

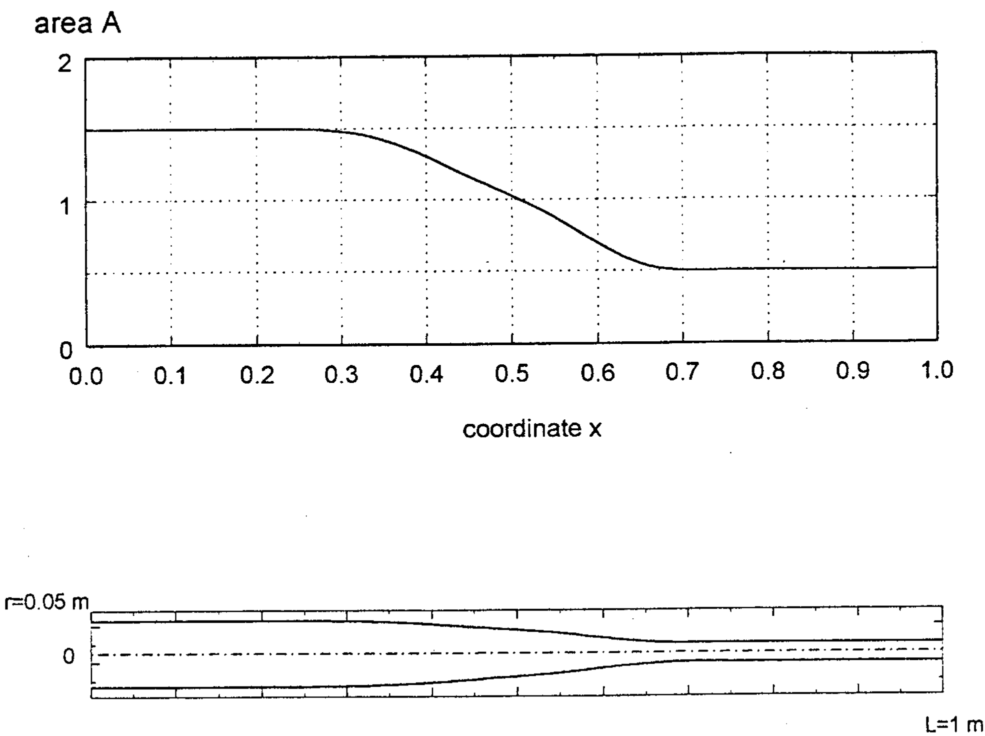

FIG. 4. Results of monocriterion optimization of the beam with respect to the maximum of μ^*. The nondimensional area A of cross-section of the beam versus the nondimensional coordinate x and the shape of the cantilever beam for $\beta_0 = 0$.

Now we will analyze the parametrically excited system in the $(\beta_t, \theta/2\omega)$ plane. If $\mu = \mu^*$, $\beta_t(h)$ equals the critical value of amplitude of oscillating part of excitation, cf. (4.4), $\beta_t^*(h)$

$$\beta_t^*(h) = 2\mu^*(h)(\beta_{\mathrm{cr}}(h) - \beta_0) = -2\mu^*(h)(1 + \beta_0 b)/b. \tag{4.32}$$

We optimize the system with respect to $\beta_t^*(h)$ – the system is most stable if $\beta_t^*(h)$ attains maximum. On the basis of simple calculations we have

$$\frac{\Delta\beta_t^*}{(\beta_t)_{\mathrm{pr}}^*} = \frac{f_{\mathrm{pr}}}{f} \frac{\left(\overline{\beta}_{\mathrm{cr}} - \overline{\beta}_0 \dfrac{4}{\pi^4}\right)}{\left(1 - \overline{\beta}_0 \dfrac{4}{\pi^2}\right)} - 1. \tag{4.33}$$

In Table 1 we present also the results of numerical calculation of $\Delta\beta_t^*/\beta_{t\mathrm{pr}}^*$ for diffrent values of constant parts of parametric excitation. The proper shaping of the beam stabilizes the object moving it away from unstable solution region. One can see that for the situation illustrated in Fig. 5, the value of β_t^* attains maximum and the system is most stable. The optimal (maximal) value of β_t^* is for $\overline{\beta}_0 = 1.0$.

