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DISCRETE-CONTINUOUS MODELS IN NONLINEAR DYNAMIC INVESTIGATIONS OF SELECTED PHYSICAL SYSTEMS

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The paper deals with dynamic investigations of nonlinear discrete-continuous models in which elastic elements are deformed torsionally, longitudinally or transversally, and the classical wave equation can be used for the description of their motion. The investigations focus on nonlinear vibrations of the discrete-continuous models with a local nonlinearity described by the polynomial of a third degree. The detailed analysis for a simple nonlinear discrete-continuous system is done. It is shown that amplitude jumps in the case of a hard characteristic and the escape phenomenon in the case of a soft characteristic can occur.

1. Introduction

The paper concerns dynamic investigations of nonlinear discrete-continuous models. As it follows from the technical literature, the discrete-continuous systems have received much attention. Such models consist of rigid bodies connected by means of ponderable elastic elements. To these models also belong those where the motion of elastic elements can be described by the classical wave equation.

The use of the classical wave equation gives some limitations for the group of the systems under considerations; on the other hand, it enables to apply the solution of the d'Alembert type leading to the equations with a retarded argument. The foundations of the wave approach one can find in [1 – 5].

The wave approach allows us to consider the systems torsionally, longitudinally or transversally deformed. In systems torsionally deformed, one may consider shafts on which various discs and gears are set. Such shafts can be found in branched systems, gear transmissions, internal combustion engines, transport drive systems, current-generating sets; mainly – in systems operating in rotary motion, having densely distributed bearings, a considerable bending stiffness and a small stiffness in torsion, [1, 3 – 14]. As longitudinally systems deformed, one may consider certain machine elements, truss members, railway cars and river barges, [1, 5, 15]. Among the systems undergoing transverse deformations, one may consider e.g. the string systems and low structures subject to transversal excitations, [1, 2, 5, 16, 17]. It should be pointed out that the utilized wave approach is verified experimentally in [18].

The present paper concerns nonlinear discrete-continuous systems with local nonlinearities. The variety of nonlinear problems is wide. It may include weak and strong nonlinearities. In [8 – 12] it is shown how various local nonlinearities, including strong ones and impacts, can be incorporated in the analysis of complex mechanical discrete-continuous systems torsionally deformed. Interesting results given there have an important practical meaning.

However, many problems concerning nonlinear vibrations of discrete-continuous systems have not been discussed yet. In the dynamics of nonlinear discrete systems such phenomena as amplitude jumps and escapes are observed, [19 – 21]. It is interesting if in nonlinear discrete-continuous systems such types of phenomena can occur. These phenomena are not noticed in dynamic investigations of nonlinear problems discussed in papers [8 – 12].

The aim of the paper is the study of nonlinear vibrations of discrete-continuous systems with local nonlinearities presented by nonlinear discrete elements. The considerations concern those nonlinear discrete-continuous systems which have nonlinear discrete elements being either springs with a nonlinear characteristic, or being modelled by means of nonlinear springs. The nonlinear discrete elements can have the nonlinear characteristic of a soft as well as of a hard type. The presence of local nonlinearities in discrete-continuous models can have important consequences for their overall dynamic behaviour. The local nonlinearity is described by the third-degree polynomial function.

Some effects of the local nonlinearity described by the third-degree polynomial in complex discrete-continuous systems have been already shown in [13, 15, 17], however for the local nonlinearity having the characteristic of a hard type. In the present paper, after short reminding of the wave approach, the detailed considerations are done for a simple nonlinear discrete-continuous system with the nonlinear spring characteristic being of a hard as well as of a soft type.

2. Short presentation of the wave approach

Consider the physical systems described by discrete-continuous models consisting of an arbitrary number of homogeneous elastic elements connected by means of a suitable number of rigid bodies. The cross-sections of elastic elements remain flat during the motion. Their lengths are finite and their cross-sections can be moderately variable, [5]. The considerations concerning nonlinear models are limited here to the constant cross-sections of the elastic elements, and their motion is described by the classical wave equation.

The displacements and velocities of all cross-sections of the elastic elements are assumed to be equal to zero at the time instant $t = 0$, and the system is loaded by the forces $P_i(t)$. External and internal damping in the considered systems is taken into account by means of an equivalent damping applied in the selected cross-sections of the elastic elements, [5].

Local nonlinearities represented by nonlinear discrete elements can be introduced into discrete-continuous models. The inclusion of such types of nonlinearities is suggested by many engineering solutions. The springs in these elements can have nonlinear characteristic of a hard as well as of a soft type. In the present paper, the force in the nonlinear spring is assumed to be expressed by

$$F(U_i) = k_1 U_i + k_3 U_i^3. \tag{2.1}$$

The function U_i is the displacement of the elastic element of the discrete-continuous system where the local nonlinearity is taken into account, and the constants k_1 and k_3 represent linear and nonlinear terms, respectively. The nonlinear function of the type (2.1) is widely exploited in the literature in dynamic investigations of nonlinear discrete systems, [19 – 21]. In the present paper the function (2.1) is adopted to the study of the behaviour of the nonlinear discrete-continuous systems.

The equation of motion for the i-th elastic element is assumed in the form

$$U_{i,tt}(x,t) - c_f U_{i,xx}(x,t) = 0, \tag{2.2}$$

where c_f is a wave speed, and the comma denotes partial differentiation.

