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ON THE MACHINE TOOL SUBSTITUTIVE MODEL CREATION, SUPPORTED BY THE FINITE ELEMENT METHOD

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A primary model is created by the rigid finite element method. The selected "rigid" part which is supposed to be flexible, is idealised by classic finite elements of various kinds. The solution to dynamics of the model introduced is a pattern which the primary model should follow. Suitable accuracy criteria are included to compare the results. If the chosen criterion is satisfied, flexibility of that "rigid' part will be distributed over some spring-damping element. If the latter is not accomplished, the "rigid" part will be subdivided into two rigid finite elements, connected by additional spring-damping element. Some illustrative examples are included.

Key Words: Discrete Modelling. Finite Element Method. Dynamics. Machine Tools. Numerical Analysis. Computation.

Notation

$\mathbf{B}_h$	diagonal damping matrix of SDE no. h
$\mathbf{B}_k$	diagonal damping matrix of SDE no. k
$\mathbf{C}_h$	diagonal stiffness matrix of SDE no. h
$\mathbf{C}_k$	diagonal stiffness matrix of SDE no. k
$i_{\kappa A}$	number of SDEs attached to RFE A
$i_{\kappa B}$	number of SDEs attached to RFE B
$i_{\eta A}$	number of nodes corresponding to RFE A
$i_{\eta B}$	number of nodes corresponding to RFE B
j	imaginary unit
$\mathbf{K}$	stiffness matrix of the whole discrete system
$\mathbf{L}$	damping matrix of the whole discrete system
l_{RFE}	number of RFEs
l_{SDE}	number of SDEs
l_{sw}	number of nodal degrees of freedom of the FE
l_w	number of nodes of the FE

M	number of harmonic components from the Fourier decomposition of excitation $\mathbf{p}(t)$
$\mathbf{M}$	inertia matrix of the whole discrete system
$\mathbf{M}_r$	inertia matrix of RFE no. r
$\mathbf{N}_{ej}$	shape function matrix of FE no. e_j, to which point j belongs, and calculated in the coordinate system of this point
$\hat{\mathbf{N}}_{ej}$	shape function matrix of FE no. e_j, to which point j belongs, calculated in the coordinate system of this point and expressed in the global coordinate system $x_1x_2x_3$ of the DFB
$\mathbf{p}$	vector of generalised forces of the model
$\mathbf{p}_r$	vector of generalised forces of RFE no. r
$\mathbf{q}$	vector of generalised displacements of the model
$\mathbf{q}_e$	vector of generalised coordinates of FE no. e
q_{hji}	generalised displacement of node no. j of the DFB along direction no. i
$\mathbf{q}_p$	vector of generalised displacements of RFE no. p
$\mathbf{q}_r$	vector of generalised displacements of RFE no. r
q_{rji}	generalised displacement of point no. j of RFE no. r along direction no. i
$\hat{\mathbf{q}}_{ej}$	vector of nodal displacements of FE no. e_j, to which point j belongs, expressed in the global coordinate system $x_1x_2x_3$ of the DFB
$\mathbf{S}_{Ah}$, $\mathbf{S}_{Bh}$	matrices of connection coordinates of SDE no h to RFEs: A and B, respectively
$\mathbf{S}_{Ak}$, $\mathbf{S}_{Bk}$	matrices of connection coordinates of SDE no. k to RFEs: A and B, respectively
$\mathbf{S}_{ekA}$, $\mathbf{S}_{ekB}$	matrices of connection coordinates of node no. i to RFEs: A and B, respectively
$\mathbf{S}_j$	matrix of coordinates of point j, which belongs to RFE no. r
$\mathbf{S}_{rk}$, $\mathbf{S}_{pk}$	matrices of connection coordinates of SDE no. k to RFEs no. r and no. p, respectively
$\hat{\mathbf{S}}_{Aj}$	matrix of connection coordinates of point j, transformed to the global coordinate system $x_1x_2x_3$ of the DFB
W_d	desired allowable value of index W
ε_{ji}	allowable difference in generalised displacement of point no. j along direction no. i
$\mathbf{\Theta}_{Ah}$, $\mathbf{\Theta}_{Bh}$	matrices of direction cosines between coordinate system of SDE no. h and those of RFE A and RFE B, respectively
$\mathbf{\Theta}_{Ak}$, $\mathbf{\Theta}_{Bk}$	matrices of direction cosines between coordinate system of SDE no. k and those of RFE A and RFE B, respectively
$\mathbf{\Theta}_{ekA}$, $\mathbf{\Theta}_{ekB}$	matrices of direction cosines between the $x_1x_2x_3$ coordinate system of the DFB and those of RFE A and RFE B, respectively
$\mathbf{\Theta}_j$	matrices of direction cosines between coordinate system of point j and those of RFE no. r
$\mathbf{\Theta}_{rk}$, $\mathbf{\Theta}_{pk}$	matrices of direction cosines between coordinate system of SDE no. k and those of RFE no. r and RFE no. p, respectively
ω_α	angular frequency of harmonic component no. α from the Fourier decomposition of excitation $\mathbf{p}(t)$

1. Introduction

A study of machine tool phenomena yields the observation that the basis is its a carrying system. It usually consists of the frame, the bracket (console milling machines), the bed (lathes, shapers), slides (transverse and longitudinal), the table (milling and drilling machines), the independently driven beam (some milling machines), the spindle head (milling machines), the spindle box (lathes) and the tool carriage (lathes, shapers). A consistence of such a structure is quite general. Because of a variety of machine tools, an explicit description is possible only for special types.

Due to extremely inconvenient working conditions (i.e. vibration, fatigue loads, thermal effects), and the necessity of high quality machining being assured (that is to say: accuracy, efficiency, reliability and industrial safety), each sub-unit should be an extremely reliable element of the structure. Thus, they are created as solid bodies, usually made of cast iron or machine steel. They are manufactured as castings or welded structures, and subsequently formed or machined. Also structures which have been composed of screwed-on parts, can be observed (e.g. frames and boxes).

Several substructures are coupled by movable connections (e.g. guides, turn-around tables, roll carriages, bearings) or by fixed joints (bolted, pin, spigot, welded, clamped guides etc). Certainly, the joints in the structure cause reduction in stiffness of the carrying system. Thus, the presence of:

- solid and rigid sub-units, usually with perfect geometric properties, and
- flexible constructional joints is an important feature of the carrying system.

The reasons above support the aim of using the conventional rigid finite element method (RFEM) [1] as an attractive tool for making a primary discrete model. It is to be emphasised that the process of subdivision is performed quite naturally, because [2]:

- several substructures of the machine tool are solid bodies with their own high rigidity, and so they may be idealised as rigid finite elements (RFEs);
- the flexibility of constructional connections (as movable as those fixed) is significant, so that they can be favourably modelled by spring-damping elements (SDEs).

In order to make the model adequate, the main goal depends upon proper estimation of the inertia parameters of RFEs and the spring-damping coefficients of SDEs. The following methods are applied for this purpose:

- analytical methods, which are usually supported by commercial computer software [3, 4];
- identification procedures, which are based on the results of measurements from a real structure [5];

• both experimental and numerical approaches simultaneously [6].

Unfortunately, the primary model of the carrying system must be verified very often. The reason is that the rigidity of some solid bodies of the model may appear to be insufficient, and then it makes the model inadequate. In the next section a new approach will be suggested, which allows to apply the classic finite element method [7, 8] in order to examine and, if necessary, to improve the quality of conventional modelling by the RFEM. As a result, more adequate model of the real structure is obtained.

2. General concept

The proposed approach depends first on application of conventional RFEM to modelling, so that machine tool substructures are considered as separate RFEs. A primary model, which is composed of such RFEs connected with each other by SDEs, is obtained in the above suggested way. A dynamic problem of the model (described below) is computed. Following that, the chosen "rigid" part, which is assumed to be flexible, is to be idealised by the classic finite elements of various kinds (Fig. 1), and the relevant dynamic problem is computed again.

If someone is not sure whether the chosen part is really flexible, the trial-and-error approach is recommended for this purpose. Besides, a thorough study on geometry and properties of the part, supported by experience and engineering intuition, are the bases of selection. In case of uncertainty or lack of experience, the approach enables the examination for any part of the primary model.

Application of the FEM does not mean that it improves the results of calculation. However, some features of the method (e.g. high density of discretization, true idealization of perfect geometry of real structures) cause that it is difficult to find any alternative approach at this stage of computation, which could be treated as a pattern.

The examination means a comparison between the results of both computations outlined above and a suitable accuracy criterion is included for this purpose. If the latter is not accomplished, the primary model should be improved. It depends upon subsequent division of the RFE previously separated into two new RFEs, which are connected by one SDE. New parameters of them must be determined and then substitutive model of the structure is derived at the end.