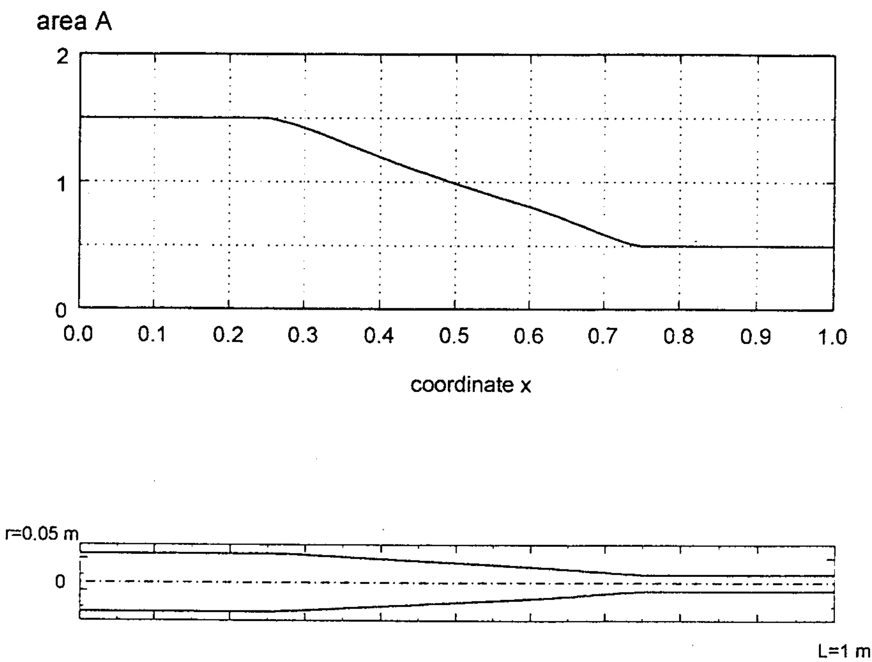

FIG. 5. Results of monocriterion optimization of the beam with respect to the maximum of μ^*. The nondimensional area A of cross-section of the beam versus the nondimensional coordinate x and the shape of the cantilever beam for $\beta_0 = 0.2$.

The problem of optimization of parametrically excited systems may be formulated in a different way by introducing, instead of the critical parameters $\mu^*(h)$, $\beta_t^*(h)$, another objective functions connected with a part of instability region.

One of them is associated with the area $S(h)$ of a part of the instability region, e.g. that enclosed between μ^* and $2\mu^*$, (4.17) and (4.18). Another measure of the instability region is determined as the width of dynamic instability region for $\mu = 2\mu^*$, (4.19) (cf. Fig. 1). It enables us to define the interval of frequency $\Delta\theta(\mu^*, 2\mu^*)$ of external excitation where the solution is unstable and determined by (4.21) and (4.22). The parametric systems are optimal if objective functions (4.17) – (4.22) attain minimum. In some special cases these objective functions are proportional to the first natural frequency. Under these assumptions, the optimization with respect to the minimum of (4.18), (4.22) is resolved into optimization problems with respect to the minimum of the first natural frequency.

Figure 7 illustrates the results of minimization of the first value of natural frequancy of the Kelvin–Voigt cantilever beam, for $\beta_0 = 0$. For constant value of the damping coefficient we have good compatibility of the results with the GRINIEV and FILLIPOV paper [16], Fig. 3. So after a simple calculation we have