Searching solutions for specific systems, we must add to Eq. (2.2) the initial conditions

$$U_i(x,0) = 0, \qquad U_{i,t}(x,0) = 0 \tag{2.3}$$

and appropriate boundary conditions. These are the conditions for the displacements of the i-th and the $i+1$-th elastic elements of the system

$$U_i(x,t) = U_{i+1}(x,t) \tag{2.4}$$

in the cross-sections of the contact of these elements, or the conditions for forces acting either in the cross-sections of the contact of the neighbouring elastic elements or in the cross-sections in which rigid bodies are attached. The rigid bodies can be loaded by the forces $P_i(t)$.

These latter conditions mostly take the form

1) for a free end

$$U_{i,x} = 0, \tag{2.5}$$

2) for the cross-section where a rigid body is connected with successive elastic elements

$$P_i + a_{1i}U_{i,tt} + a_{2i}U_{i,x} + a_{3i}U_{i+1,x} + a_{4i}U_{i,t} + a_{5i}U_{i,xt} + F(U_i) = 0, \tag{2.6}$$

3) for the contact cross-section of successive elastic elements where a rigid body is not attached

$$a_{2i}U_{i,x} + a_{3i}U_{i+1,x} = 0, \tag{2.7}$$

4) when in the left-hand end of the elastic element only a rigid body is attached

$$P_i - a_{1i}U_{i,tt} + a_{2i}U_{i,x} - a_{4i}U_{i,t} + a_{5i}U_{i,xt} = 0, \tag{2.8}$$

5) when in the right-hand end of the elastic element a rigid body is attached

$$P_i + a_{1i}U_{i,tt} + a_{2i}U_{i,x} + a_{4i}U_{i,t} + a_{5i}U_{i,xt} = 0, \tag{2.9}$$

where a_{1i} are determined by the masses of rigid bodies or by their mass moments of inertia, a_{2i} and a_{3i} are determined by the material constants of the elements, a_{4i} represent the coefficients of the equivalent external damping while a_{5i} represent the coefficients of the equivalent internal damping. It should be pointed out that taking into account the equivalent damping applied in the selected cross-sections made it possible to assume equations of motion (2.2) in which damping is neglected. The local nonlinearities represented by the function (2.1) may be introduced in any contact cross-section.

If the appropriate coefficients a_{ji}, forces P_i and $F(U_i)$ are equal to zero, then the boundary conditions (2.5) – (2.9) contain numerous particular cases of the physical systems torsionally, longitudinally or transversally deformed, e.g. the nonlinear systems considered in [13, 15, 17] and the linear systems discussed in [1 – 7]. The boundary conditions (2.4) – (2.9) are valid directly for uniaxial discrete-continuous systems. When the elastic elements do not have the common axis then the boundary conditions (2.4) – (2.9) have to be slighty modified, as it is done for a single gear transmission in [4, 7, 14] and for the plane truss members in [15]. Appropriate boundary conditions for various complex discrete-continuous mechanical systems concerning the problems of a practical meaning can be found in [8 – 12, 16, 18].

The solution of Eq. (2.2) is sought in the form

$$U_i(x,t) = F_i(c_f(t-t_{0i}) - x + x_{0i}) + G_i(c_f(t-t_{0i}) + x - x_{0i}), \tag{2.10}$$

where the functions F_i and G_i represent disturbances caused by the external forces in the i-th elastic elements of the considered systems, in a direction consistent and opposite to the direction of the x-axis, respectively. The constants t_{0i} and x_{0i} in the arguments of these functions denote the time instant and the location of one of the ends of the i-th element in which the first disturbance reaches this element. Moreover, the functions F_i and G_i are equal to zero for negative arguments.

The functions F_i and G_i are the functions of a single variable. Their forms are determined by the boundary conditions of a specific problem. Substituting the solution (2.10) into suitable boundary conditions, a set of ordinary differential equations with a retarded argument is obtained for the functions F_i, G_i. For linear systems these equations can be solved analytically or numerically, however in nonlinear cases only numerical solutions are possible. The class of the unknown functions F_i and G_i is also determined by the boundary conditions of a specific problem.

Numerical calculations for nonlinear discrete-continuous systems with the local nonlinearity (2.1) are much more laborious than the linear cases. The nonlinear models describing appropriate real complex systems are characterized by different parameters, so they have different resonant regions and different numerical results, [13, 15, 17]. However, the suitable solutions for nonlinear vibrations have some similar properties. These properties one can notice also in the case of a simple nonlinear discrete-continuous system discussed below. Such a system can be treated as a particular case for all nonlinear discrete-continuous systems studied in [13, 15, 17]. For this reason, detailed analytical and numerical analysis may be done e.g. for this simple nonlinear system. Moreover, in [13, 15, 17] the local nonlinearity of a hard type is considered while in the present paper the hard as well as the soft characteristic cases are taken into account.

3. Nonlinear vibrations of a simple discrete-continuous system

As an example, a simple physical system shown in Fig. 1 is considered. The system consists of an elastic element, a rigid body and a nonlinear discrete element. The elastic element can be subject to longitudinal, torsional or transverse deformations with the motion described by the classical wave equation. Here it is assumed that the system is longitudinally deformed and is characterized by the length l_0, the constant cross-section A, the density ρ and the Young's modulus E. The left-hand end of the elastic element is connected to a rigid body having the mass m_0, and to a discrete element consisting of a nonlinear spring and a damper of the viscous type with the coefficient d_0. The right-hand end of the elastic element is fixed. The force in the nonlinear spring according to (2.1) is expressed by the nonlinear function

$$F(t) = k_1 u(x,t) + k_3 u^3(x,t) \qquad \text{for} \quad x = 0, \tag{3.1}$$

where $u(x,t)$ is the displacement of the cross-sections of the elastic element. The x-axis is parallel to the axis of the elastic element and its origin coincides with the location of the left-hand end of the elastic element at $t = 0$. Displacements and velocities of all cross-sections are assumed to be equal to zero at $t = 0$. The formula (3.1) contains the linear case with $k_3 = 0$, the hard characteristic case for $k_3 > 0$ and the soft characteristic case with $k_3 < 0$.