Let us analyse a dynamic equation of a primary model, which is composed of RFEs, connected by SDEs (Fig. 2), that is:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{L}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{p}(t), \tag{2.1}$$

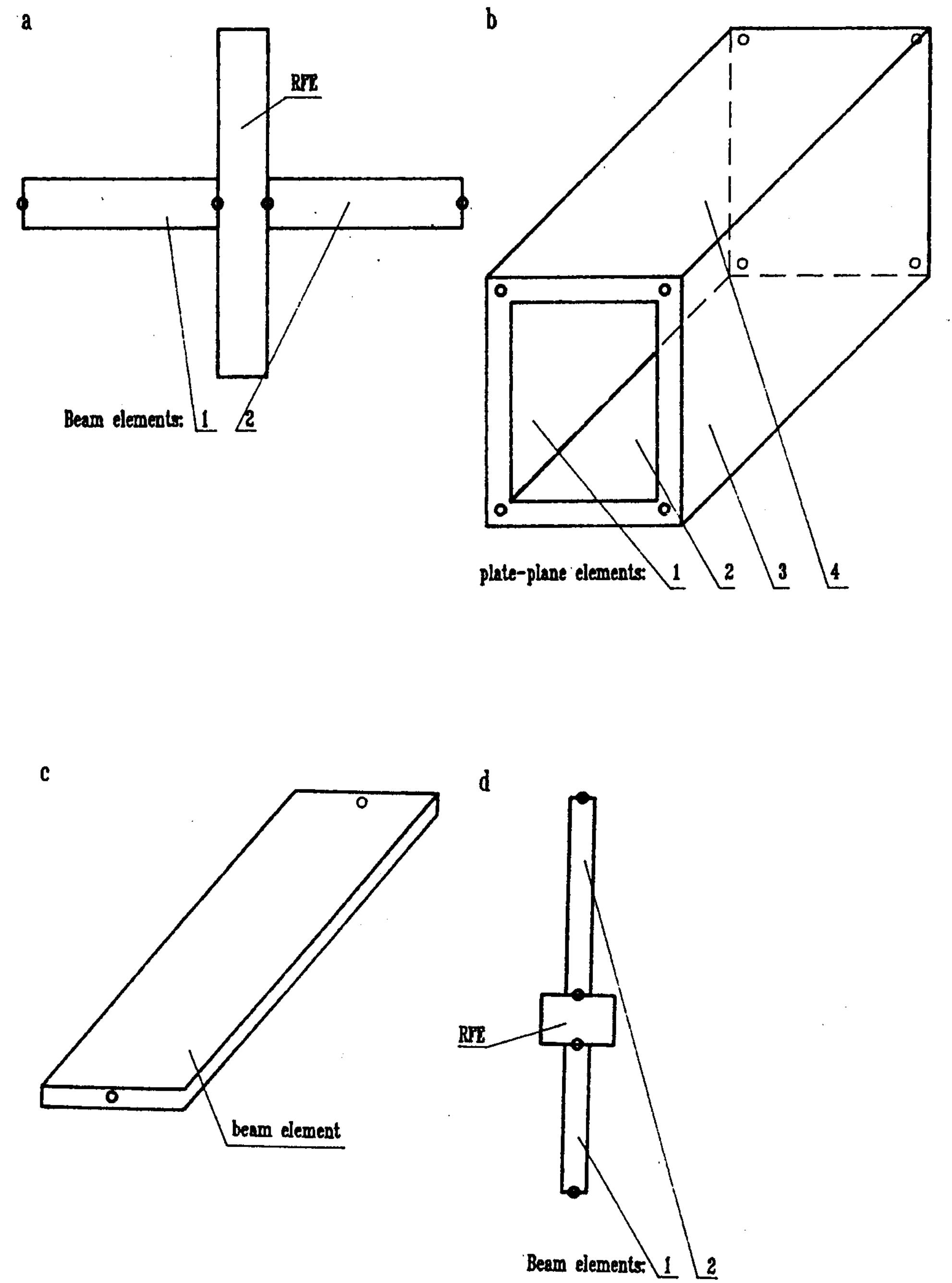

FIG. 1. Some typical machine tool substructures, which could be idealised by RFEs and FEs.

where:

$$\mathbf{q} = \mathrm{col}(\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_r, ..., \mathbf{q}_{l_{\mathrm{RFE}}}), \tag{2.2}$$

$$\mathbf{p} = \mathrm{col}(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_r, ..., \mathbf{p}_{l_{\mathrm{RFE}}}), \tag{2.3}$$

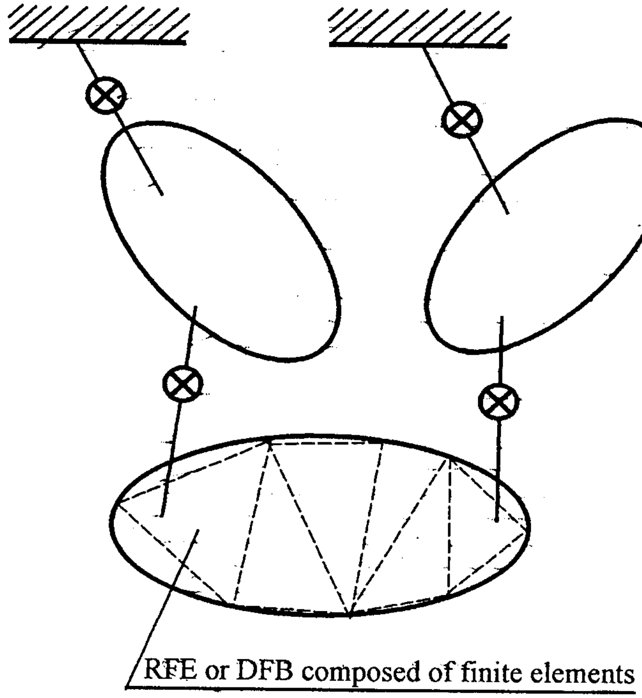

FIG. 2. RFE no. r, which is replaced by DFB composed of a number of FEs.

For a steady state, the dynamic problem (which is described by Eq. (2.1)) is solved using a Fourier transformation [2]. As a result, we get

$$\sum_{\alpha=0}^{M-1} \left(\mathbf{K} - \mathbf{M}\omega_\alpha^2 + \mathrm{j}\omega_\alpha \mathbf{L}\right) \mathbf{q}(\mathrm{j}\omega_\alpha) = \sum_{\alpha=0}^{M-1} \mathbf{p}(\mathrm{j}\omega_\alpha), \tag{2.4}$$

The future research will refer only to dynamics of the system for harmonic component α, and the accompanying amplitudes of generalised displacements. However, a range of validity of the method proposed is restricted by no means. The frequency value ω_α sought for should be selected from the real spectrum, observed during machining. Thus, we obtain

$$\left(\mathbf{K} - \mathbf{M}\omega_\alpha^2 + \mathrm{j}\omega_\alpha \mathbf{L}\right) \mathbf{q}(\mathrm{j}\omega_\alpha) = \mathbf{p}(\mathrm{j}\omega_\alpha). \tag{2.5}$$

The solution of the above equation contains the vector function of angular frequency ω_α

$$\mathbf{q} = \mathbf{q}(\mathrm{j}\omega_\alpha). \tag{2.6}$$

In the case where RFE no. r is supposed to be insufficiently rigid, subsequent idealisation is necessary using some classic finite elements (Fig. 2). A new creation, which is obtained in such a way, is called a discrete flexible body (DFB). Equations of dynamics of the model have a similar form, as described by (2.1). The difference is in the $\mathbf{q}$ vector, in which generalized coordinate vector $\mathbf{q}_r$ of RFE no. r is replaced by the $\mathbf{q}_h$ generalized coordinate vector of the DFB, that is

$$\mathbf{q} = \mathrm{col}(\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_h, ..., \mathbf{q}_{l_{\mathrm{RFE}}}), \tag{2.7}$$

where:

$$\mathbf{q}_h = \mathrm{col}(q_{hs}), \qquad s = 1, 2, ..., l_w l_{sw}. \tag{2.8}$$

The generalised displacements of point j of RFE no. r (i.e. rigid body) are determined using the expression

$$\mathbf{q}_{rj} = \mathbf{\Theta}_j \mathbf{S}_j \mathbf{q}_r. \tag{2.9}$$

If we replace the RFE by the DFB, which is composed of deformable elements, the generalised displacements of point j are determined by the expression

$$\mathbf{q}_{hj} = \mathbf{N}_{ej} \mathbf{q}_e. \tag{2.10}$$

In order to assess the rigidity of RFE no. r, the displacements of chosen points (which are determined by Eqs. (2.9) and (2.10)) should be compared with each other. Some accuracy criteria to be described in the next section, must be introduced for this purpose.