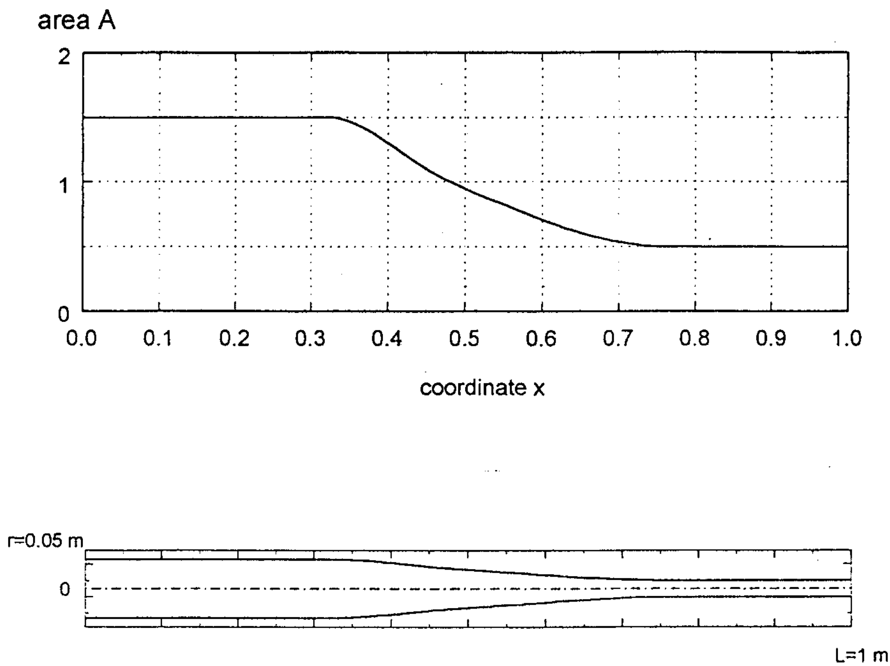

FIG. 6. Results of monocriterion optimization of the beam with respect to the maximum of μ^*. The nondimensional area A of cross-section of the beam versus the nondimensional coordinate x and the shape of the cantilever beam for $\beta_0 = 0.6$.

$$\frac{\Delta\omega}{\omega} = \left|\frac{f_{\mathrm{pr}} - f}{f}\right| 100\% = 52.6\%, \tag{4.34}$$

$$\frac{\Delta S}{S_{\mathrm{pr}}} = \frac{\Delta(\Delta\theta)}{(\Delta\theta)_{\mathrm{pr}}} = \left|\frac{f_{\mathrm{pr}}^2 - f^2}{f^2}\right| 100\% = 77.5\%. \tag{4.35}$$

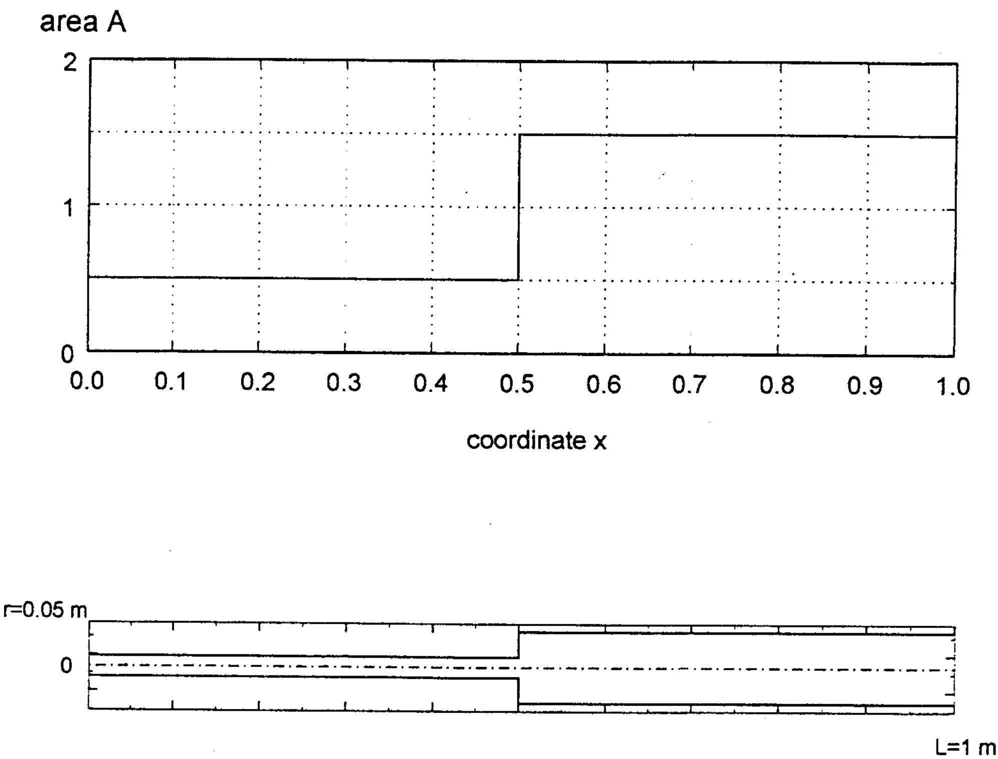

FIG. 7. Results of minimization of μ^* for the cantilever beam for $\beta_0 = 0.6$.