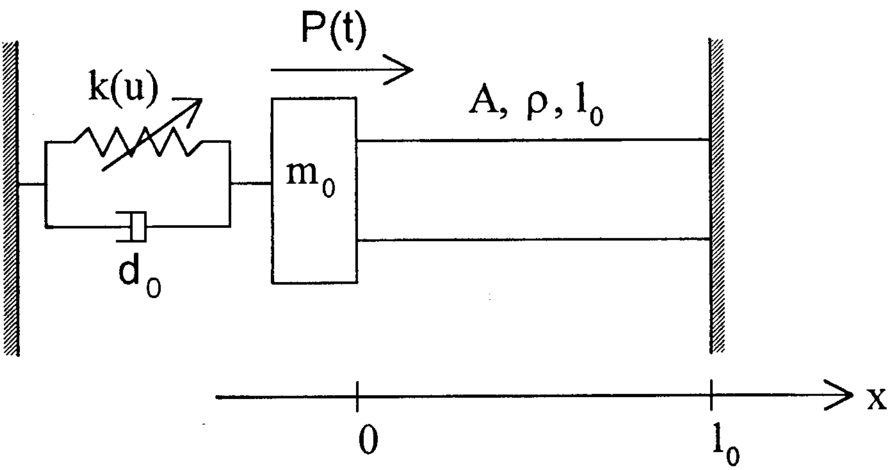

FIG. 1. A simple nonlinear discrete-continuous system.

An external loading applied to the rigid body can be described by various time functions. In the analogy to the nonlinear discrete models, it is assumed in the form

$$P(t) = a_0 \sin(pt), \tag{3.2}$$

where a_0 and p are constant.

The problem of determining the displacements, strains and velocities in the cross-sections of the elastic element for the analyzed discrete-continuous system, under the above assumptions, is reduced to solving the classical wave equation

$$u_{,tt} - a^2 u_{,xx} = 0 \tag{3.3}$$

with the zero initial conditions

$$u(x,0) = u_{,t}(x,0) = 0 \tag{3.4}$$

and the following boundary conditions:

$$\begin{aligned} &m_0 u_{,tt} + d_0 u_{,t} + k_1 u + k_3 u^3 - AE(D_0 u_{,xt} + u_{,x}) = a_0 \sin(pt) \quad \text{for} \quad x = 0, \\ &u(x,t) = 0 \qquad \text{for} \quad x = l_0, \end{aligned} \tag{3.5}$$

where $a^2 = E/\rho$, and the coefficient D_0 represents the internal damping, [5].

Upon the introduction of the nondimensional quantities

$$\begin{aligned} \overline{x} &= x/l_0, \quad \overline{t} = at/l_0, \ \overline{u} = u/u_0, \ \overline{d}_0 = d_0 l_0/a m_0, \ \overline{D} = aD_0/l_0, \\ \overline{k}_1 &= k_1 l_0^2/m_0 a^2, \ \overline{k}_3 = k_3 l_0^2 u_0^2/m_0 a^2, \ K_0 = A\rho l_0/m_0, \\ \overline{a}_0 &= a_0 l_0^2/m_0 a^2 u_0, \ \overline{p} = p l_0/a \end{aligned} \tag{3.6}$$

the relations (3.3) – (3.5) are as follows:

$$u_{,tt} - u_{,xx} = 0 \tag{3.7}$$

$$u(x,0) = u_{,t}(x,0) = 0 \tag{3.8}$$

$$\begin{aligned} &u_{,tt} + d_0 u_{,t} + k_1 u + k_3 u^3 - K_0(D_0 u_{,xt} + u_{,x}) = a_0 \sin(pt) \quad \text{for} \quad x = 0, \\ &u(x,t) = 0 \qquad \text{for} \quad x = 1, \end{aligned} \tag{3.9}$$

where u_0 is a constant displacement, and the bars are omitted for convenience.

According to (2.10), the solution of the equation of motion (3.7) is sought in the form

$$u(x,t) = f(x-t) + g(t+x). \tag{3.10}$$

Substituting (3.10) into the boundary conditions (3.9), the following ordinary differential equations for the functions $f(z)$ and $g(z)$ are obtained:

$$\begin{aligned} &g(z) = -f(z-2) \\ &r_1 f''(z) + r_2 f'(z) + k_1[f(z) + g(z)] + k_3[f(z) + g(z)]^3 + r_3 g''(z) \\ &\qquad + r_4 g'(z) = a_0 \sin(pt) \end{aligned} \tag{3.11}$$

where

$$r_1 = 1 + K_0 D_0, \quad r_2 = d_0 + K_0, \quad r_3 = 1 - K_0 D_0, \quad r_4 = d_0 - K_0. \tag{3.12}$$

The Equations (3.11) are solved numerically by means of the Runge-Kutta method.

4. Numerical results

In numerical calculations for the nonlinear system shown in Fig. 1, the hard as well as the soft characteristics of the spring are taken into account. These calculations focus on the determination of appropriate amplitude-frequency curves.

4.1. A hard characteristic case

Exemplary numerical calculations concerning the solutions in steady states are carried out for the system shown in Fig. 1 with the spring having the hard characteristic for the following constant parameters:

$$\begin{aligned} K_0 &= 0.3, \quad d_0 = D_0 = 0, 0.05, 0.1, 0.15, \quad a_0 = 1, \\ k_1 &= 0.05, \quad k_3 = 0, 0.001, 0.005, 0.01, 0.1, 1.0. \end{aligned} \tag{4.1}$$

The first two frequencies of free vibrations of the considered system are $\omega_1 = 0.564$, $\omega_2 = 3.235$.

