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FREE VIBRATION OF THE SYSTEM OF TWO TIMOSHENKO BEAMS COUPLED BY A VISCOELASTIC INTERLAYER

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In this paper the uniform analytical method [3] has been used for solving a problem of free vibrations of continuous sandwich beam with damping. External layers are modelled as Timoshenko beams, while the internal layer possesses the characteristics of a viscoelastic, one-directional Winkler foundation. The phenomenon of free vibration has been described using a homogenous system of coupled partial differential equations. After separation of variables in the system of differential equations, the boundary problem has been solved and four complex sequences have been obtained: the sequences of frequencies, and the sequences of free vibration modes. Then, the property of orthogonality of complex free vibration modes has been demonstrated. The free vibration problem has been solved for arbitrarily assumed initial conditions.

1. Introduction

Some mechanical and building structural elements are treated as beams. Different models of beams are applied, depending on the complexity of the structure and the requirements [5]. In the last years the Bernoulli-Euler [1, 2, 4, 10, 14, 23] and the Timoshenko [7, 8, 11, 17, 18, 19, 20 – 22, 24 – 28] models for laminar beams or different sandwich beams [23] have been considered. Replacement of the Bernoulli-Euler model with the Timoshenko model, gives a result which is closer to the scientific results in the field of theory of elasticity [22], especially for very thick beams [5].

For the first time the influence of transverse forces and rotational inertia in a beam has been demonstrated in the paper [25], where the shear coefficient $k' = 2/3$ has been obtained. In the papers [7, 8] the criteria of choice of the shear coefficient in plates of medium thickness have been considered. Natural frequencies for continuous Timoshenko beams have been demonstrated in the paper [30], and for discrete-continuous Timoshenko models – in the papers [11, 17]. In the paper [20] the motion equations of the Timoshenko beam resting on two-parametric elastic WINKLER [31] foundation have been derived.

The property of orthogonality of complex free vibration modes for discrete systems with damping has been demonstrated in the paper [12], for discrete-continuous systems with damping – in the paper [29], and for continuous systems with damping – in the papers [1 – 4].

In the paper [10], the problem of free vibrations of a system of two Bernoulli-Euler beams, transversally coupled with discrete springs, without damping, has been considered. Influences of axial forces in beams have been taken into consideration. In the paper [3], the general method of solving problems of free vibration for complex, continuous, one-dimensional systems with damping, for various boundary conditions and different initial conditions has been presented. The mathematical analysis has been presented for a system of two strings with a viscoelastic interlayer.

The sandwich beam consists of two external layers and one internal layer, i.e. an interlayer connecting the external layers. The external layers are modelled as Bernoulli-Euler [1, 2, 4, 10, 14, 23] or Timoshenko beams. An interlayer is a one- or two-directional viscoelastic Winkler layer, but it can also be the multiparametric viscoelastic layer [32, 33].

The aim of this paper is to perform a dynamic analysis of the sandwich beam, in which external layers have been modelled as Timoshenko beams, while an internal layer corresponds to the characteristics of a continuous, viscoelastic, one-directional Winkler foundation [31]. Then, a mathematical analysis of a solution of the problem concerning free vibration of sandwich beam for a continuous system with damping has been developed.

2. Formulation of the problem

The physical model of a structural system consists of two homogenous, elastic, parallel Timoshenko beams of equal length, coupled together by a viscoelastic interlayer (Fig. 1). The beams are supported at their ends. The viscoelastic interlayer corresponds to the characteristics of a homogenous, continuous, one-directional WINKLER foundation [31] and has been described by the Voigt-Kelvin model [9, 13, 16].

The mathematical model of the problem constitutes a system of the following coupled partial differential equations, describing small transverse vibration of the physical system in the following form:

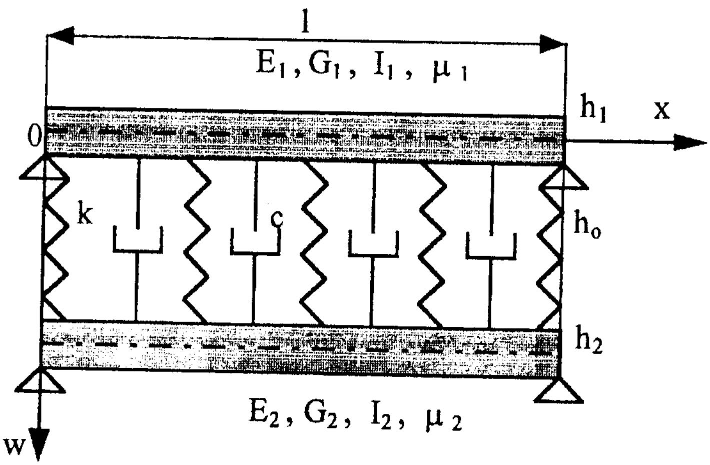

FIG. 1. Dynamic model of two Timoshenko beams coupled by a viscoelastic interlayer.