5. Multidegree of freedom parametric systems

In some problems the parametrically excited "non-prismatic" objects may be described by coupled differential equations with variable coefficients (cf. [11]) in the form

$$\frac{d^2 f_k}{dt^2} + 2\varepsilon_k(\mathbf{h})\frac{df_k}{dt} + \omega_k^2(\mathbf{h})\left[f_k + \beta(t)\sum_{j=1}^{N} b_{kj} f_j\right] = 0, \quad k = 1, 2, ..., N. \tag{5.1}$$

In the system (5.1) the periodic parametric and combination parametric resonances may occur. If elements b_{ii} dominate in the matrix $\{b_{ij}\}$, the periodic parametric resonance is the important resonance [17]. We restrict ourselves to this type of instability. On the boundaries of instability regions the solutions have the form (3.1) and (3.2). Inserting (3.1) and (3.2) into (5.1) and equating to zero the proper determinant we have the equations enabling us to determine the boundary of instability regions in the form

$$W_{\infty}^{(2T)} = |F(\theta, \beta_t)| = 0, \qquad W_{\infty}^{T} = |F(\theta, \beta)_t| = 0. \tag{5.2}$$

On the basis of nonlinear equations of motion we have the amplitude A determined by a proper determinant. Near the first more important instability region we have

$$W_{\infty}^{(2T)} = |F(\theta, A, \beta_t)|\,. \tag{5.3}$$

In many problems, the parametrically excited "non-prismatic" objects may be described by N non-coupled Mathieu–Hill differential equations [13], in the form

$$\frac{d^2 f_k}{dt^2} + 2\varepsilon_k(\mathbf{h})\frac{df_k}{dt} + \omega_k^2(\mathbf{h})[f_k + \beta(t) b_k f_k] = 0, \qquad k = 1, 2, \ldots, N. \tag{5.4}$$

We restrict ourselves to the first type of instability and the first most important instability region. On the boundaries of instability regions the solutions of (5.4) for k-th mode have the form

$$q_1^{(k)} = a_1^{(k)} \cos\frac{\theta t}{2} + b_1^{(k)} \sin\frac{\theta t}{2}, \qquad \text{where } k = 1, 2, \ldots, N. \tag{5.5}$$

Substituting (5.5) into (5.4) and repeating the procedure of Sec. 4.1 we have N critical values of exciting parameters

$$\mu^{*(k)}(\mathbf{h}) = \frac{\Delta^{(k)}}{\pi} = \frac{\tau J_2^{(k)}}{\left[J_1^{(k)}(J_2^{(k)} + \beta_0 J_3^{(k)})\right]^{\frac{1}{2}}}, \tag{5.6}$$

where J_1, J_2, J_3 are introduced in (2.7).

5.1. Objective function and optimization

For the systems described by coupled differential equations with variable coefficients, the proper objective functions in optimization procedure are the amplitude of oscillating part of external parametric excitation β_t and amplitude of vibration A, determined by (5.2) and (5.3), respectively. Because the objective functions have no explicit form, we restrict ourselves, in these cases, only to parametric optimization. Papers [18, 19] are devoted to examples of parametric optimization of parametrically excited systems.

The parametric optimization can by formulated as follows: we look for such parameters of optimization κ_i for which minimal (critical) value of $\beta_t(\kappa_i), \min\beta_t(\kappa_i) = \beta_t^*(\kappa_i)$ has the maximum

$$\max(\min \beta_t(\kappa_i)) = \max \beta_t^*(\kappa_i) \tag{5.7}$$

with constraints: $V = \text{const}$, $l = \text{const}$, $|F(\theta, \beta_t)| = 0$, $\theta \cong 2\omega$, where ω is the first natural frequency.

Similarly, we may formulate the optimization with respect to the minimum of amplitude in parametric periodic resonance:

$$\min A(\kappa_i) \tag{5.8}$$

with $|F(\theta, A, \beta_t)| = 0, \quad \theta \cong 2\omega$.

When the parametrically excited "non-prismatic" object is described by N non-coupled Mathieu–Hill differential equations, the following vector objective functions

$$J(\mathbf{h}) = \mu^*(\mathbf{h}) = \left(\mu^{*(1)}(\mathbf{h}), \mu^{*(2)}(\mathbf{h}), \ldots, \mu^{*(N)}(\mathbf{h})\right) \tag{5.9}$$

can characterize the parametrically excited N degrees of freedom systems.

We initially consider the monocriterial optimization problem (cf. Sec. 4.2). Let $\mu^{*(k)}(h)$ be the objective function in monocriterial optimization. We look for such a cross-sectional function that maximizes the k-th functional $\mu^{*(k)}(h)$. So we have in N monocriterial optimization problems the N cross-sectional functions: $\left[h_0^{(1)} \ldots\ldots h_0^{(N)}\right]$ that maximize N critical values of exciting parameters $\mu_0^{*(k)} = \max_{\mathbf{h}} \mu^{*(k)}(\mathbf{h}^{(k)})$, $k = 1 \ldots N$.