3. Accuracy criteria

Criterion 1.

This concerns the determination of the generalised displacements of all nodes of the DFB. Following that, the results are to be compared with those points of the RFE, which refer to nodes of the DFB. The criterion for RFE no. r is assured, when:

$$\big||q_{hji}| - |q_{rji}|\big| < \varepsilon_{ji}, \quad j = 1, ..., l_w, \ i = 1, ..., l_{sw}. \tag{3.1}$$

Criterion 2.

This concerns a comparison of relative displacements of those points which refer to all nodes of the DFB. The performance index is defined below. In this case:

$$W = \left(\sqrt{\frac{\sum_j \sum_i (|q_{hji}| - |q_{rji}|)^2}{\sum_j \sum_i |q_{hji}|^2}} \right) \cdot 100\%. \tag{3.2}$$

If suitable generalised displacements of the referred points of the RFE and DFB were equal (i.e. RFE with infinite rigidity), the index value should be $W = 0$. This is its minimum value. We consider that RFE no. r is satisfactorily rigid, when

$$W \leq W_d. \tag{3.3}$$

Criterion 3.

This concerns a comparison of the relative displacements which refer only to chosen points of the DFB. These are usually points in which SDEs are attached to RFE no. r.

4. A METHOD OF DISTRIBUTION OF THE FLEXIBILITY

If the selected criterion of rigidity is fulfilled, a subdivision of RFE no. r is not relevant. However, in order to improve the accuracy of modelling, it is suggested that the flexibility of RFE no. r should be distributed over the SDEs, which are attached to it. This means that the stiffness coefficients of these SDEs have to be corrected.

Let us consider a part of a primary model, which contains two RFEs no. r and p, connected by SDE no. k (Fig. 3). The deformations of SDE no. k are described by the relationship

$$\Delta \mathbf{w}_k = \left[\mathbf{\Theta}_{pk} \mathbf{S}_{pk} \vdots - \mathbf{\Theta}_{rk} \mathbf{S}_{rk} \right] \begin{bmatrix} \mathbf{q}_p \\ \mathbf{q}_r \end{bmatrix}, \tag{4.1}$$

where

$$\Delta \mathbf{w}_k = \mathrm{col}(\Delta w_{ki}), \qquad i = 1, ..., 6.$$

The following matrices are explicitly known in the model:

$$\mathbf{C}_k = \mathrm{diag}[c_{ki}], \qquad i = 1, ..., 6,$$

$$\mathbf{B}_k = \mathrm{diag}[b_{ki}], \qquad i = 1, ..., 6,$$

$$\mathbf{M}_r = \mathrm{diag}[m_{ri}], \qquad i = 1, ..., 6.$$

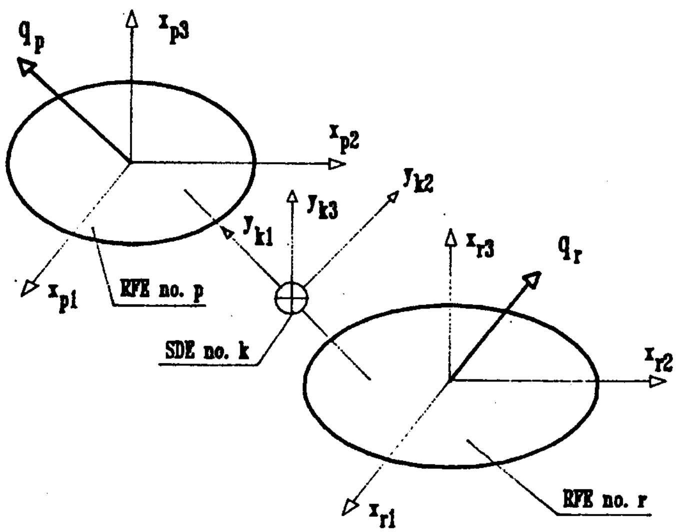

FIG. 3. A part of primary model, which contains RFEs no. r and p and SDE no. k.

Now we substitute RFE no. r with a set of deformable elements (i.e. DFB, Fig. 4). After solving the dynamic Eq. (2.1) again, the generalised displacements $\mathbf{q}'_p$ of RFE no. p and the generalised displacements of the DFB nodes are derived.

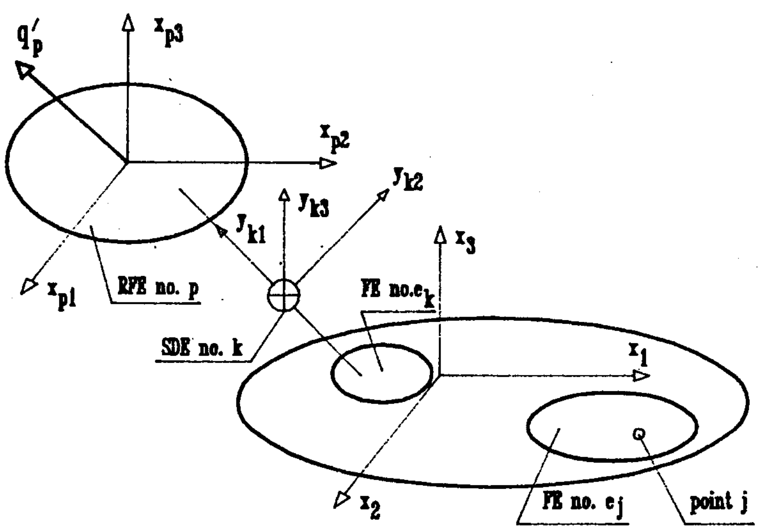

FIG. 4. A part of a model, which contains RFE no. p, SDE no. k and the DFB.

Because we supposed the stiffness criterion to be fulfilled, subdivision of RFE no. r is not advised. It still remains one rigid body (Fig. 5), but now it is called RFE no. r'. However, the vector of its generalised displacements $\mathbf{q}'_r$ should be defined such as to idealise the true image of motion of the DFB. Thus, we choose l points j of the DFB (Fig. 4) and define their displacements in the form

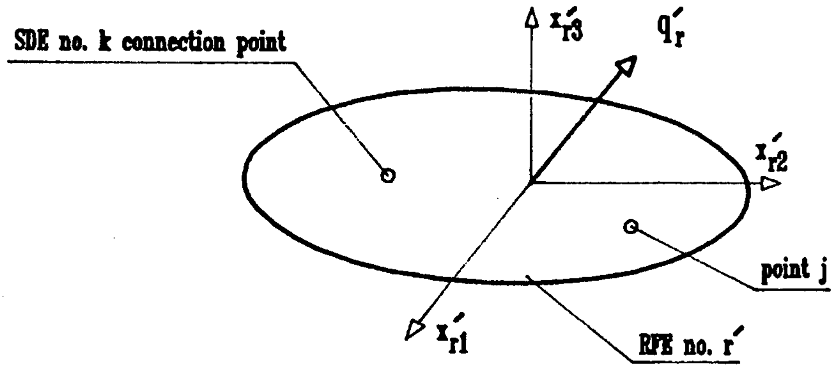

FIG. 5. The RFE, which replaces the DFB.

$$\mathbf{q}_j = \hat{\mathbf{N}}_{ej}\hat{\mathbf{q}}_{ej}. \tag{4.2}$$

Next, the displacements of point j are expressed as a function of vector $\mathbf{q}'_r$ of RFE no. r', that is

$$\mathbf{q}'_{rj} = \mathbf{S}_j\mathbf{q}'_r. \tag{4.3}$$

Since

$$\begin{aligned} \mathbf{q}_j &= \mathrm{col}(q_{ji}), \qquad i = 1,...,6, \\ \mathbf{q}'_{rj} &= \mathrm{col}(q'_{rji}), \quad i = 1,...,6, \end{aligned}$$

it is required that

$$\sum_{j=1}^{l}\sum_{i=1}^{6}(q_{ji} - q'_{rji})^2 = \min. \tag{4.4}$$

Since

$$q'_{rji} = \sum_{\kappa=1}^{6} S_{ji\kappa}q'_{r\kappa}, \tag{4.5}$$

we obtain the equation

$$\sum_{j=1}^{l}\sum_{i=1}^{6}(q_{ji} - \sum_{\kappa}^{6} S_{ji\kappa}q'_{r\kappa})^2 = \min, \tag{4.6}$$

which allows the components of vector $\mathbf{q}'_r$ to be determined by the least square method. Following this, a new vector of deformation of SDE no. k is expressed in the form

$$\Delta\mathbf{w}'_k = \left[\mathbf{\Theta}_{pk}\mathbf{S}_{pk} \vdots - \mathbf{\Theta}_{rk}\mathbf{S}_{rk}\right]\left[\begin{array}{c}\mathbf{q}'_p\\ \mathbf{q}'_r\end{array}\right], \tag{4.7}$$

where

$$\Delta\mathbf{w}'_k = \mathrm{col}(\Delta w'_{ki}), \qquad i = 1, ..., 6.$$

If we apply the potential energy conservation law to SDE no. k, improved values of its stiffness coefficients are calculated in the form

$$c'_{ki} = \left(\frac{|\Delta w_{ki}|}{|\Delta w'_{ki}|}\right)^2 c_{ki}, \qquad i = 1, ..., 6. \tag{4.8}$$