The system under consideration can be also studied using Galerkin's method leading to Duffing's equation, [5, 20]. So, it seems to be desirable to perform certain comparative calculations for the particular case of the considered system using the wave and Galerkin's approach. Such comparisons are carried out for amplitude-frequency curves obtained from the appropriate Duffing's equation and from (3.11), for the cross-section $x = 0$ with $d_0 = D_0 = 0$ in the first resonant region.

Seeking a single modal solution for the Eq. (3.7) in the form

$$u(x,t) = X_1(x)T(t) \tag{4.2}$$

where

$$X_1(x) = (k_1 - \omega_1^2)\sin(\omega_1 x)/(\omega_1 K_0) + \cos(\omega_1 x)\,, \tag{4.3}$$

is the first eigenfunction of the linear case with $X_1(0) = 1$, according to the Galerkin's method the following nonlinear equation for an unknown function $T(t)$ from (3.7) multiplied by K_0 and from (3.9) is obtained, [5, 20],

$$\ddot{T} + \omega_1^2 T + k_3' T^3 = a_0' \sin(pt), \tag{4.4}$$

where

$$k_3' = k_3 X_1^4(0)/S_0, \quad a_0' = a_0 X_1(0)/S_0, \quad S_0 = K_0 \int_0^1 X_1^2 dx + X_1^2(0).$$

The first approximation of the solution for (4.4) according to Duffing's method is sought in the form $T_1 = A\sin(pt - \gamma)$, [5, 20], where γ is a phase angle. It leads to the following relation between the amplitude a'_0 of the external moment and the amplitude A of the solution for nonlinear vibrations:

$$3k'_3A^3/4 + (\omega_1^2 - p^2)A = a'_0. \tag{4.5}$$

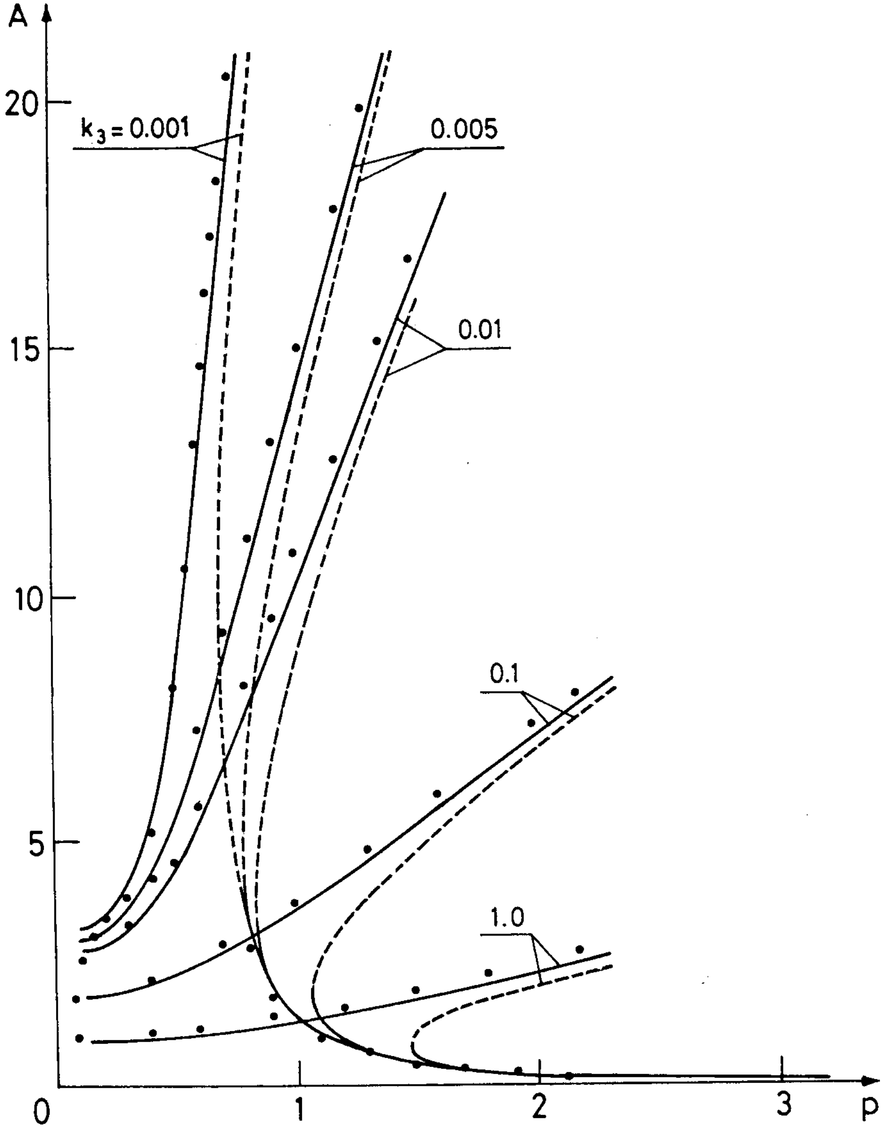

FIG. 2. Amplitude frequency curves according to Duffing's equation (continuous lines and dashed lines) and according to the wave approach in $x = 0$ (dotted lines) neglecting damping.

