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FABRIC TENSORS IN BONE MECHANICS

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Mechanical properties of cancellous and cortical bone have been investigated. The fabric tensors used in the relevant literature have been discussed. Nonlinear elastic and elastic-perfectly plastic constitutive relationships have been proposed within the framework of small deformations. To this end the theory of representation of tensor functions has been used. It has been shown that the fabric tensor plays the role of a parametric tensor. Orthotropic linear elasticity has been carefully examined from the point of view of interrelations of classical material constants with the proposed material parameters and eigenvalues of the fabric tensor. Hoffman's strength criterion has been extended by incorporating the fabric tensor. Anisotropic properties of human cancellous and cortical bones have been investigated by using the relations derived.

1. Introduction

Some materials such as woods, granular materials, bones and plastics exhibit elastic, plastic and locking behaviour under compressive stresses. The stress-deformation curves are then strongly influenced by the density of a material, cf. Figs. 10.3 and 11.5 in [12]. Cancellous (spongy or trabecular) bone is quite porous; often more than half of the bone volume is occupied by pores [8, 11, 12], cf. also Fig. 1 a. The cellular structure of cancellous bone consists of an interconnected network of rods or plates. A network of rods produces low-density open cells, while one of the plates gives higher-density, virtually closed cells. There are some theoretical models for the elastic modulus and strength dependence upon the structural density of very high porosity open cell or closed cell materials. These models help to explain the obvious trends in the properties with density [12]. Cancellous bone structure is anisotropic as well as porous and inhomogeneous. In the mechanics of porous materials, it is recognized that porosity is the primary measure of local material microstructure. There appears to be a general agreement that a tensor is the best second rank measure of local material microstructure in many porous and composite materials. Following the work of ODA [26], this tensor is generally called the *fabric tensor*. The definition of the fabric tensor varies with the type of material and the investigator. For example, KANATANI [24] expands the distribution density function in spherical harmonics and obtains an infinite series of even rank tensors. The first of these tensors is a second rank tensor.

In Sec. 2 of our contribution we shall discuss structural tensors currently used in the bone mechanics. Moreover, RYCHLEWSKI and ZHANG'S [31] anisotropy measure will be applied. Constitutive equations for geometrically linear elastic materials characterized by a positive definite structural tensor will be introduced in Sec. 3. In particular, linear relationships will be given. In contrast to the papers [5 – 10, 13, 39], the structural tensor will be treated as a parametric tensor, and not as a variable. Such an approach is consistent with the general theory of anisotropic tensor functions [2, 21 – 23, 41]. In Sec. 4 we shall generalize HOFFMAN'S [18] strength criterion, well known in the composite mechanics, in a manner suitable for defining a yield condition for the trabecular bone as well as constitutive relationships for elastic-perfectly plastic materials within the framework of small deformations. In contrast to Tsai and Wu's criterion [33 – 36], which was used in the papers [3, 4, 6, 8] as a strength criterion for bones, Hoffman's criterion requires carrying out only standard strength tests. The last criterion requires no additional hypotheses concerning the determination of material parameters by performing multi-axial tests.

We observe that mechanical properties of bones have been studied in many books and papers, cf. [8, 11, 12] and the references cited therein. The review paper by KEAVENY and HAYES [25] summarizes the state-of-the art in the trabecular bone mechanics.

2. THE FABRIC TENSOR

In the present contribution we shall provide a general framework for elastic and elastic-plastic orthotropic materials, provided that structural anisotropy is described by a second-order tensor, called the fabric tensor, cf. [4 – 6, 26].

Let us introduce this tensor. First, however, following WHITEHOUSE [40] we recall the notion of the mean intercept length L. This author measured L in cancellous bone as a function of direction on polished plane sections. Then L is the distance between two bone/marrow interfaces measured along a line. The value of L is a function of the slope Θ of the line along which the measurement is made. WHITEHOUSE [40] showed that when $L(\Theta)$ is plotted in the polar coordinates then the polar diagram produced ellipses. If the test lines are rotated through several values of Θ and the corresponding values of $L(\Theta)$ are measured, the results are found to fit the following equation of an ellipse, cf. Fig. 1 b

$$\frac{1}{L(\Theta)} = M_{11}\cos^2\Theta + M_{22}\sin^2\Theta + 2M_{12}\sin\Theta\cos\Theta, \tag{2.1}$$

where M_{11}, M_{22} and M_{12} are constants, provided that the reference line from which the angle Θ is measured is kept constant.