If we assume that the energy dissipation function of SDE no. k does not change [1], it is easy to find the improved values of the damping coefficients in the form

$$b'_{ki} = \left(\frac{|\Delta \dot{w}_{ki}|}{|\Delta \dot{w}'_{ki}|}\right)^2 b_{ki} = \left(\frac{|\Delta w_{ki}|}{|\Delta w'_{ki}|}\right)^2 b_{ki}, \qquad i = 1, ..., 6, \tag{4.9}$$

and, if we apply the kinetic energy conservation law to RFEs no. r and r', values of the inertia coefficients of RFE no. r' are calculated in the form

$$m'_{ri} = \left(\frac{|\dot{q}_{ri}|}{|\dot{q}'_{ri}|}\right)^2 m_{ri} = \left(\frac{|q_{ri}|}{|q'_{ri}|}\right)^2 m_{ri}, \qquad i = 1, ..., 6, \tag{4.10}$$

where

$$\Delta\dot{w}'_{ki} = \mathrm{j}\omega_\alpha \Delta w_{ki}, \qquad \dot{q}_{ri} = \mathrm{j}\omega_\alpha q_{ri}, \qquad \dot{q}'_{ri} = \mathrm{j}\omega_\alpha q'_{ri}.$$

5. The method of subdivision using RFEs

If the chosen criterion of rigidity is not satisfied, a subdivision of RFE no. r into two rigid finite elements connected by an additional spring-damping element is advised. Thus, the DFB is replaced by RFEs A and B, which are connected by SDE no. h (Fig. 6). Both these RFEs should idealise a true image of the DFB motion, so that the respective vectors of generalised displacements $\mathbf{q}_A$ and $\mathbf{q}_B$ are introduced. For this purpose let us choose l points j of the DFB, which belong to RFE A as well. Then we describe their generalized displacements, referred to the global coordinate system $x_1x_2x_3$, i.e.

$$\mathbf{q}_j = \hat{\mathbf{N}}_{ej}\hat{\mathbf{q}}_{ej}, \tag{5.1}$$

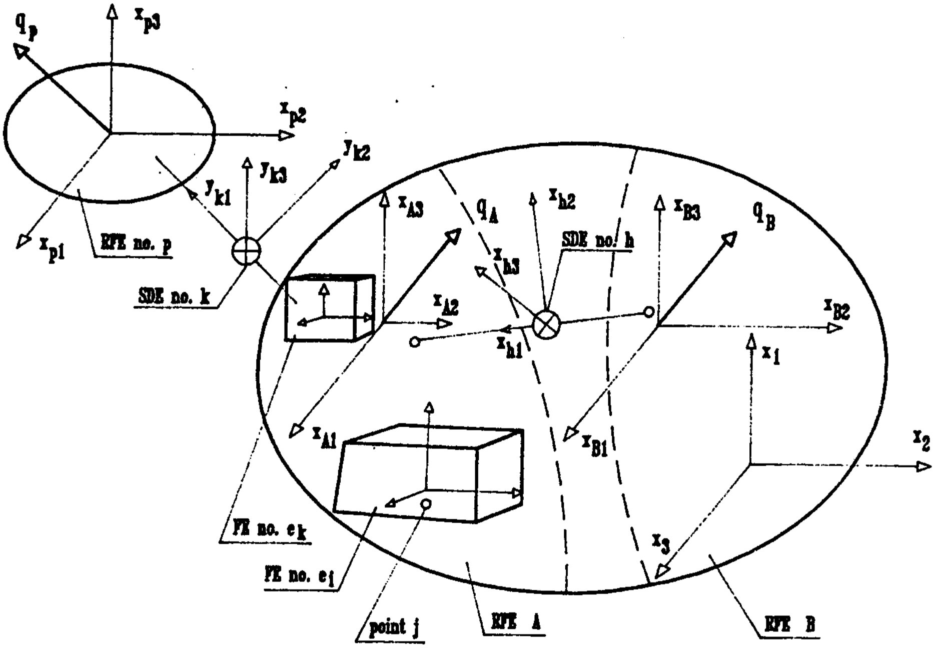

FIG. 6. Division of the DFB into two RFEs, which are connected by SDE no. h.

where

$$\mathbf{q}_j = \mathrm{col}(q_{ji}), \qquad i = 1, ..., 6.$$

If we assume that point j is that of RFE A (solid body), its generalised displacements, referred to the global coordinate system, can now be described in the form

$$\mathbf{q}_j^A = \hat{\mathbf{S}}_{Aj}\mathbf{q}_A. \tag{5.2}$$

Since

$$\begin{aligned} \mathbf{q}_j^A &= \mathrm{col}(q_{ji}^A), \qquad i = 1, ..., 6, \\ \mathbf{q}_j &= \mathrm{col}(q_{ji}), \qquad i = 1, ..., 6, \\ q_{ji}^A &= \sum_{\kappa=1}^{6} \hat{S}_{Aji\kappa} q_{A\kappa}, \end{aligned}$$

it is required that

$$\sum_{j=1}^{l}\sum_{i=1}^{6}\left(q_{ji} - \sum_{\kappa=1}^{6} \hat{S}_{Aji\kappa} q_{A\kappa}\right)^2 = \min. \tag{5.3}$$

Components of the $\mathbf{q}_A$ vector are determined using the least square method. We obtain

$$\mathbf{q}_A = \mathrm{col}(q_{A\kappa}).$$

Similarly, vector $\mathbf{q}_B$ can also be determined.

Suppose that the kinetic energies of RFE no. r and of the couple of RFEs A and B, are equal. This assumption allows us to determine the diagonal matrices $\mathbf{M}_A$ and $\mathbf{M}_B$ of inertia coefficients of RFEs A and B, as follows:

$$\mathbf{M}_{AB} = \mathbf{T}_{\dot{q}}^{-1}\dot{\mathbf{q}}_{AB}\dot{\mathbf{q}}_r^T\mathbf{M}_r\dot{\mathbf{q}}_r\dot{\mathbf{q}}_{AB}^T\mathbf{T}_{\dot{q}}^{-1}, \tag{5.4}$$

where:

$$\mathbf{M}_{AB} = \begin{bmatrix} \mathbf{M}_A & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_B \end{bmatrix}, \quad \dot{\mathbf{q}}_{AB} = \mathrm{col}(\dot{\mathbf{q}}_A, \dot{\mathbf{q}}_B), \qquad \dot{\mathbf{q}}_A = \mathrm{j}\omega_\alpha\mathbf{q}_A,$$

$$\dot{\mathbf{q}}_B = \mathrm{j}\omega_\alpha\mathbf{q}_B, \qquad \mathbf{T}_{\dot{q}} = \dot{\mathbf{q}}_{AB}\dot{\mathbf{q}}_{AB}^T.$$

The deformations of SDE no. h are described by the vector

$$\Delta\mathbf{w}_h = \mathbf{T}_h \begin{bmatrix} \mathbf{q}_A \\ \mathbf{q}_B \end{bmatrix}, \tag{5.5}$$

where

$$\mathbf{T}_h = [\mathbf{\Theta}_{Ah}\mathbf{S}_{Ah} \; -\mathbf{\Theta}_{Bh}\mathbf{S}_{Bh}], \qquad \Delta\mathbf{w}_h = \mathrm{col}(\Delta w_{hi}), \qquad i = 1,...,6. \tag{5.6}$$

In accordance with the Kelvin-Voigt rheological model [1, 2], the vector of interaction forces of SDE no. h is expressed by the form:

$$\mathbf{f}_h(\mathrm{j}\omega_\alpha) = -(\mathbf{C}_h\Delta\mathbf{w}_h(\mathrm{j}\omega_\alpha) + \mathbf{B}_h\Delta\dot{\mathbf{w}}_h(\mathrm{j}\omega_\alpha)), \tag{5.7}$$

or, alternatively:

$$\mathbf{f}_h(\mathrm{j}\omega_\alpha) = -(\mathbf{C}_h + \mathrm{j}\omega_\alpha\mathbf{B}_h)\Delta\mathbf{w}_h(\mathrm{j}\omega_\alpha). \tag{5.8}$$

Here

$$\mathbf{C}_h = \mathrm{diag}[c_{hi}], \qquad \mathbf{B}_h = \mathrm{diag}[b_{hi}].$$