In Fig. 2 amplitudes A as functions of the excitation frequency p determined from (4.5) for selected values of k_3 are marked by continuous and dashed lines for stable and unstable branches of solutions, respectively. Displacement amplitudes u_A of the cross-section $x = 0$ (dotted lines in Fig. 2) are determined by solving Eq. (3.11) with the zero initial conditions, from $z = 0$ until the steady state is reached for the displacements expressed by (3.10). During numerical calculations it appears that for every considered parameter k_3 there exists a value p_0 for which displacement amplitudes jump from the upper to the down curves. However, solving Eq. (3.11) for $p > p_0$ with nonzero initial conditions we obtain displacement amplitudes lying on the extension of the upper amplitude-frequency curves up to the next jump, similarly to the case of the nonlinear discrete model, [19]. The nonzero initial conditions are expressed by the known values of functions $f(z)$, $g(z)$ and their derivatives, taking into account the shift of the argument in $(3.11)_1$. As an example, for $k_3 = 0.001$, 0.005, 0.01 we have $p_0 = 0.71$, 0.81, 0.87, respectively. The comparable results shown in Fig. 2 concern only the first resonant region.

Further numerical results are presented for the discrete-continuous system. They concern the effect of damping coefficients and of the parameter k_3 on amplitude-frequency curves in selected cross-sections of the system.

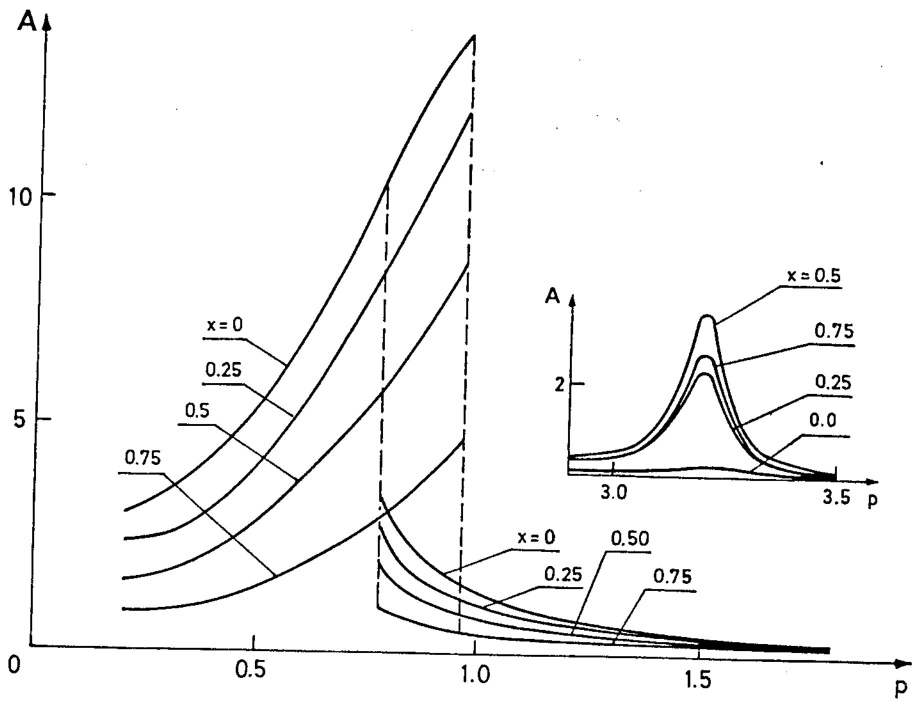

FIG. 3. Amplitude-frequency curves for $d_0 = D_0 = 0.05$.

In Figs. 3 – 5 are plotted the diagrams of amplitude-frequency curves in the cross-sections $x = 0$, 0.25, 0.5 and 0.75 for $k_3 = 0.005$ and for the coefficients of damping $d_0 = D_0 = 0.05$, 0.1 and 0.15, respectively. From Figs. 3 and 4 it follows that for $d_0 = D_0 = 0.05$ and 0.1, there exist two values of frequency p of the external loading $P(t)$ for which the displacement amplitude jumps occur in the first resonant region. The distance between these jump frequencies is constant for all considered cross-sections of the elastic element, when the coefficients d_0 and D_0 are fixed. From Figs. 3 and 4 it also follows that in the first resonant region the largest displacement amplitudes occur in the cross-section $x = 0$, and the lowest one in $x = 0.75$. The effect of damping is significant. If we increase damping, the displacement amplitudes and the distance between the jump frequencies of the external loading decrease. In the the second resonant region, the displacement amplitude jumps do not occur and each curve in this region is rather symmetric with the respect to the vertical axis at $p = \omega_2 = 3.235$, i.e., for the second frequency of free vibration. Displacement amplitudes also do not jump for the damping coefficients $d_0 = D_0 = 0.15$, Fig. 5.