$$
(2.1)\quad
\begin{aligned}
&R_1 \frac{\partial^2 \psi_1}{\partial x^2} + N_1 \left(\frac{\partial w_1}{\partial x} - \psi_1 \right) - \Xi_1 \frac{\partial^2 \psi_1}{\partial t^2} = 0, \\
&\mu_1 \frac{\partial^2 w_1}{\partial t^2} - N_1 \left(\frac{\partial^2 w_1}{\partial x^2} - \frac{\partial \psi_1}{\partial x} \right) + (w_1 - w_2)k + c\frac{\partial}{\partial t}(w_1 - w_2) = 0. \\
&R_2 \frac{\partial^2 \Psi_2}{\partial x^2} + N_2 \left(\frac{\partial w_2}{\partial x} - \psi_2 \right) - \Xi_2 \frac{\partial^2 \Psi_2}{\partial t^2} = 0, \\
&\mu_2 \frac{\partial^2 w_2}{\partial t^2} - N_2 \left(\frac{\partial^2 w_2}{\partial x^2} - \frac{\partial \psi_2}{\partial x} \right) - (w_1 - w_2)\, k - c\frac{\partial}{\partial t}(w_1 - w_2) = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
&R_1 = E_1 I_1, \quad R_2 = E_2 I_1, \quad N_1 = k' G_1 F_1, \quad N_2 = k' G_2 F_2, \\
&\mu_1 = \rho_1 F_1, \quad \mu_2 = \rho_2 F_2, \quad \Xi_1 = \rho_1 I_1, \quad \Xi_2 = \rho I_2,
\end{aligned}
$$

and: $w_1 = w_1(x,t)$, $w_2 = w_2(x,t)$ – transverse deflections of beams I and II, $\psi_1 = \psi_1(x,t)$, $\psi_2 = \psi_2(x,t)$ – angle of rotation of cross-sections of beams I and II, E_1 , E_2 – Young's modulus of the material for beams I and II, I_1, I_2 – moments of inertia of cross-section of beams I and II, F_1, F_2 – areas of cross-section of beams I and II, G_1, G_2 – Kirchhoff's modulus of the material for beams I and II, ρ_1, ρ_2 – mass density of the material of beams I and II, k' – shear coefficient, k – coefficient of elasticity of the interlayer, c – coefficient of viscosity of the interlayer, h_1, h_2 – heights of beams I and II, h_0 – height of the interlayer, l – lengths of beams I and II.

Bending moment and transversal force are described by the following equations [24 – 28]:

$$M_1 = -R_1 \frac{\partial \Psi_1}{\partial x}, \quad Q_1 = k' G_1 F_1 \gamma_1, \quad M_2 = -R_2 \frac{\partial \psi_2}{\partial x}, \quad Q_2 = k' G_2 F_2 \gamma_2, \tag{2.2}$$

where

$$\frac{\partial w_1}{\partial x} = \psi_1 + \gamma_1, \quad \frac{\partial w_2}{\partial x} = \Psi_2 + \gamma_2,$$

and $\gamma_1 = \gamma_1(x,t)$, $\gamma_2(x,t)$ – angle of shear in beams I and II.

3. Solution of the boundary value problem

By substituting (3.1) and (3.2) [12, 13, 15, 17, 29, 1 – 5] in the system of differential equations (2.1):

$$w_1 = W_1(x)\exp(i\nu t), \qquad w_2 = W_2(x)\exp(i\nu t), \tag{3.1}$$

$$\psi_1 = \Psi_1(x)\exp(i\nu t), \qquad \psi_2 = \Psi_2(x)\exp(i\nu t), \tag{3.2}$$

the homogenous system of coupled ordinary differential equations describing the complex modes of vibration of Timoshenko beams shall be obtained:

$$\begin{aligned}
&R_1 \frac{d^2\Psi_1}{dx^2} + N_1\left(\frac{dW_1}{dx} - \Psi_1\right) + \Xi_1 \Psi_1 \nu^2 = 0,\\
&N_1\left(\frac{d^2 W_1}{dx^2} - \frac{d\Psi_1}{dx}\right) - k(W_1 - W_2) - ic\nu(W_1 - W_2) + \mu_1 W_1 \nu^2 = 0,\\
&R_2 \frac{d^2\Psi_2}{dx^2} + N_2\left(\frac{dW_2}{dx} - \Psi_2\right) + \Xi_2 \Psi_2 \nu^2 = 0,\\
&N_2\left(\frac{d^2 W_2}{dx^2} - \frac{d\Psi_2}{dx}\right) + k(W_1 - W_2) + ic\nu(W_1 - W_2) + \mu_2 W_2 \nu^2 = 0,
\end{aligned} \tag{3.3}$$

where: $W_1 = W_1(x)$, $W_2 = W_2(x)$ – the complex transverse vibrational modes of Timoshenko beams I and II, $\Psi_1 = \Psi_1(x)$, $\Psi_2 = \Psi_2(x)$ – the complex rotational modes of Timoshenko beams I and II, ν – the complex frequency of vibration of the beams I and II, t – time.

Searching for a particular solution of the system of differential equations (3.3) in the form of [6]:

(3.4) $$W_1 = A\exp(rx), \qquad W_2 = B\exp(rx),$$

(3.5) $$\Psi_1 = \Theta\exp(rx), \qquad \Psi_2 = \Gamma\exp(rx),$$

the homogeneous system of linear algebraic equations is obtained:

(3.6)
$$\begin{aligned}
&\Theta\left[R_1r^2 - N_1 + \Xi_1\nu^2\right] + AN_1r = 0,\\
&\Gamma\left[R_2r^2 - N_2 + \Xi_2\nu^2\right] + BN_2r = 0,\\
&A\left[N_1r^2 - k - ic\nu + \mu_1\nu^2\right] + B[k + ic\nu] - \Theta N_1r = 0,\\
&A[k + ic\nu] + B\left[N_2r^2 - k - ic\nu + \mu_2\nu^2\right] - \Gamma N_2r = 0.
\end{aligned}$$

Constructing the determinant of the characteristic matrix of the system of equations (3.6) and equating it to zero