After transforming the $\mathbf{f}_h$ force acting onto the mass centres of RFEs A and B (i.e. points A and B), we get

$$\mathbf{f}_h^{AB} = -\mathbf{T}_h^T(\mathbf{C}_h + \mathrm{j}\omega_\alpha\mathbf{B}_h)\Delta\mathbf{w}_h. \tag{5.9}$$

A vector of interaction forces of SDE no. k has the form

$$\mathbf{f}_k = -(\mathbf{C}_k + \mathrm{j}\omega_\alpha\mathbf{B}_k)\Delta\mathbf{w}_k, \tag{5.10}$$

and, after transforming the above to point A:

$$\mathbf{f}_k^A = \mathbf{S}_{Ak}^T\mathbf{\Theta}_{Ak}^T\mathbf{f}_k, \tag{5.11}$$

or, after transforming to point B:

$$\mathbf{f}_k^B = \mathbf{S}_{Bk}^T \mathbf{\Theta}_{Bk}^T \mathbf{f}_k. \tag{5.12}$$

A vector of external forces of finite element (FE) no. e_k, which contains n_{ek} nodes, has the form

$$\mathbf{f}_{ek} = \mathrm{col}(\mathbf{f}_{eki}), \qquad i = 1, ..., n_{ek}. \tag{5.13}$$

A vector of external forces $\mathbf{f}_{eki}$ of node no. i of the FE no. e_k (denoted by $\hat{\mathbf{f}}_{eki}$ in the global coordinate system), after transforming it to point A, has the form:

$$\mathbf{f}_{eki}^A = \mathbf{S}_{ekA}^T \mathbf{\Theta}_{ekA}^T \hat{\mathbf{f}}_{eki}, \tag{5.14}$$

or, transforming to point B:

$$\mathbf{f}_{eki}^B = \mathbf{S}_{ekB}^T \mathbf{\Theta}_{ekB}^T \hat{\mathbf{f}}_{eki}. \tag{5.15}$$

Once the generalised equation of dynamics is applied to point A, we get:

$$\mathbf{f}_A = \sum_{\kappa=1}^{i_{\kappa A}} \mathbf{f}_{k(\kappa)}^A + \sum_{\eta=1}^{i_{\eta A}} \mathbf{f}_{ek(\eta)i}^A - \mathbf{M}_A \ddot{\mathbf{q}}_A, \tag{5.16}$$

or, if applied to point B, we get:

$$\mathbf{f}_B = \sum_{\kappa=1}^{i_{\kappa B}} \mathbf{f}_{k(\kappa)}^B + \sum_{\eta=1}^{i_{\eta B}} \mathbf{f}_{ek(\eta)i}^B - \mathbf{M}_B \ddot{\mathbf{q}}_B. \tag{5.17}$$

Now, it is time to construct suitable vectors of forces of RFEs A and B, i.e.:

$$\mathbf{f}^{AB} = \mathrm{col}(\mathbf{f}_A \;\; \mathbf{f}_B), \tag{5.18}$$

and to formulate an equation of equilibrium:

$$\mathbf{f}^{AB} + \mathbf{f}_h^{AB} = \mathbf{0}. \tag{5.19}$$

Making use of Eq. (5.9) and performing some matrix transformations, we obtain

$$\mathbf{f}^* = \mathbf{C}_h^* \Delta \mathbf{w}_h, \tag{5.20}$$

where

$$\mathbf{f}^* = (\mathbf{T}_h \mathbf{T}_h^T)^{-1} \mathbf{T}_h \mathbf{f}^{AB}, \qquad \mathbf{C}_h^* = \mathbf{C}_h + j\omega_\alpha \mathbf{B}_h.$$

Moreover,

$$\mathbf{f}^* = \mathrm{col}(f_i^*), \qquad i = 1, ..., 6,$$

and

$$\mathbf{C}_h^* = \mathrm{diag}[c_{hi}^*], \qquad i = 1, ..., 6.$$

Once vectors $\mathbf{f}^*$ and $\Delta \mathbf{w}_h$ are explicitly known, solution of Eq. (5.20) provides the $\mathbf{C}_h^*$ matrix components. Subsequently, all the stiffness and damping coefficients of SDE no. h are calculated using the relationships given below:

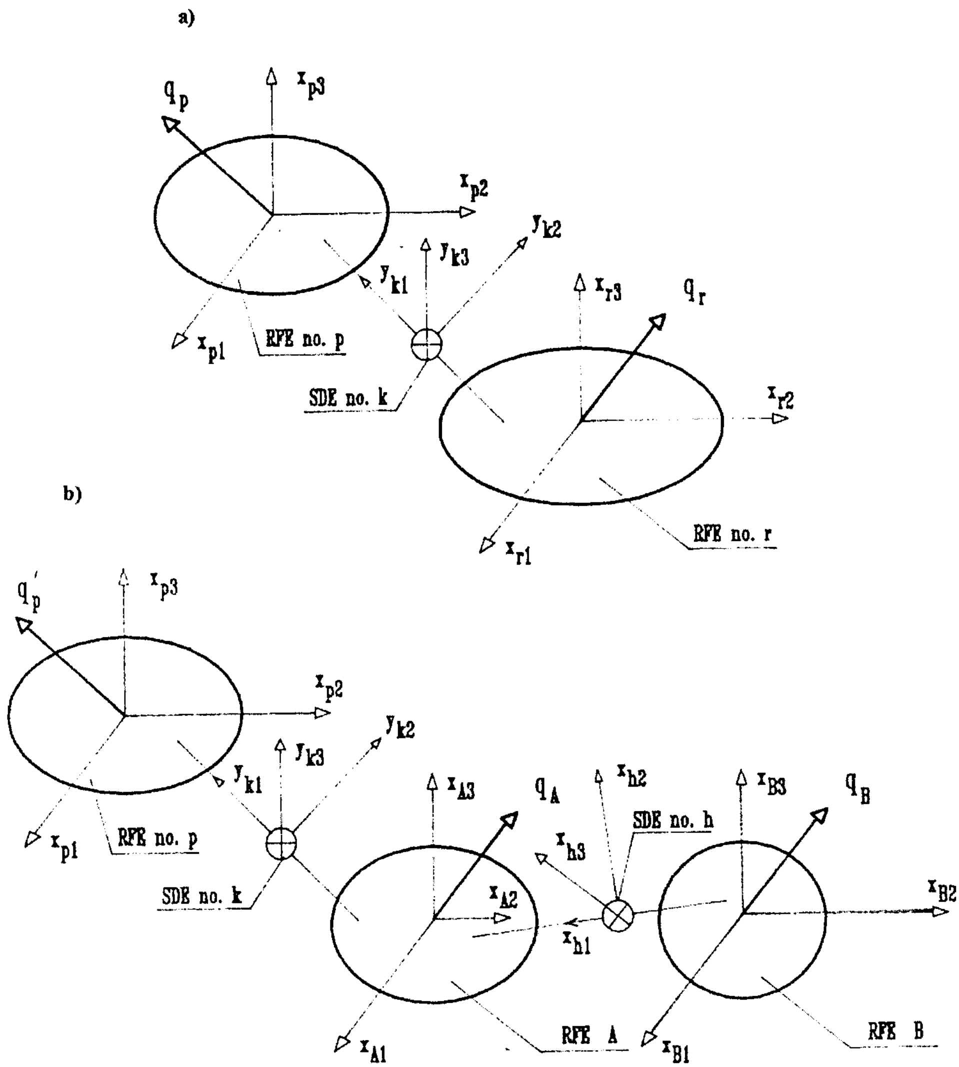

FIG. 7. A part of the model with SDEs, whose parameters are to be improved: a) the model before division, b) the model after division.

(5.21) $$c_{hi} = \mathrm{Re}(c_{hi}^*), \qquad i = 1, ..., 6,$$

(5.22) $$b_{hi} = \frac{1}{\omega_\alpha}\mathrm{Im}(c_{hi}^*), \qquad i = 1, ..., 6,$$

where

$$c^*_{hi} = f^*_i / \Delta w_{hi}, \qquad i = 1, ..., 6. \tag{5.23}$$

Moreover, as it was explained in Sec. 4, the stiffness and damping coefficients of all the SDEs which are attached to RFEs A and B, should be improved. For the part (i.e. RFE no. r) before its subdivision (Fig. 7a), a vector of deformation of SDE no. k is described by Eq. (4.1), while for the same part after the subdivision (Fig. 7b), this vector will have the form:

$$\Delta \mathbf{w}'_k = \left[\boldsymbol{\Theta}_{pk}\mathbf{S}_{pk} \vdots - \boldsymbol{\Theta}_{Ak}\mathbf{S}_{Ak} \right] \begin{bmatrix} \mathbf{q}'_p \\ \mathbf{q}_A \end{bmatrix}, \tag{5.24}$$

if SDE no. k is attached to RFE A, or

$$\Delta \mathbf{w}'_k = \left[\boldsymbol{\Theta}_{pk}\mathbf{S}_{pk} \vdots - \boldsymbol{\Theta}_{Bk}\mathbf{S}_{Bk} \right] \begin{bmatrix} \mathbf{q}'_p \\ \mathbf{q}_B \end{bmatrix}, \tag{5.25}$$

if SDE no. k is attached to RFE B. Improved stiffness coefficients of SDE no. k are determined by Eq. (4.8), but improved damping coefficients of SDE no. k found from Eq. (4.9).