Next we introduce the scalar objective function:

$$P(J(\mathbf{h})) = \left(\sum_{i=1}^{N} \left| \frac{1}{\mu^{*(i)}(\mathbf{h})} - \frac{1}{\mu_0^{*(i)}} \right|^2 \right)^{\frac{1}{2}} \to \min. \tag{5.10}$$

So the multicriterial optimization problem with respect to vector objective function is resolved into an association optimization problem with respect to the scalar objective function (5.10). We look for a control function that minimizes the scalar functional (5.6).

For the Kelvin–Voigt viscoelastic material and for $\beta_0 = 0$, the associated multicritrial optimization problems with respect to minimization of scalar objective function (5.10) are resolved into an association optimization problem with respect to minimization of the scalar objective function in the form

$$P(J(\mathbf{h})) = \left(\sum_{i=1}^{N} \left| \frac{1}{\omega^{(i)}(\mathbf{h})} - \frac{1}{\omega_0^{(i)}} \right|^2 \right)^{\frac{1}{2}} \to \min. \tag{5.11}$$

The optimization of the beams with respect to the square of natural frequency for k-mode are well-known problems (cf. OLHOFF [15]) and enable us to obtain numerical results of our multicriterial optimization problem with respect to the minimum of scalar objective function.

5.2. *Multidegree of freedom-examples*

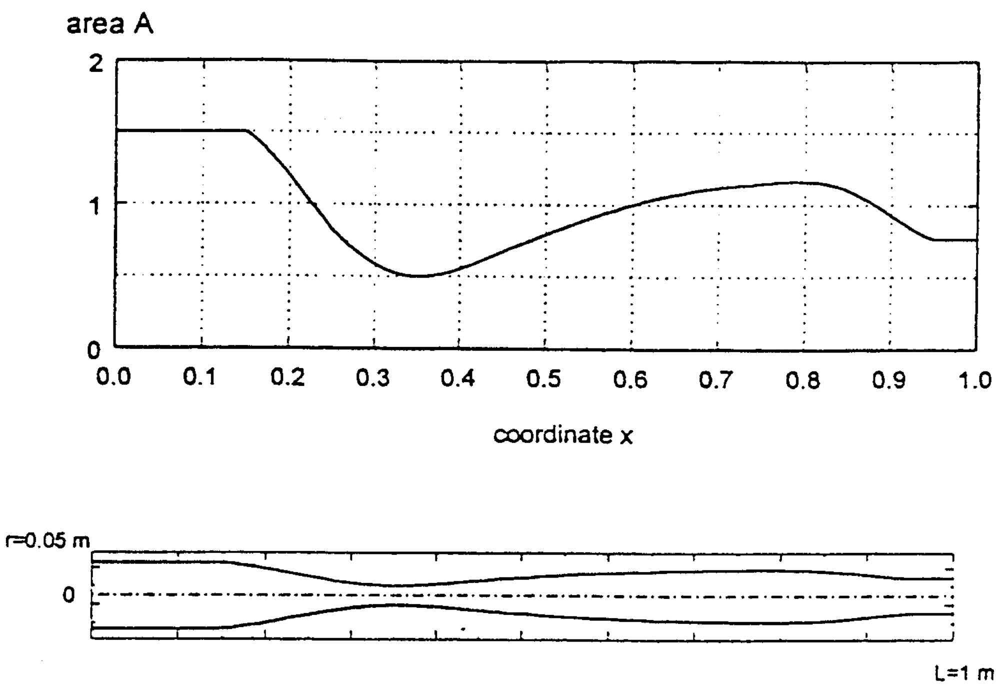

FIG. 8. Results of multicriterial optimization of the beam with respect to the maximum of μ^*. The nondimensional area A of cross-section of the beam versus the nondimensional coordinate x and the shape of simply supported-fixed beam for $\beta_0 = 0$.