(3.7)
$$\begin{vmatrix}
N_1r & 0 & R_1r^2 - N_1 + \Xi_1\nu^2 & 0\\
N_1r^2 - n_1 & k + ic\nu & -N_1r & 0\\
0 & N_2r & 0 & R_2r^2 - N_2 + \Xi_2\nu^2\\
k + ic\nu & N_2r^2 - n_2 & 0 & -N_2r
\end{vmatrix} = 0,$$

we obtain the characteristic equation in the form of the following algebraical equation:

(3.8) $$r^8 + a_{11}r^6 + a_{22}r^4 + a_{33}r^2 + a_{44} = 0,$$

with the following roots $r_j = (-1)^{j-1}i\lambda_\xi$, $j = (2\xi - 1)$, 2χ, $\chi = 1, 2, 3, 4$, where $n_1 = k + ic\nu - \mu_1\nu^2$, $n_2 = k + ic\nu - \mu_2\nu^2$, a_{11}, a_{22}, a_{33}, a_{44} are constant coefficients.

After applying the Euler formulas, the solution of the system of differential equations (3.3) consists of the fundamental system of solutions:

(3.9)
$$\begin{aligned}
W_1(x) &= \sum_{\chi=1}^{4} A_\chi^* \sin\lambda_\chi x + A_\chi^{**}\cos\lambda_\chi x\\
\Psi_1(x) &= \sum_{\chi=1}^{4} \Theta_\chi^* \cos\lambda_\chi x + \Theta_\chi^{**}\sin\lambda_\chi x,\\
W_2(x) &= \sum_{\chi=1}^{4} B_\chi^* \sin\lambda_\xi + B_\chi^{**}\cos\lambda_\chi x,\\
\Psi_2(x) &= \sum_{\chi=1}^{4} \Gamma_\chi^* \cos\lambda_\chi + \Gamma_\chi^{**}\sin\lambda_\chi x,
\end{aligned}$$

where A_χ^*, A_χ^{**}, B_χ^*, B_χ^{**}, Θ_χ^*, Θ_χ^{**}, Γ_χ^*, Γ_χ^{**} are constants, $\lambda_\xi = \alpha_\xi + i\beta_\xi$ is a parameter describing roots of the characteristic equation (3.8).

In agreement with (3.6), the following relations exist between constants of (3.9):

$$
\begin{aligned}
a_\chi^* &= \frac{B_\chi^*}{A_\chi^*}, & a_\chi^{**} &= \frac{B_\chi^{**}}{A_\chi^{**}},\\
b_\chi^* &= \frac{\Gamma_\chi^*}{\Theta_\chi^*}, & b_\chi^{**} &= \frac{\Gamma_\chi^{**}}{\Theta_\chi^{**}},\\
c_\chi^* &= \frac{\Theta_\chi^*}{A_\chi^*}, & c_\chi^{**} &= \frac{\Theta_\chi^{**}}{A_\chi^{**}},\\
d_\chi^* &= \frac{\Gamma_\chi^*}{A_\chi^*}, & d_\chi^{**} &= \frac{\Gamma_\chi^{**}}{A_\chi^{**}},
\end{aligned}
\tag{3.10}
$$

where

$$
\begin{aligned}
a_\chi^* &= a_\chi^{**} = a_\chi = -\frac{NN_1 + RR_1 nn_1}{k + ic\nu} = -\frac{k + ic\nu}{NN_2 + RR_2 nn_2},\\
b_\chi^* &= b_\chi^{**} = b_\chi = a_\chi \frac{RR_2}{RR_1},\\
c_\chi^* &= c_\chi = RR_1 i, \quad c_\chi^* = -c_\chi^{**},\\
d_\chi^* &= d_\chi = RR_2 i, \quad d_\chi^* = -d_\chi^{**},
\end{aligned}
\tag{3.11}
$$

and:

$$
\begin{aligned}
NN_1 &= -N_1\lambda_\chi^2 - k - ic\nu + \mu_1\nu^2,\\
NN_2 &= -N_2\lambda_\chi^2 - k - ic\nu + \mu_2\nu^2,\\
nn_1 &= iN_1\lambda_\chi, \quad nn_2 = iN_2\lambda_\chi,\\
RR_1 &= i\frac{N_1\lambda_\chi}{-R_1\lambda_\chi^2 - N_1 + \Xi_1\nu^2},\\
RR_2 &= i\frac{N_2\lambda_\chi}{-R_2\lambda_\chi^2 - N_2 + \Xi_2\nu^2}.
\end{aligned}
\tag{3.12}
$$

After substituting (3.10) in (3.9), the general solution of the system of differential equations (3.3) takes the following form:

(3.13)
$$\begin{aligned}
W_1(x) &= \sum_{\chi=1}^{4} A_\chi^* \sin\lambda_\chi + A_\chi^{**}\cos\lambda_\chi x,\\
\Psi_1(x) &= \sum_{\chi=1}^{4} c_\chi \left(A_\chi^* \cos\lambda_\chi x - A_\chi^{**}\sin\lambda_\chi x\right),\\
W_2(x) &= \sum_{\chi=1}^{4} a_\chi \left(A_\chi^* \sin\lambda_\chi x + A_\chi^{**}\cos\lambda_\chi x\right),\\
\Psi_2(x) &= \sum_{\chi=1}^{4} d_\chi \left(A_\chi^* \cos\lambda_\chi x - A_\chi^{**}\sin\lambda_\chi x\right).
\end{aligned}$$

In order to solve the boundary-value problem, the following boundary conditions have been applied:

(3.14)
$$\begin{aligned}
W_1(0) &= 0, & W_1(l) &= 0,\\
W_2(0) &= 0, & W_2(l) &= 0,\\
\frac{d\Psi_1}{dx}(0) &= 0, & \frac{d\Psi_1}{dx}(l) &= 0,\\
\frac{d\Psi_2}{dx}(0) &= 0, & \frac{d\Psi_2}{dx}(l) &= 0.
\end{aligned}$$

By substituting (3.13) in (3.14), the homogenous system of linear algebraic equations shall be obtained, which in the matrix notation has the following form:

(3.15)
$$\mathbf{YX} = 0,$$

where $\mathbf{X} = [A_1^*, A_2^*, A_3^*, A_4^*, A_1^{**}, A_2^{**}, A_3^{**}, A_4^{**}]^T$ is a vector of unknowns of the system of equations, and

(3.16)
$$\mathbf{Y} = [Y_{i*J}]_{8*8}$$

is the characteristic matrix of the system of equations (3.15).

(3.17)
$$\begin{bmatrix} 1 & 1 & 1 & 1\\ c_1 & c_2 & c_3 & c_4\\ a_1 & a_2 & a_3 & a_4\\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} \begin{bmatrix} A_1^{**}\\ A_2^{**}\\ A_3^{**}\\ A_4^{**} \end{bmatrix} = 0.$$

From the system of equations (3.17) we obtain $A_1^{**} = A_2^{**} = A_3^{**} = A_4^{**} = 0$. The remaining four equations (3.15) give the following system of equations:

$$\begin{bmatrix} \sin\lambda_1 l & \sin\lambda_2 l & \sin\lambda_3 l & \sin\lambda_4 l \\ -\lambda_1 c_1 \sin\lambda_1 l & -\lambda_2 c_2 \sin\lambda_2 l & -\lambda_3 c_3 \sin\lambda_3 l & -\lambda_4 c_4 \sin\lambda_4 l \\ a_1 \sin\lambda_1 l & a_2 \sin\lambda_2 l & a_3 \sin\lambda_3 l & a_4 \sin\lambda_4 l \\ -\lambda_1 d_1 \sin\lambda_1 l & -\lambda_2 d_2 \sin\lambda_2 l & -\lambda_3 d_3 \sin\lambda_3 l & -\lambda_4 d_4 \sin\lambda_4 l \end{bmatrix} \times \begin{bmatrix} A_1^* \\ A_2^* \\ A_3^* \\ A_4^* \end{bmatrix} = 0. \tag{3.18}$$

The condition of solving the system of equations (3.18) is vanishing of the characteristic determinant, i.e.

$$\begin{vmatrix} \sin\lambda_1 l & \sin\lambda_2 l & \sin\lambda_3 l & \sin\lambda_4 l \\ -\lambda_1 c_1 \sin\lambda_1 l & -\lambda_2 c_2 \sin\lambda_2 l & -\lambda_3 c_3 \sin\lambda_3 l & -\lambda_4 c_4 \sin\lambda_4 l \\ a_1 \sin\lambda_1 l & a_2 \sin\lambda_2 l & a_3 \sin\lambda_3 l & a_4 \sin\lambda_4 l \\ -\lambda_1 d_1 \sin\lambda_1 l & -\lambda_2 d_2 \sin\lambda_2 l & -\lambda_3 d_3 \sin\lambda_3 l & -\lambda_4 d_4 \sin\lambda_4 l \end{vmatrix} = 0. \tag{3.19}$$

Expanding the determinant (3.19), the following characteristic equation has been obtained:

$$\sin\lambda_1 l \sin\lambda_2 l \sin\lambda_3 l \sin\lambda_4 l = 0, \tag{3.20}$$

where $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$.
The characteristic equation (3.20) may be rewritten in the form:

$$\sin\lambda l = 0, \tag{3.21}$$

where

$$\lambda = \alpha + i\beta \tag{3.22}$$

in the general case are complex numbers.

Substituting (3.22) in (3.21), the following equation has been obtained:

$$\sin\alpha l Ch\beta l + i\cos\alpha l Sh\beta l = 0, \tag{3.23}$$

which has the following roots:

$$\alpha_s = \frac{s\pi}{l}, \qquad \beta_s = 0, \qquad s = 1, 2, 3 \ldots. \tag{3.24}$$

Taking into consideration (3.24) in (3.22), the following identity has been obtained:

$$\lambda_s = \alpha_s = \frac{s\pi}{l}. \tag{3.25}$$

By substituting $r = i\lambda_s$ in the equation (3.8) and carrying out all the transformations, the following equation of frequency is obtained:

$$\nu^8 + b_{11}\nu^7 + b_{22}\nu^6 + b_{33}\nu^5 + b_{44}\nu^4 + b_{55}\nu^3 + b_{66}\nu^2 + b_{77}\nu + b_{88} = 0, \tag{3.26}$$

from which a sequence of complex natural frequencies will be determined

$$\nu_n = i\eta_n \pm \omega_n, \tag{3.27}$$

where $n = (4s-3)$, $(4s-2)$, $(4s-1)$, $4s$, and b_{11}, b_{22}, b_{33}, b_{44}, b_{55}, b_{66}, b_{77}, b_{88} are constant coefficients.