6. Illustrative examples

The examples below illustrate the application of methodology for creating substitute models of the selected real machine tool substructures, as described in previous sections. Stiffness parameters of SDE no. h are computed, and stiffness values of the other SDEs are improved as well.

The FWD 32J milling machine spindle

A discrete model of the main driving system, whose parameters are reduced to the spindle [2], is considered as a primary model for calculating the FWD 32-J milling machine spindle. Thus, the model composed of 4 RFEs and 8 SDEs, is obtained (Fig. 8a). The dimensions of several RFEs are not comparative, because the RFE, which idealises the spindle end, the arbor and the face cutter, differs considerably in length from the others. This raises a question whether this part of the spindle should be really idealised by only one solid body, in view of the accuracy criteria assumed. For this purpose, the RFE introduced above is replaced by a single beam deformable finite element (FE 4) [2, 7], while the other elements of the structure, geometry of connections and external loads remain unchanged. The mixed model of RFEs, SDEs and DFB is obtained as a result. Here, a criterion of accuracy is not fulfilled. Therefore, FE 4 is replaced by two RFEs $4A$ and $4B$, which are connected by SDE no. h (Fig. 8b). If we suppose that positions of the connecting points of SDE no. h are to be established, the task depends upon determination of 6 stiffness coefficients of this SDE.

The basic input data for the FWD 32J milling machine spindle model is presented in Table 1, while results of computation of stiffness coefficients of the SDE no. h, and of the SDEs which are attached to DFB, are presented in Table 2. It is seen that the improved values of SDEs no. 7, 11 and 12 are different from the original ones.

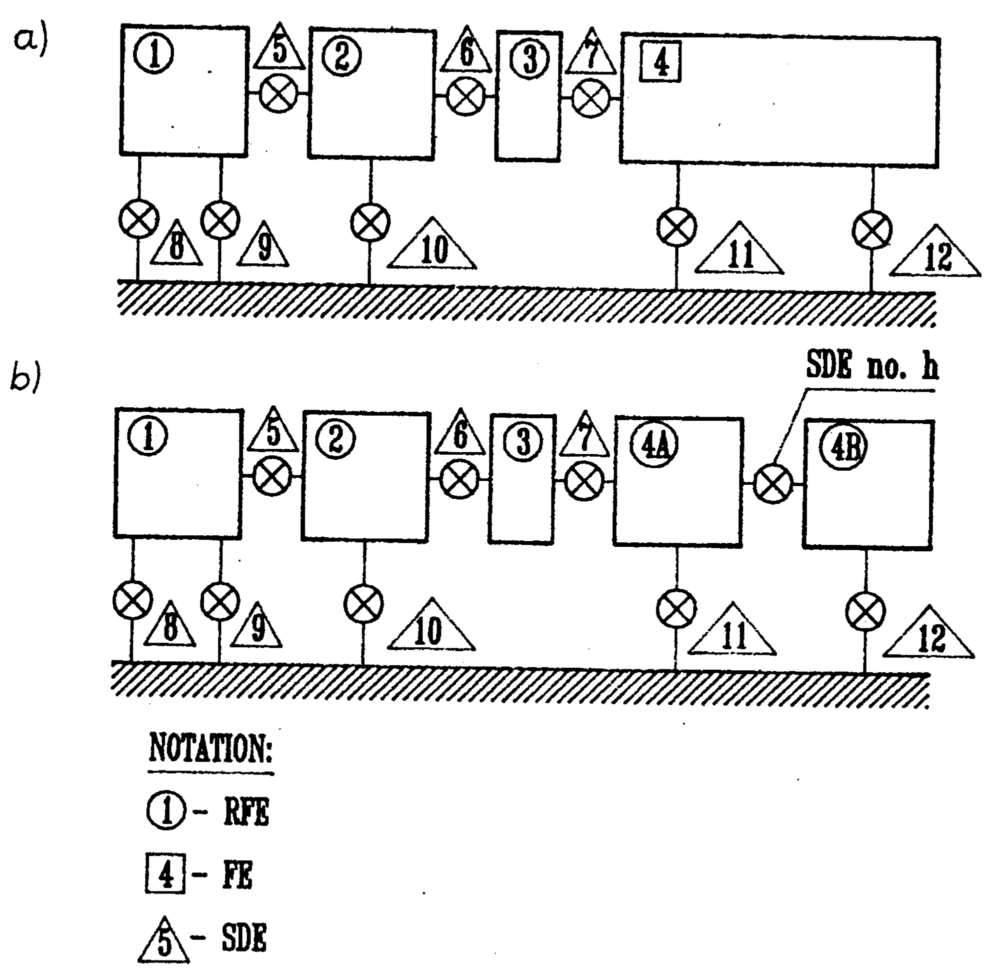

FIG. 8. The FWD 32J milling machine spindle: a) primary model, which is composed of RFEs and SDEs only, b) substitutive model, which contains the couple of new RFEs.

Table 1. Basic input data for the FWD 32J milling machine spindle model

STIFFNESS COEFFICIENTS OF ALL SDEs						
SDE no.	c_{k1}	c_{k2}	c_{k3}	c_{k4}	c_{k5}	c_{k6}
	[daN/cm]			[daNcm]		
5	2.4500×10^6	8.2750×10^5	8.2750×10^5	2.0190×10^5	2.6660×10^5	2.6660×10^5
6	1.1680×10^8	3.9440×10^7	3.9440×10^7	3.3900×10^7	1.0400×10^9	1.0400×10^9
7	6.9730×10^5	2.3550×10^5	2.3550×10^5	1.2790×10^6	1.9250×10^6	1.9250×10^6
8	2.4860×10^5	8.5060×10^4	8.5060×10^4	0.0000	0.0000	0.0000
9	0.0000	3.7830×10^3	3.7830×10^3	0.0000	0.0000	0.0000
10	0.0000	0.0000	0.0000	5.4326×10^4	0.0000	0.0000
11	0.0000	6.0021×10^4	6.0021×10^4	0.0000	0.0000	0.0000
12	0.0000	2.1940×10^5	2.1940×10^5	0.0000	0.0000	0.0000

Table 2. Calculation results for the FWD 32J milling machine spindle model

STIFFNESS COEFFICIENTS OF SDE NO. h						
	c_{h1}	c_{h2}	c_{h3}	c_{h4}	c_{h5}	c_{h6}
	[daN/ cm]			[daNcm]		
	1.8529×10^6	1.4566×10^5	1.4566×10^5	2.4012×10^6	3.1848×10^6	3.1848×10^6
IMPROVED STIFFNESS COEFFICIENTS OF SDEs ATTACHED TO THE DFB						
SDE no.	c_{k1}	c_{k2}	c_{k3}	c_{k4}	c_{k5}	c_{k6}
	[daN/ cm]			[daNcm]		
7	5.0327×10^5	2.0525×10^4	2.0525×10^4	8.1770×10^5	6.8940×10^6	6.8940×10^6
11	0.0000	3.9526×10^4	3.9526×10^4	0.0000	0.0000	0.0000
12	0.0000	5.0166×10^5	5.0166×10^5	0.0000	0.0000	0.0000

Furthermore, a steady harmonic vibration of the spindle end along two mutually perpendicular directions (i.e. direction 1 and 2) for various values of angular frequency ω is computed. Suitable resonant curves for both the basic model and the one after division are shown in Fig. 9 and 10. Although it is possible to

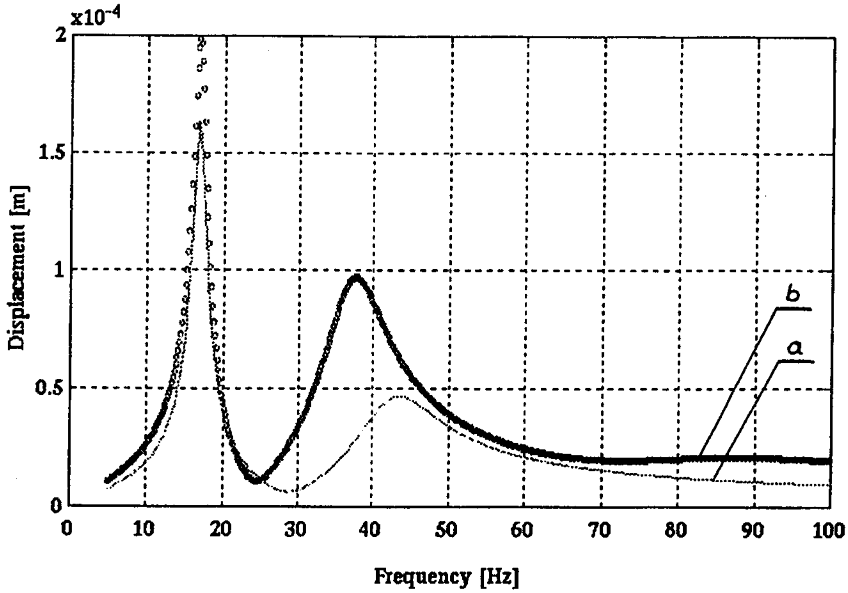

FIG. 9. Resonant curves of the FWD 32J milling machine spindle end along direction 1 for a) the primary model, b) the substitutive model.