In paper [13] the authors considered the multicriterial optimization of the parametrically excited Kelvin–Voigt viscoelastic beam with respect to the maximum of vector objective function (5.11), the vector of the critical values of exciting parameter $\mu^{*(k)}$. This problem is reduced to monocriterial optimization with respect to the minimum of scalar objective function (5.10). Making the assumptions: $\beta_0 = 0$, $(\beta(t) = \beta_t \cos\theta t)$, the scalar objective function has the form (5.11). The numerical calculations of optimal shape of the rod were made by the method described in [13]. The function h which describes the area of cross-section is approximated on the basis of the values of the area in eleven equidistant points of the beam. The quantities $A_0, \dots, A_{10}$ are the parameters of optimization. In calculations, the geometrical constraints are adopted $h_1 \le h(x) \le h_2$. The length and volume of the beam and its material parameters are fixed. The beams are assumed to be of circular cross-section. Numerical calculations were performed for fixed and simple supported beam (Fig. 8). On the graphs in Fig. 8 (cf. [13]) the following quantities are shown:

i) the nondimensional area of cross-section A ($A \to \dfrac{A}{A_0}$, A_0 is the area of cross-section for prismatic beam) versus nondimensional coordinate x ($x \to x/l$) of the beam;

ii) the shape of the beam.

6. Conclusions

The paper is a contribution to the study of the optimization of a beam in a steady state of periodic parametric resonance. A significant point in the optimization procedure consists in determining and introducing appropriate measures of the considered phenomena, i.e. optimization criteria – some objective functions. Example of mono- and multicriterial optimization of parametrically loaded beams with respect to the loss of stability is solved and presented. The multicriterial optimization problem of the system according to Pareto, is reduced to the problem of system minimization with respect to scalar objective functionals of the associated problem. So the optimization can be carried out taking into account a few aspects of the problem and various cost functions in mono- and multicriterial optimization.

The presented results indicate the importance of optimization in parametric resonance. Practical applications of these results may be anticipated due to the fact that beams are elements of many structures and machines. The results can be used for systems in which one aims at elimination of the parametric effects.

References

1. W. W. Bolotin, *Dynamic stability of elastic systems*, (translation from the Russian), San Francisco: Holden-Day 1964.
2. M. P. Cartmell, *Introduction to linear, parametric and nonlinear vibrations*, London Chapman and Hall 1990.
3. R. A. Ibrahim *et al.*, *Parametric vibrations*, The Shock of Vibration Digest, **10**, 1, 1978: Part I: *Mechanics of linear problems*, **10**, 2, 15–29; Part II: *Mechanics of nonlinear problems*, **10**, 3, 9–14; Part III: *Current problems (1)*, **10**, 4, 41–57; Part IV: *Current problems (2)*, **10**, 4, 19–47, 1978.
4. L. Adler, *Parametric phenomena in physics*, Am. J. Phys., **39**, 1522–1527, 1971.
5. W. Szemplińska-Stupnicka, *Problems of parametrical vibrations in dynamics of machines*, [in:] Contemporary problems of machine dynamics, PAN 1976.
6. L. Ruby, *Applications of Mathieu equation*, Am. J. Phys., **64**, 1, 1996.
7. H. Winter and H. W. Ortjohann, *Simple demonstration of storing macroscopic particles in a "Paul trap"*, Am. J. Phys., **59**, 9, 1991.
8. C. Sackett, E. Cornell *et. al.*, *A magnet suspension system for atoms and bar magnets*, Am. J. Phys., **61**, 4, 1993.
9. A. Foryś, *On an objective function in optimization problems with a loss of dynamic stability*, J. Theor. Appl. Mech., **30**, 1, 1992.
10. A. Foryś, *Variational optimization problems against a loss of dynamic stability*, Structural Optimization, **8**, 1994.
11. A. Foryś, *Optimization of mechanical systems in conditions of parametric resonance and in autoparametric resonances*, Cracow University of Technology, 1996.
12. A. Foryś, *Optimization of parametrically excited mechanical systems against loss of dynamic stability*, J. Sound Vibr. (in print), 1998.
13. A. Foryś and J. Snamina, *Multicriterial optimization of parametrically excited systems against loss of dynamic stability*, Structural Optimization, **16**, 4, 1998.
14. N. W. Banichuk, S. Ivanova and A. W. Sharanyuk, *Dynamics of structures. Analysis and optimization* [in Russian], Nauka, Moskva 1989.
15. N. Olhoff, *Optimal design against structural vibration and instability*, Technical University of Denmark, Lyngby 1978.
16. W. B. Griniev and A. P. Filippov, *Optimization of bars with respect of angenvalues* [in Russian], 1979.
17. W. Szemplińska-Stupnicka, *The behaviour of non-linear vibrating systems*, Dordrecht 1990.
18. A. S. Foryś and A. Gajewski, *Parametric optimization of a visco-elastic rod in view of its dynamic stability* [in Polish], Engng. Trans., **35**, 1987.
19. A. S. Foryś, *Parametric optimization of a rod at different support conditions* [in Polish], Engng. Trans., **38**, 1990.

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