Substituting equation (3.27) in equations (3.11) and (3.12), the following formulas for coefficients of amplitudes are obtained:

$$\begin{aligned} a_n &= -\frac{NN_1 + RR_1 nn_1}{k + ic\nu_n} = -\frac{k + ic\nu_n}{NN_2 + RR_2 nn_2}, \\ b_n &= a_n \frac{RR_2}{RR_1}, \\ c_n &= RR_1 i, \\ d_n &= RR_2 i, \end{aligned} \tag{3.28}$$

$$\begin{aligned} NN_1 &= -N_1\lambda_s^2 - k - ic\nu_n + \mu_1\nu_n^2, \\ NN_2 &= -N_2\lambda_s^2 - k - ic\nu_n + \mu_2\nu_n^2, \\ nn_1 &= iN_1\lambda_s, \qquad nn_2 = iN_2\lambda_s, \\ RR_1 &= i\frac{N_1\lambda_s}{-R_1\lambda_s^2 - N_1 + \Xi_1\nu_n^2}, \\ RR_2 &= i\frac{N_2\lambda_s}{-R_2\lambda_s^2 - N_2 + \Xi_2\nu_n^2}. \end{aligned} \tag{3.29}$$

Substituting the sequences λ_s and a_n, c_n, d_n in (3.13), the four following sequences of modes of free vibration for two Timoshenko beams are obtained:

$$\begin{aligned} W_{1n}(x) &= \sin\lambda_s x, \\ \Psi_{1n}(x) &= c_n \cos\lambda_s x, \\ W_{2n}(x) &= a_n \sin\lambda_s x, \\ \Psi_{2n}(x) &= d_n \cos\lambda_s x. \end{aligned} \tag{3.30}$$

Free vibration for two Bernoulli-Euler beams are described (3.30), under the assumption that integration constants Θ^*_χ, Θ^{**}_χ, Γ^*_χ, Γ^{**}_χ, are zero [1, 4].

4. Solution of the initial value problem

The complex equation of motion

$$T = \Phi \exp(i\nu t), \tag{4.1}$$

in the case of $\nu = \nu_n$ can be written in the following form:

$$T_n = \Phi_n \exp(i\nu_n t), \tag{4.2}$$

where Φ_n – the Fourier coefficient.

Free vibration of Timoshenko beams has been presented in the form of the Fourier series based on the complex eigenfunctions [15], i.e.:

$$\begin{aligned} w_{1n} &= \sum_{n=1}^{\infty} W_{1n}(x)\Phi_n \exp(i\nu_n t), \quad w_{2n} = \sum_{n=1}^{\infty} W_{2n}(x)\Phi_n \exp(i\nu_n t), \\ \Psi_{1n} &= \sum_{n=1}^{\infty} \Psi_{1n}(x)\Phi_n \exp(i\nu_n t), \quad \Psi_{2n} = \sum_{n=1}^{\infty} \Psi_{2n}(x)\Phi_n \exp(i\nu_n t). \end{aligned} \tag{4.3}$$

From the system of equations (3.3), performing some algebraical transformations, adding the equations together, and then integrating them on both sides in limits from 0 to l, the property of orthogonality of eigenfunctions for two Timoshenko beams coupled together by a viscoelastic interlayer has been obtained

$$\begin{aligned} \int_0^l \Big[&\xi_1 (W_{1m}V_{1n} + W_{1n}V_{1m}) + \xi_2 (W_{2n}V_{2n} + W_{2n}V_{2m}) \\ &+ \zeta_1 (\Psi_{1m}Q_{1n} + \Psi_{1n}Q_{1m}) + \zeta_2 (\Psi_{2m}Q_{2n} + \Psi_{2n}Q_{2m}) \\ &+ 2\eta (W_{1n} - W_{2n})(W_{1m} - W_{2m}) \Big] dx = N_n \delta_{mn}, \end{aligned} \tag{4.4}$$

where: δ_{mn} – Kronecker delta,

(4.5)
$$N_n = 2\int_0^l [\xi_1 W_{1n} V_{1n} + \xi_2 W_{2n} V_{2n} + \zeta_1 \Psi_{1n} Q_{1n} + \zeta_2 \Psi_{2n} Q_{2n} + \eta (W_{1n} - W_{2m})^2]\, dx,$$

(4.6)
$$\begin{aligned} V_{1n} &= i\nu_n W_{1n}(x), & V_{2n} &= i\nu_n W_{2n}(x),\\ V_{1m} &= i\nu_m W_{1m}(x), & V_{2m} &= i\nu_m W_{2m}(x),\\ Q_{1n} &= i\nu_n \Psi_{1n}(x), & Q_{2n} &= i\nu_n \Psi_{2n}(x),\\ Q_{1m} &= i\nu_m \Psi_{1m}(x), & Q_{2m} &= i\nu_m \Psi_{2m}(x), \end{aligned}$$
$$\xi_1 = \frac{\mu_1}{\mu},\quad \xi_2 = \frac{\mu_2}{\mu},\qquad \zeta_1 = \frac{\Xi_1}{\mu},\quad \zeta_2 = \frac{\Xi_2}{\mu}.$$

In case of a system of two strings [3], two Bernoulli-Euler beams [1, 4], and a system of a string and a Bernoulli-Euler beam [2]:

(4.7)
$$\Psi_{1n} = \frac{dW_{1n}}{dx},\quad \Psi_{2n} = \frac{dW_{2n}}{dx},\quad \Psi_{1m} = \frac{dW_{1m}}{dx},\quad \Psi_{2m} = \frac{dW_{2m}}{dx},$$
$$\zeta_1 = \zeta_2 = 0.$$

The following initial conditions are the basis for solving the problem of free vibrations:

(4.8)
$$\begin{aligned} w_1(x,0) &= w_{01},\ w_2(x,0) = w_{02},\ \Psi_1(x,0) = \Psi_{01},\ \Psi_2(x,0) = \Psi_{02},\\ \dot w_1(x,0) &= \dot w_{01},\ \dot w_2(x,0) = \dot w_{02},\ \Psi_1(x,0) = \Psi_{01},\ \Psi_2(x,0) = \Psi_{02}. \end{aligned}$$