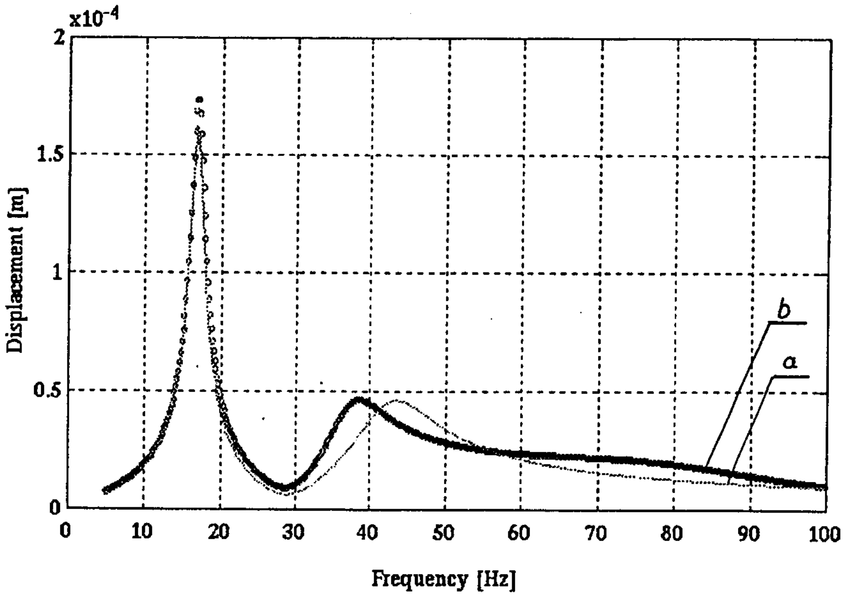

FIG. 10. Resonant curves of the FWD 32J milling machine spindle end along direction 2 for a) the primary model, b) the substitutive model.

find some intervals in which the plots are compatible, we cannot observe it for a wide frequency band. It is to be emphasised that significant differences are noticed near the resonant peaks. They concern the value of the second resonant frequency, but in case of direction 1 – also all resonant amplitudes. The results for both these models diverge considerably from each other. It means that the behaviour of improved model differs from the original, which renders the substitutive model reasonable. It seems to be a promising approach, which makes discrete idealization of the machine tool more reliable. Although FEM's idealization and substitutive model yield the same number of degrees of freedom, inertia matrices in case of the latter model remain diagonal. It makes the model more convenient for dynamic calculations.

The closed profile box frame

Here a calculation is made for a box frame, whose primary model is composed of 4 deformable plate-plane elements (i.e. FEs no. 1, 2, 3 and 4) [7, 8] and total number of six-degree-of-freedom nodes is 8. The elements are connected with one another at their nodes, and are attached to the ground by 4 SDEs (Fig. 11a). A suitable accuracy criterion is not satisfied in this case either. Thus, the model is replaced by two RFEs A and B, which are connected by SDE no. h (Fig. 11b). The other elements of the structure, connection geometry and external loads remain unchanged. If we suppose that positions of connecting points of SDE no. h are known, then the task consists in determination of 6 stiffness coefficients of this SDE.

The basic input data for the initial model is presented in Table 3 and the results of the calculation of stiffness coefficients of the SDE no. h, and of the SDEs which are attached to the DFB, are presented in Table 4. The values of parameters of the resultant model significantly differ from those of the primary one.

Table 3. Basic input data for the box frame.

STIFFNESS COEFFICIENTS OF ALL SDEs						
SDE no.	c_{k1}	c_{k2}	c_{k3}	c_{k4}	c_{k5}	c_{k6}
	[daN/cm]			[daNcm]		
5	9.1000×10^8	9.1000×10^8	9.1000×10^8	2.8000×10^9	2.8000×10^9	2.8000×10^9
6	9.1000×10^8	9.1000×10^8	9.1000×10^8	2.8000×10^8	2.8000×10^8	2.8000×10^8
7	9.1000×10^8	9.1000×10^8	9.1000×10^8	2.8000×10^8	2.8000×10^8	2.8000×10^8
8	9.1000×10^8	9.1000×10^8	9.1000×10^8	2.8000×10^8	2.8000×10^8	2.8000×10^8

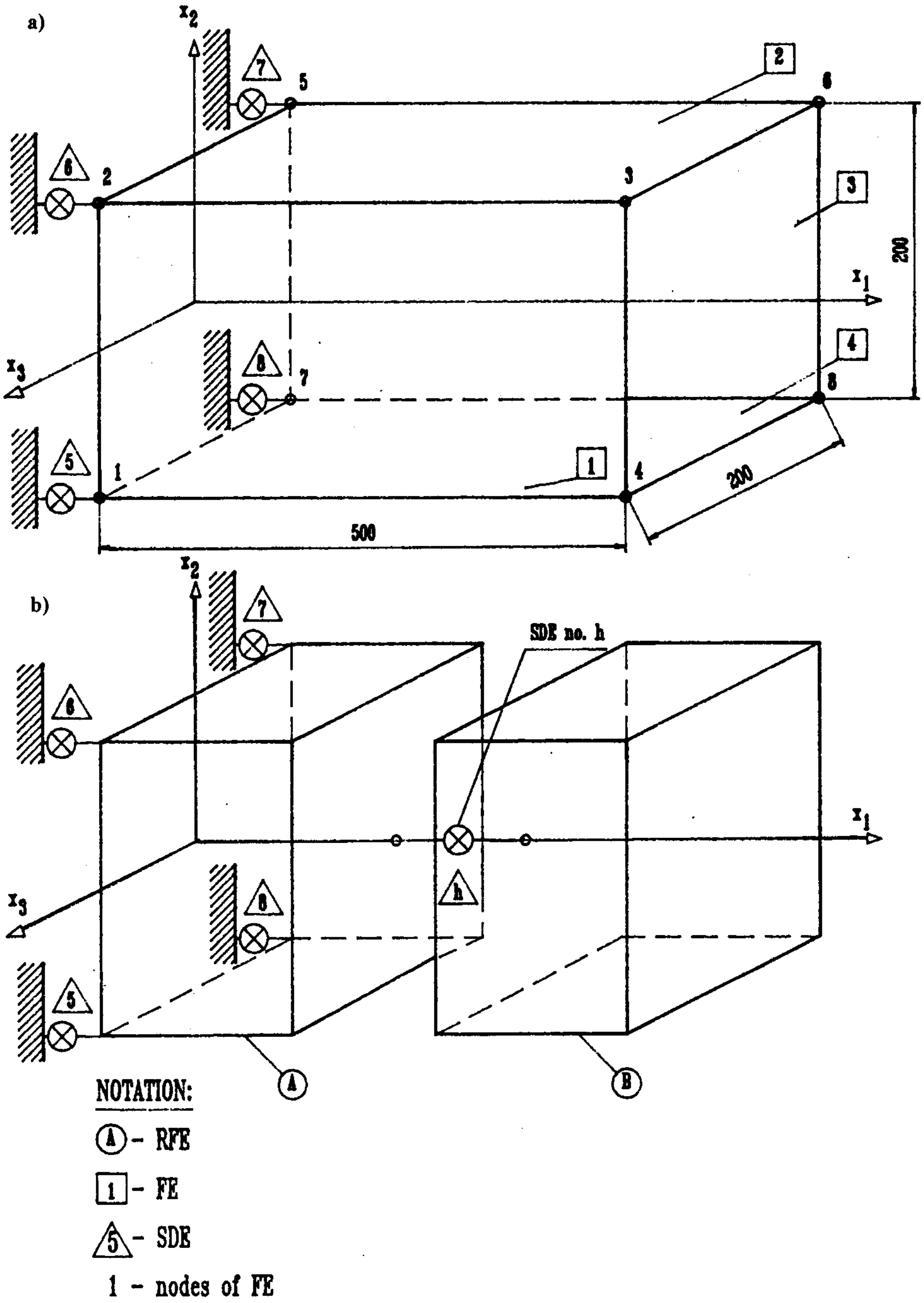

FIG. 11. The box frame: a) primary model, which is composed of FEs and SDEs, b) substitutive model, which contains the couple of two RFEs.