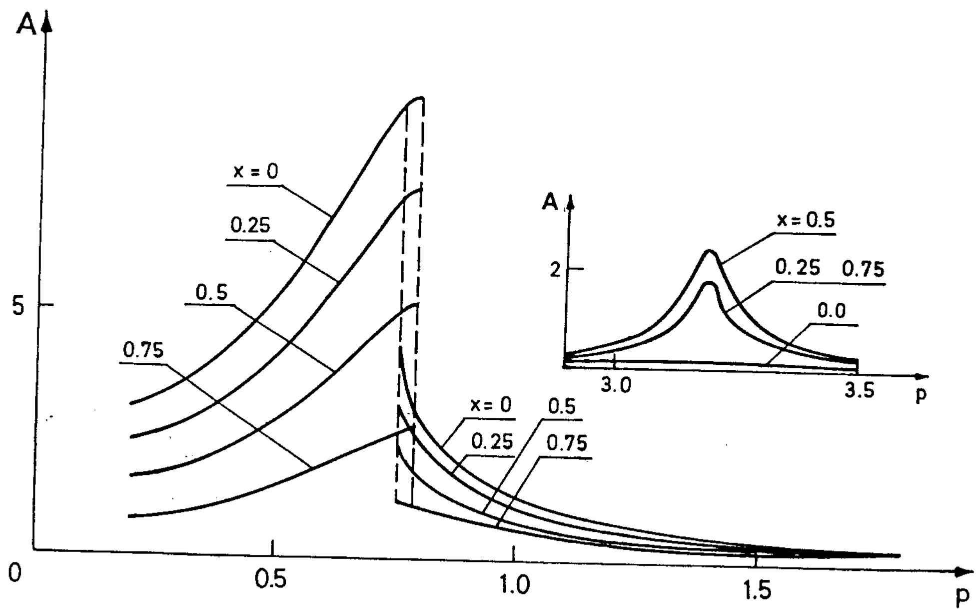

FIG. 4. Amplitude-frequency curves for $d_0 = D_0 = 0.1$.

The effect of the local nonlinearity is directly expressed by the parameter k_3 in (3.1). In Figs. 6 and 7 are plotted the diagrams of amplitude-frequency curves in the cross-sections $x = 0$ and $x = 0.5$ for $d_0 = D_0 = 0.1$, and for $k_3 = 0$, 0.001, 0.005, 0.01 in the first and the second resonant regions. The effect of the coefficient k_3 is observed only in the first resonant region. It can be noted that displacement

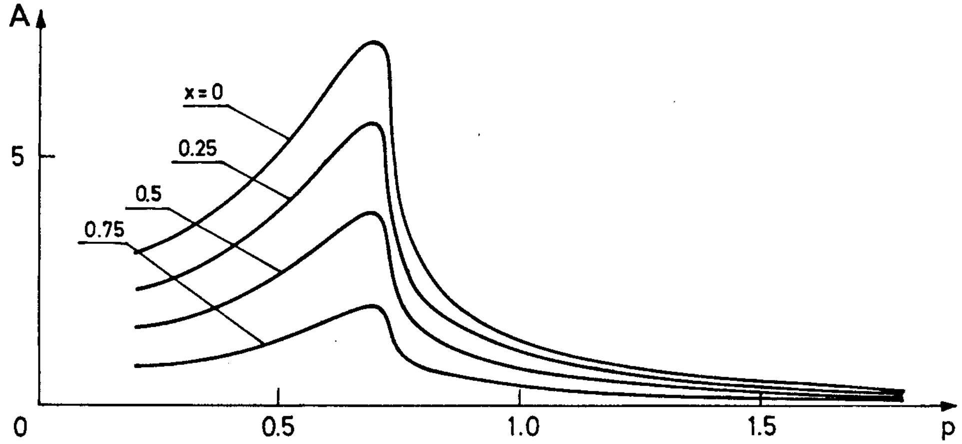

FIG. 5. Amplitude-frequency curves for $d_0 = D_0 = 0.15$.

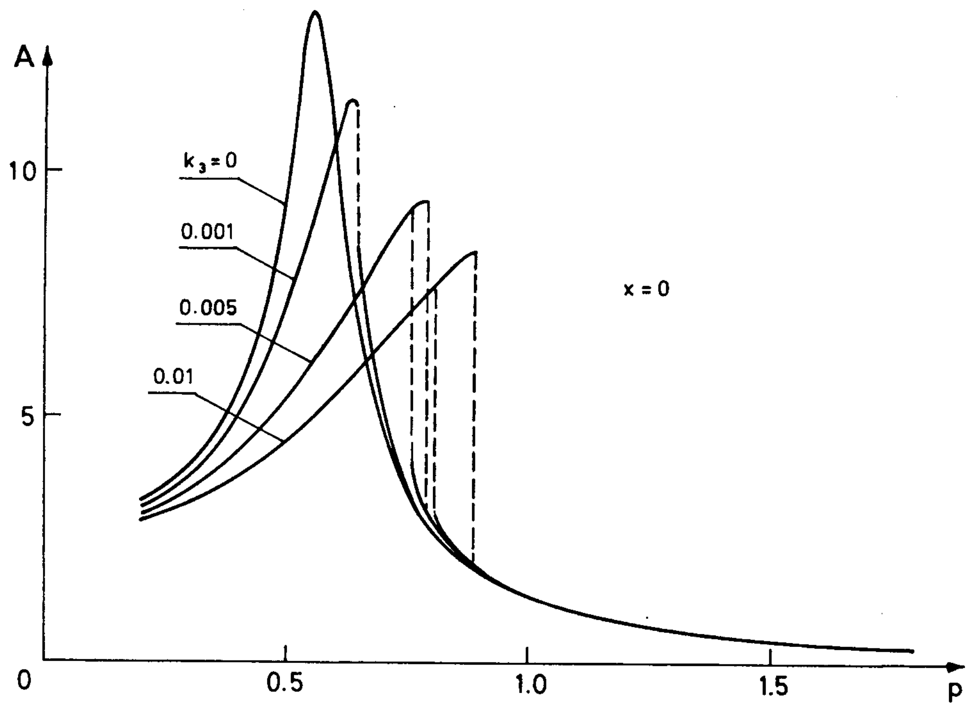

FIG. 6. Amplitude-frequency curves for $x = 0$ and $k_3 = 0$, 0.001, 0.005, 0.01.

amplitudes and the distance between the frequencies of the external loading $P(t)$ for which the jumps occur are dependent on k_3. The amplitudes increase and the distance between jump frequencies decreases with the decrease of k_3 in the whole considered cross-sections. For $k_3 = 0$ the amplitude-frequency curves concern the linear system. From Figs. 6 and 7 it follows that the highest displacement amplitudes occur in the first resonant region. In the second resonant region the displacement amplitudes in the cross-section $x = 0$ are very small and for this reason they are not shown in Fig. 6.