Applying conditions (4.8) in the series (4.3) and taking into account the property of orthogonality (4.4), the formula for the Fourier coefficient is obtained,

(4.9)
$$\begin{aligned} \Phi_n = \frac{1}{N_n}\int_0^l \{&\xi_1 (V_{1n} w_{01} + W_{1n} \dot w_{01}) + \xi_2 (V_{2n} w_{02} + W_{2n} \dot w_{02})\\ &+ \zeta_1 \left(Q_{1n} \psi_{01} + \Psi_{1n} \dot\psi_{01}\right) + \zeta_2 \left(Q_{2n} \psi_{02} + \Psi_{2n} \dot\psi_{02}\right)\\ &+ 2\eta \left[(W_{1n} - W_{2n}) (w_{01} - w_{02})\right]\}\, dx. \end{aligned}$$

After substituting (3.30), (4.2) and (4.9) in (4.3) and performing the trigonometrical and algebraic transformations, the final form of free vibration of Timoshenko beams is obtained:

$$
\begin{aligned}
w_1 &= \sum_{n=1}^{\infty} e^{-\eta_n t} |W_{1n}| |\Phi_n| \cos(\omega_n t + \varphi_n + \xi_{1n}),\\
\psi_1 &= \sum_{n=1}^{\infty} e^{-\eta_n t} |\Psi_{1n}| |\Phi_n| \cos(\omega_n t + \varphi_n + \theta_{1n}),\\
w_2 &= \sum_{n=1}^{\infty} e^{-\eta_n t} |W_{2n}| |\Phi_n| \cos(\omega_n t + \varphi_n + \chi_{2n}),\\
\psi_2 &= \sum_{n=2}^{\infty} e^{-\eta_n t} |\Psi_{2n}| |\Phi_n| \cos(\omega_n t + \varphi_n + \theta_{2n}),
\end{aligned}
\tag{4.10}
$$

where

$$
\begin{aligned}
|W_{1n}| &= \sqrt{X_{1n}^2 + Y_{1n}^2}, \quad |W_{2n}| = \sqrt{X_{2n}^2 + Y_{2n}^2},\\
|\Psi_{1n}| &= \sqrt{\Lambda_{1n}^2 + \Omega_{1n}^2}, \quad |\Psi_{2n}| = \sqrt{\Lambda_{2n}^2 + \Omega_{2n}^2},\\
\chi_{1n} &= \arg W_{1n}, \quad \chi_{2n} = \arg W_{2n},\\
\theta_{1n} &= \arg \Psi_{1n}, \quad \theta_{2n} = \arg \Psi_{2n},\\
|\Phi_n| &= \sqrt{C_n^2 + D_n^2}, \quad \varphi_n = \arg \Phi_n,
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
X_{1n} &= \mathrm{re} W_{1n}, \quad Y_{1n} = \mathrm{im} W_{1n}, \quad X_{2n} = \mathrm{re} W_{2n}, \quad Y_{2n} = \mathrm{im} W_{2m},\\
\Lambda_{1n} &= \mathrm{re} \Psi_{1n}, \quad \Omega_{1n} = \mathrm{im} \Psi_{1n}, \quad \Lambda_{2n} = \mathrm{re} \Psi_{2n}, \quad \Omega_{2n} = \mathrm{im} \Psi_{2n},\\
C_n &= \mathrm{re} \Phi_n, \quad D_n = \mathrm{im} \Phi_n.
\end{aligned}
\tag{4.12}
$$

Free vibrations of Bernoulli-Euler beams are described in the form of (4.10), where $\psi_1 = \dfrac{dw_1}{dx}$, $\psi_2 = \dfrac{dw_2}{dx}$ [1, 4].

5. Numerical results

Computer calculations have been carried out for the following data: $E_1 = E_2 = E = 2.1 * 10^{11}$ [$\mathrm{Nm^{-2}}$], $E_0 = 10^8$ [$\mathrm{Nm^{-2}}$], $k = (E_0 b_0)/h_0$, $k' = 0.84$, $\nu_0 = 0.2$, $c = 0.75$ [$\mathrm{N\ s\ m^{-2}}$], $\rho_1 = \rho_2 = 7.8{*}10^3$ [$\mathrm{N\ s^2 m^{-4}}$], $b_1 = b_2 = b_0 = b$,