Table 4. Calculation results for the box frame.

STIFFNESS COEFFICIENTS OF SDE NO. h						
	c_h	c_{h2}	c_{h3}	c_{h4}	c_{h5}	c_{h6}
	[daN/cm]			[daNcm]		
	4.9223×10^6	6.6387×10^5	6.4144×10^5	1.5376×10^8	2.4328×10^8	1.9696×10^8
IMPROVED STIFFNESS COEFFICIENTS OF SDEs ATTACHED TO THE DFB						
SDE no.	c_{k1}	c_{k2}	c_{k3}	c_{k4}	c_{k5}	c_{k6}
	[daN/cm]			[daNcm]		
5	3.0087×10^9	4.6872×10^9	4.6872×10^9	2.0510×10^7	5.1686×10^9	6.8445×10^9
6	2.1428×10^9	4.6872×10^9	4.0104×10^8	2.0510×10^6	5.1686×10^8	6.8445×10^8
7	3.2540×10^8	4.0104×10^8	4.0104×10^8	2.0510×10^6	5.1686×10^8	6.8445×10^8
8	1.7772×10^9	4.0104×10^8	4.6872×10^9	2.0510×10^6	5.1686×10^8	6.8445×10^8

The substitutive model has only 12 degrees of freedom. Due to the number of them in the classic FEM approach, which is equal to 48 (i.e. 8×6), the cost of computation is reduced by the factor of (approximately) $(48/12)^3 = 64$. The reasons explained above make the substitutive model fully justified.

The FWD32J milling machine's beam

As a result of the analysis of the FWD32J milling machine's carrying system [2, 5], its natural discrete model has been created. Suitable primary model is obtained in such a way and it is a classic RFEM's representation. One component of the structure is an independently driven beam. Its structure is mainly designed of some frame boxes and is connected with the machine tool frame by four slideways.

Rough analysis of beam geometry allows us to suspect that its substitution by one RFE may lead to significant errors of modelling. Therefore, primary model (i.e. RFE which idealises the beam) has been replaced by a set of 21 deformable plate-plane elements (i.e. FEs), whose total number of six-degree-of-freedom nodes is 32 (Fig. 12a). Suitable accuracy criterion has been examined and it is sure that the examination is not successful. Following that, a couple of two RFEs, connected by SDE no h has been introduced as a substitution (Fig. 12b).

The latter has only 12 degrees of freedom. Due to a considerably higher number in the classic FEM approach (i.e. $32 \times 6 = 192$), cost of computation

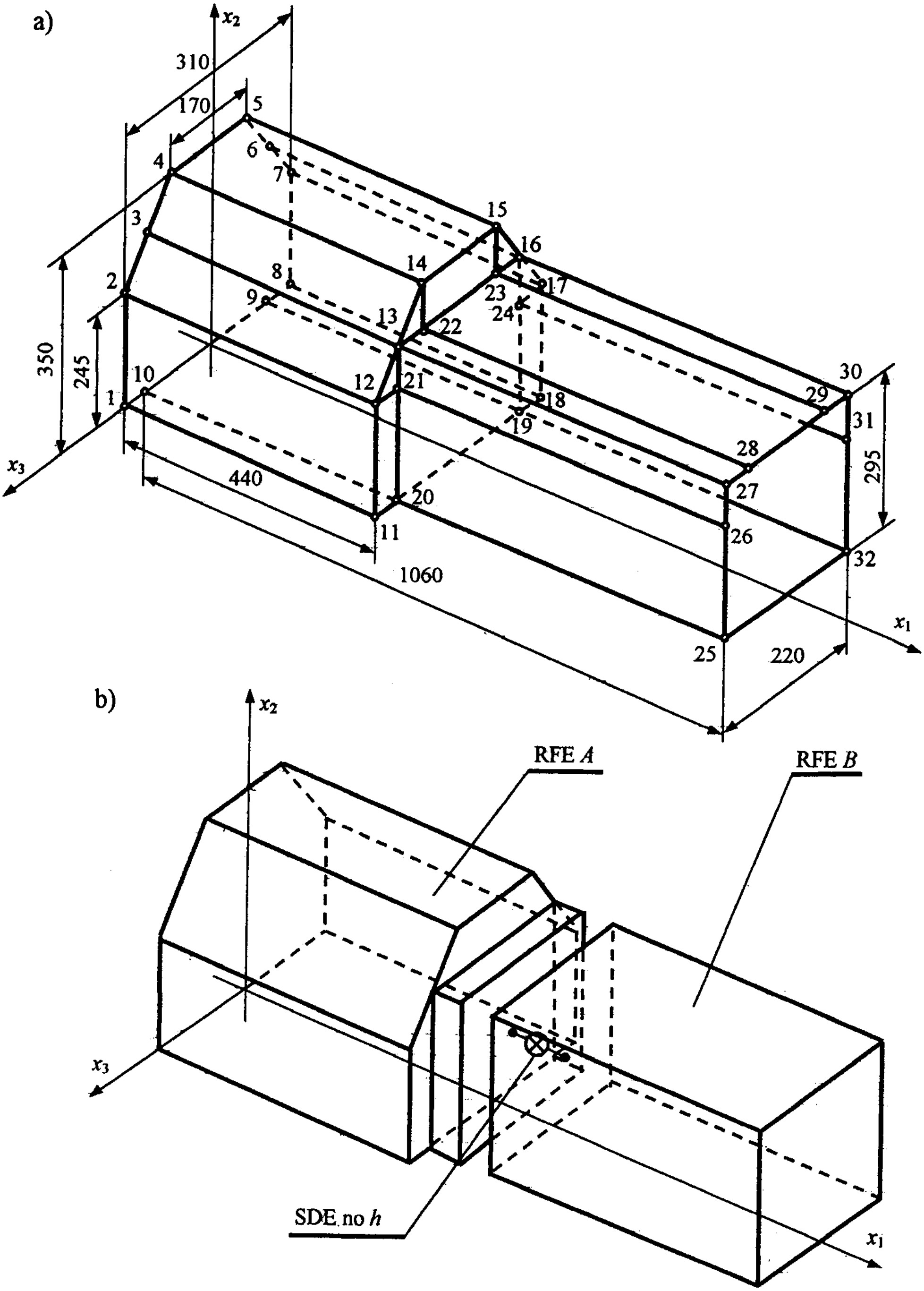

FIG. 12. The FWD 32J milling machine's beam: a) primary model, which is composed of FEs, b) substitutive model, which contains the couple of two RFEs.

is greatly reduced (i.e. approximately $(192/12)^3 = 4096$ times). It really proves the practical meaning of the method proposed in the paper.

7. Conclusions

We have proposed the approach to improve the discrete model, which was primarily created by the rigid finite element method. The reason is that the primary model rarely satisfies the accuracy criterion, and then their parts usually require further subdivision. An improved model still remains a set of similar elements. It also seems to allow to save inertia matrices being diagonal, which is extremely applicable for dynamic calculation. Following that, the dynamics may be computed using the same software. It is a very important feature, because many kinds of computer tools involve various implementations of the rigid finite element method.

The calculation results shown in the paper evidently support the approach. It has been proved that corrected values of the parameters differ significantly from the previous ones. A comparison between resonant curves of primary and improved model yields significant differences. Therefore it is advised that every discrete structure which is initially idealised by the rigid finite element method, should still be examined with respect to the accuracy criterion being fulfilled, even if it can be imagined as entirely rigid.

There are many approaches to the problem of determination of the structural parameters. However, in order to make dynamic calculations successful it is not enough to determine correct values of parameters of the discrete models. It is more important to produce the model whose performance is an extremely detailed image of the real structure. The approach proposed satisfactorily accomplishes such a requirement.

Results of deflections for both the models (i.e. FEM's one and substitute) really converge to each other. But if we define the efficiency factor as a quotient of accuracy and cost of computation, the method of substitutive model is much more effective than the classic FEM. High efficiency of modelling by the approach proposed has been proved.

One alternative approach is worthy of being mentioned at the end. It consists in reduction of the number of degrees of freedom in large discrete systems (e.g. idealised by the FEM) [7]. The approach produces accurate results in case of calculating the excited vibration problems. However, it is required to make the reduction on-line (i.e. during the main process of calculation). Therefore, despite reducing the discrete system size, it does not result in a noticeable shortening of the calculation time. The method proposed in the paper is free of such disadvantages.

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