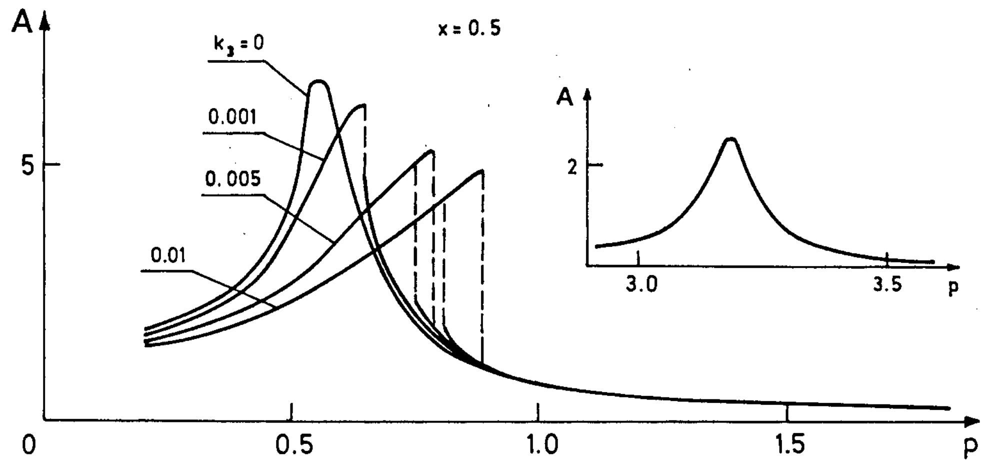

FIG. 7. Amplitude-frequency curves for $x = 0.5$ and $k_3 = 0$, 0.001, 0.005, 0.01.

4.2. A soft characteristic case

Numerical results for the nonlinear discrete-continuous system with the spring characteristic of a soft type are presented for

$$(4.6)\qquad \begin{aligned} K_0 &= 0.3, \quad d_0 = D_0 = 0.1, 0.15, \quad a_0 = 0.2, 0.3,\\ k_1 &= 0.05, \quad k_3 = -0.001, -0.005, -0.01. \end{aligned}$$

Diagrams in Figs. 3 – 7 inform that numerical solutions can be obtained in an arbitrary cross-section of the elastic elements. Here the results are shown only for the cross-section $x = 0$.

In Fig. 8 the amplitude-frequency curves are plotted for $k_3 = -0.001$, $-0.005, -0.01$, $d_0 = D_0 = 0.1$, $a_0 = 0.2$, 0.3 and $p < 0.9$. They contain only the first resonant region because in further resonant regions, similarly as in the case of the hard characteristic, no effects of the local nonlinearity were observed.

One would expect that with the increase of the amplitude of the external loading, the amplitudes of the diplacements should increase for each p. It is true up to the frequency p for which the function (3.1) approaches the maximum value postulated by the constant k_3. Then the solutions begin to diverge to infinity, and that is connected with the properties of the potential of the function (3.1). The escape phenomenon is known in nonlinear discrete systems with the nonlinearity of the type (3.1), [21]. Thus, this phenomenon is noticed also in the study of nonlinear discrete-continuous systems. Two extreme values of p for which harmonic solutions can be obtained are marked by points in Fig. 8. The points determine the interval of p where the polynomial function (3.1) is rather not useful for the description of the nonlinear characteristic of a soft type.