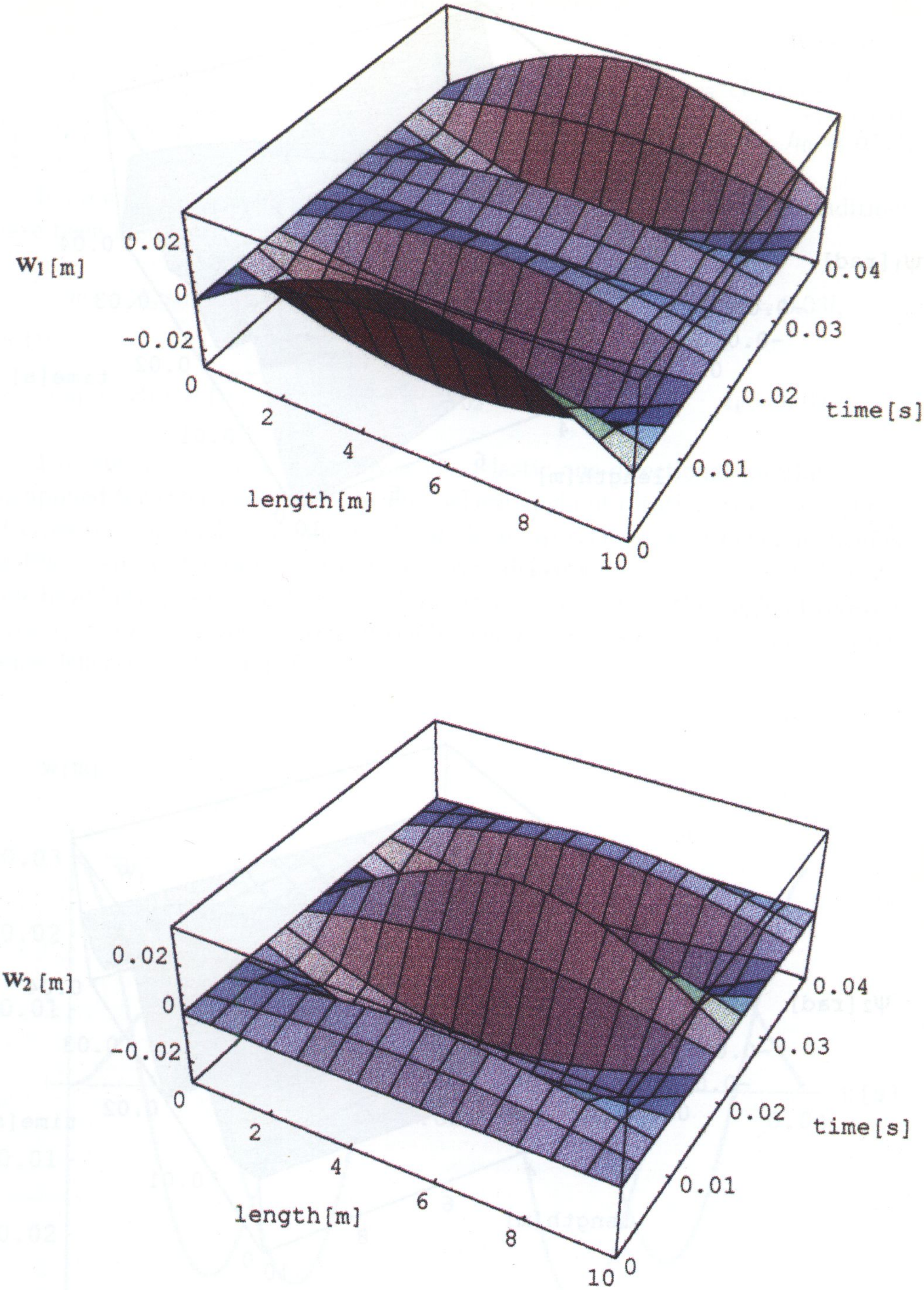

FIG. 2. Transverse deflections $w_1(x,t)$, $w_2(x,t)$ of beams I and II.