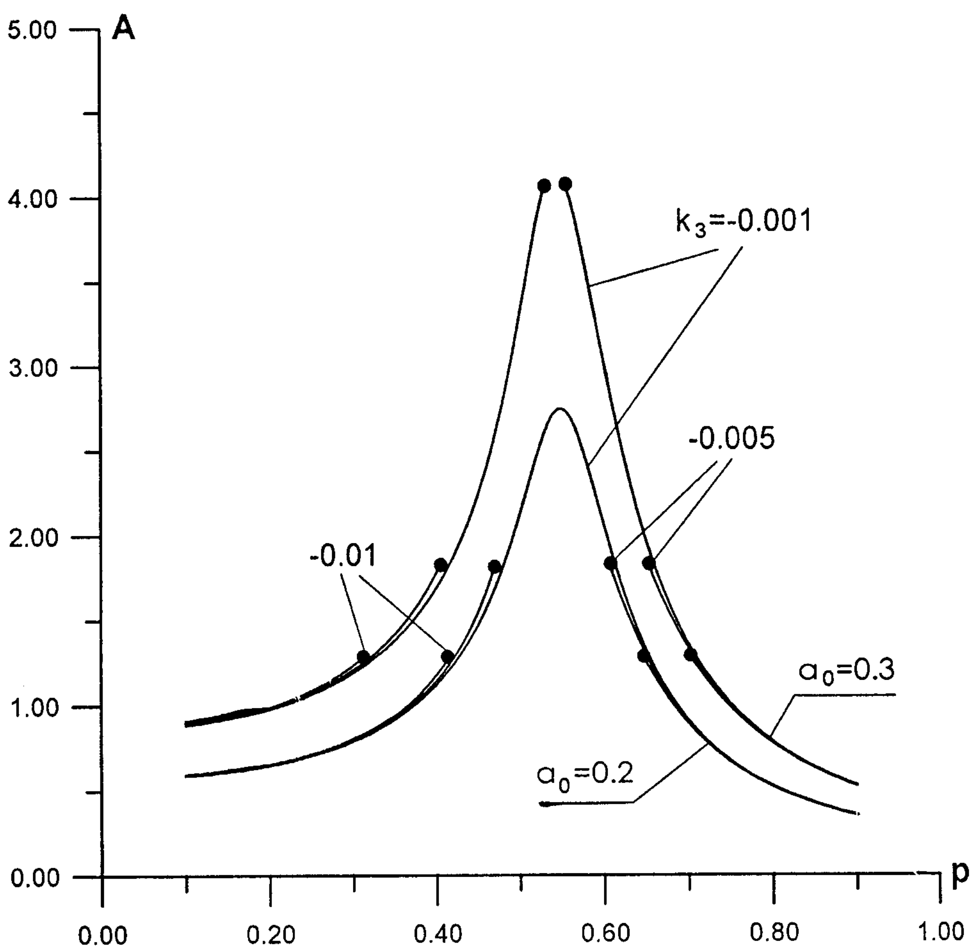

FIG. 8. Amplitude-frequency curves for $k_3 = -0.001, -0.005, -0.01$ and $a_0 = 0.2, 0.3$.

Diagrams in Fig. 9 show the maximum values of the amplitude a_0 of the external loading giving harmonic numerical solutions as functions of the frequency p of the external loading. They are done for $k_3 = -0.001, -0.005, -0.01$ and $d_0 = D_0 = 0.1, 0.15$. Tracing the diagrams in Fig. 9 one can see that the application ranges of the nonlinear function (3.1) become narrower with the decrease of the parameter k_3 representing the local nonlinearity in the system, and with the decrease of damping. From Fig. 9 it follows that the strongest restrictions concern the neighbourhood of the resonance.

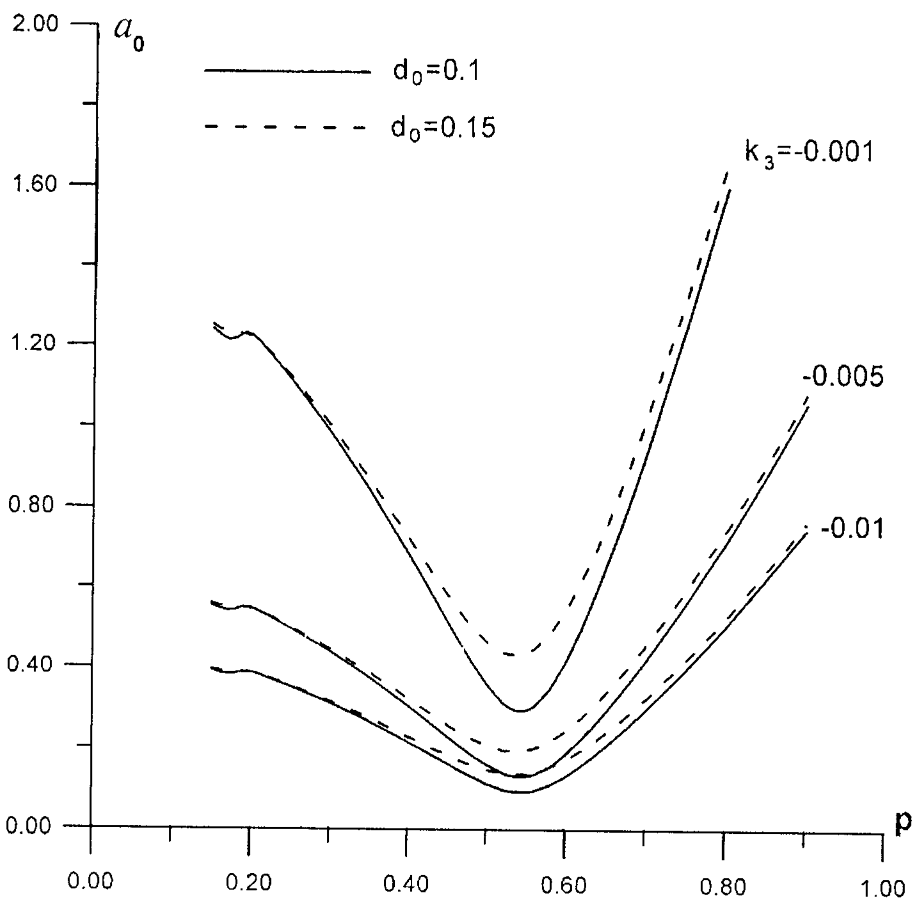

FIG. 9. Application ranges of the function (3.1) in the case of a soft characteristic.

5. Final remarks

From the above considerations it follows that the introduction of local nonlinearities, described by the third-degree polynomial function, in discrete-continuous systems has a significant influence on the dynamic behaviour of these systems. From the investigations for the simple nonlinear system it follows that in the first resonant regions, two jumps of the displacement amplitudes can occur for the local nonlinearity with the characteristic of a hard type. The distance between jump amplitudes decreases with the increase of damping and with the decrease of the parameter k_3. When the local nonlinearity with the characteristic of a soft type has to be taken into account, one may expect that the assumed polynomial function has some restrictions for its application for the description of the local nonlinearity. This case needs more investigations, however they are beyond the aim of the present paper.

The nonlinear discrete-continuous system shown in Fig. 1 is more complex than nonlinear discrete systems considered e.g., in [19 – 21]. More complicated nonlinear discrete-continuous systems with local nonlinearities represented by the third-degree polynomial are given in [13, 15, 17]. The results presented there for the local nonlinearity with the characteristic of a hard type lead to similar conclusions as those given in the present paper and concern more resonant regions.

Nonlinear discrete-continuous systems with nonlinearities described by other nonlinear functions one can find in [9 – 12, 18]. In these papers the wave approach is also applied.

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