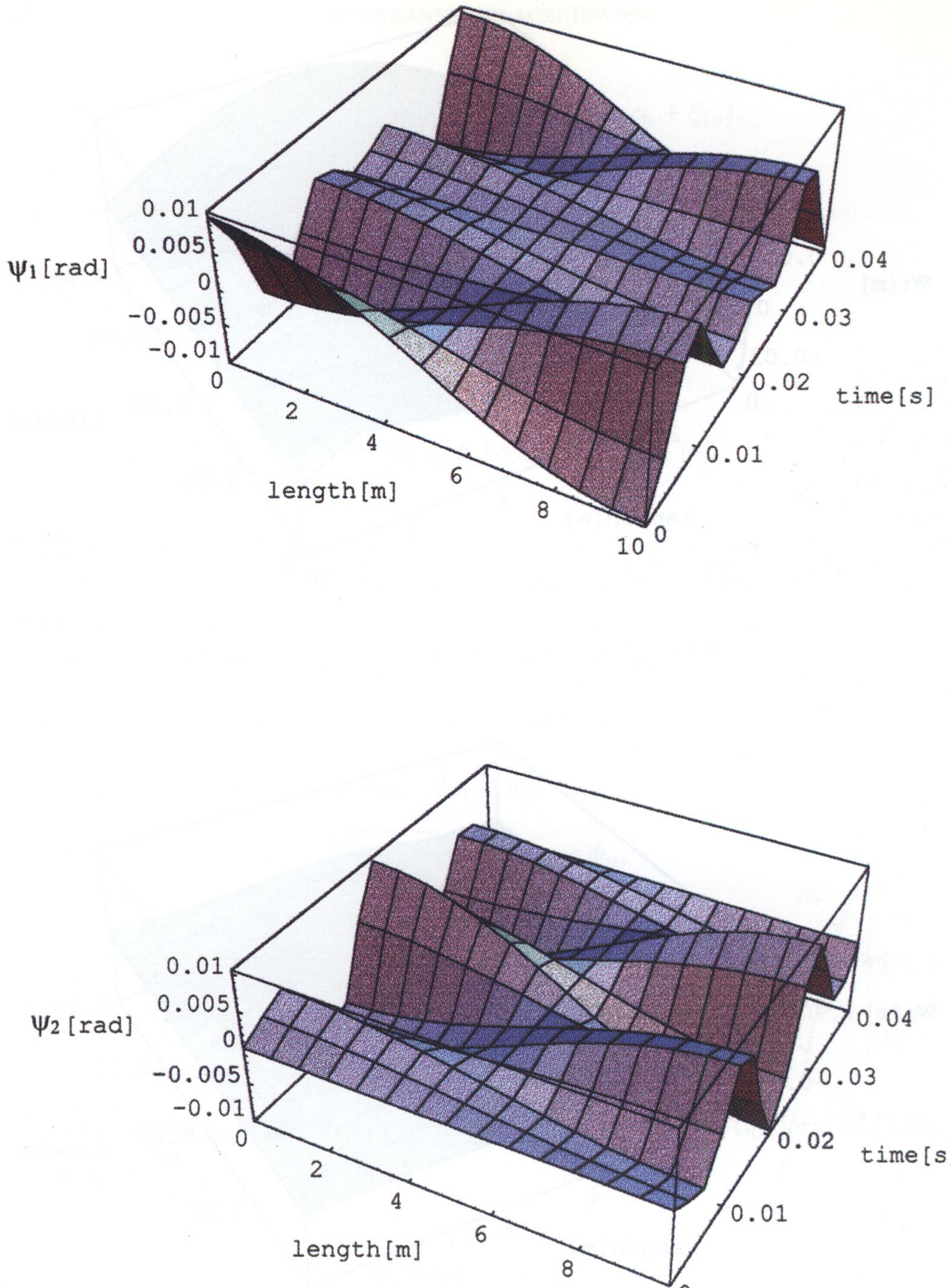

FIG. 3. Angle of rotation of cross-section of beams I and II.

$h_1 = h_2 = h_0 = h$, $b = 0.6$ [m], $1 = 10$ [m], $G_1 = G_2 = G = \dfrac{E}{2(1+\nu_o)}$, $F_1 = F_2 = bh$, $I_1 = I_2 = (bh^3)/12$, $H/1 = 0.3$, $H = h_1 + h_2 + h_0$, $h_0 = \alpha^* H$, $\alpha^* = 0.2$.

In order to find the Fourier coefficient Φ_n (4.9), the following initial conditions have been assumed:

$$w = A_s \sin\left(\frac{\pi x}{l}\right), \quad \dot{w}_{01} = 0, \quad w_{02} = 0^+, \quad \dot{w}_{02} = 0, \quad A_s = 0.03l,$$

(5.1)

$$\psi_{01} = A_k \cos\left(\frac{\pi x}{l}\right), \quad \dot{\psi}_{01} = 0, \quad \psi_{02} = 0^+, \quad \dot{\psi}_{02} = 0, \quad A_k = 0.03\pi.$$

Timoshenko beams coupled by a viscoelastic one-directional interlayer may be applied for the case of thick beams, where angle of rotation $\psi_1(x,t)$, $\psi_2(x,t)$ of cross-sections of beams and angle of shear $\gamma_1(x,t)$, $\gamma_2(x,t)$ occur in beams. In Fig. 2 space diagrams of the transversal deflections $w_1(x,t)$, $w_2(x,t)$ of beams have been presented. Figure 3 shows space diagrams of the angle of rotation $\psi_1(x,t)$, $\psi_2(x,t)$ of cross-section of the beams. Figure 4 shows the dynamic transverse deflections of beams for $x = 0.5$ 1.

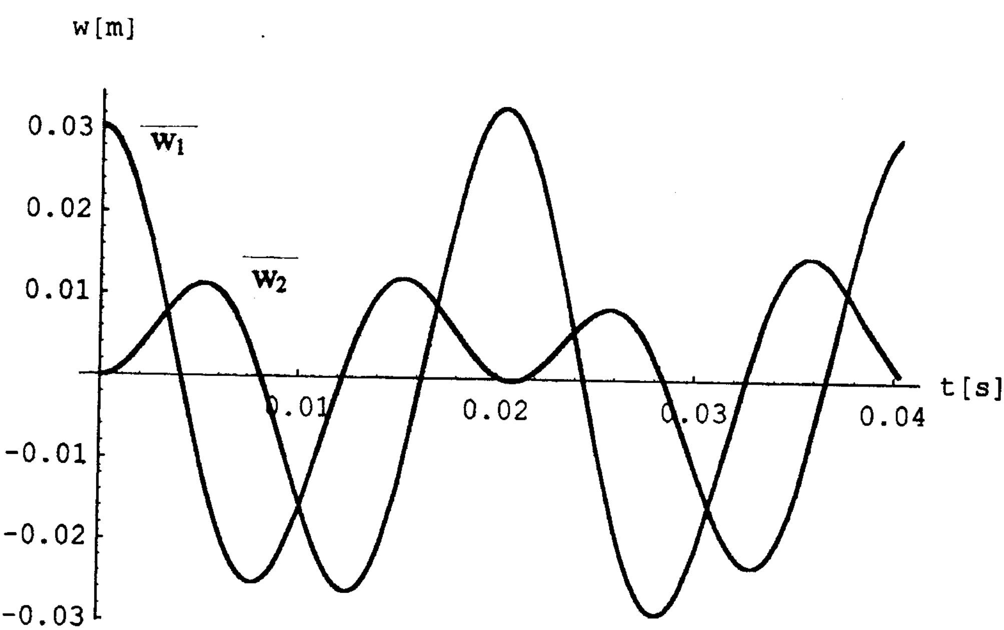

FIG. 4. Transverse deflections $w_1(t)$, $w_2(t)$ of beams I and II in the middle of their length.

6. Conclusions

In this paper complex frequencies and modes of free vibrations as well as complex motion function, with any initial conditions, have been the basis of an analytical solution of a problem concerning free vibrations of a sandwich beam with damping. The derived property of orthogonality of complex modes of free vibrations plays a crucial role in the applied method. The method presented in this paper may be also applied for solving a problem of free and forced vibrations of various engineering systems consisting of sandwich beams with damping.

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