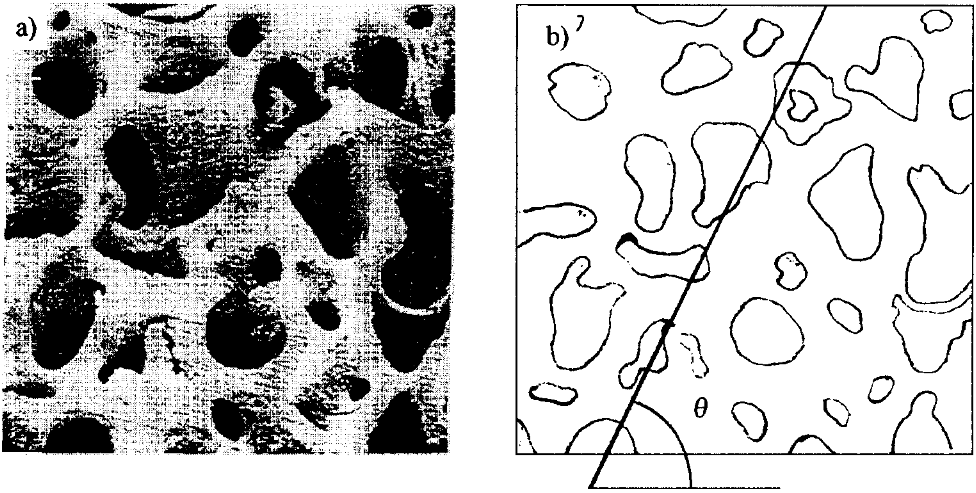

FIG. 1. a. Scanning electron micrographs showing the cellular structure of spongy bone (a specimen from the femoral head), after [12]. b. Test lines superimposed on a cancellous bone specimen. The test line are oriented at angle Θ, which is varied to obtain the mean intercept length $L(\Theta)$, after [40].

HARRIGAN and MANN [15] extended Whitehouse's approach to the three-dimensional case and showed that $L(\mathbf{n})$, as a function of a direction $\mathbf{n}$, would be represented by ellipsoids and would therefore be equivalent to a positive definite second-order tensor $\mathbf{M}$ defined by

$$\frac{1}{L(\Theta)} = M_{ij}n_i n_j \,, \tag{2.2}$$

where $\mathbf{n}$ is the unit vector in the direction of the test line.

COWIN [5 – 10] defined the fabric tensor of cancellous bone to be the inverse square root of the mean intercept length tensor $\mathbf{M}$:

$$\mathbf{H} = \frac{1}{\sqrt{\mathbf{M}}} \,. \tag{2.3}$$

Obviously, $\mathbf{H}$ is well defined because $\mathbf{M}$ is a positive definite and symmetric tensor. The components of $\mathbf{M}$ or the mean intercept ellipsoid can be measured by using the techniques described by HARRIGAN and MAN [15] for a cubic specimen.

GOULET *et al.* [13] applied the concept of the mean intercept length to investigate the relationships between the structural parameters for cancellous bone, to determine their correlation with the mechanical properties and to evaluate which parameters are important for maintaining the bone strength and integrity.

The fabric tensor $\mathbf{H}$, as defined by (2.3), is an isotropic tensor function of $\mathbf{M}$, say $\widehat{\mathbf{H}}(\mathbf{M})$. It means that

$$\forall\, \mathbf{Q} \in O(3) \quad \mathbf{Q}\widehat{\mathbf{H}}(\mathbf{M})\mathbf{Q}^T = \widehat{\mathbf{H}}(\mathbf{Q}\,\mathbf{M}\,\mathbf{Q}^T) = \mathbf{Q}\frac{1}{\sqrt{\mathbf{M}}}\mathbf{Q}^T. \tag{2.4}$$

Here $O(3)$ stands for the full orthogonal group:

$$O(3) \equiv \left\{\mathbf{Q} \middle| \quad \mathbf{Q}\,\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}\right\}, \tag{2.5}$$

where $\mathbf{I}$ is the identity tensor; moreover $\mathbf{Q}^T$ is the transpose of $\mathbf{Q}$.

Let us pass to the determination of the function

$$\mathbf{H} = \widehat{\mathbf{H}}(\mathbf{M}) = \frac{1}{\sqrt{\mathbf{M}}}. \tag{2.6}$$

Recalling that $\mathbf{M}$ is the symmetric positive definite tensor, by applying the spectral theorem we may write

$$\mathbf{M} = M_1\mathbf{i}_1 \otimes \mathbf{i}_1 + M_2\mathbf{i}_2 \otimes \mathbf{i}_2 + M_3\mathbf{i}_3 \otimes \mathbf{i}_3\,, \tag{2.7}$$

where M_j $(j = 1,2,3)$ are eigenvalues of the tensor $\mathbf{M}$, and $\mathbf{i}_j$ its eigenvectors. It is assumed that

$$M_1 \geq M_2 \geq M_3\,, \tag{2.8}$$

where

$$M_i = \frac{1}{3}\mathrm{I}_M + \frac{2}{3}\sqrt{\mathrm{I}_M^2 - 3\mathrm{II}_M}\cos\left[\frac{2}{3}\pi(i-1) - \varphi\right], \qquad i = 1,2,3 \tag{2.9}$$

and

$$\cos 3\varphi = \frac{2\mathrm{I}_M^3 - 9\mathrm{I}_M\mathrm{II}_M + 27\mathrm{III}_M}{\sqrt{2(\mathrm{I}_M^2 - 3\mathrm{II}_M)^3}}. \tag{2.10}$$

The basic invariants of $\mathbf{M}$ are given by

$$\begin{aligned} \mathrm{I}_M &= \mathrm{tr}\,\mathbf{M}, \qquad \mathrm{II}_M = \frac{1}{2}\left(\mathrm{tr}^2\mathbf{M} - \mathrm{tr}\,\mathbf{M}^2\right), \\ \mathrm{III}_M &= \det\mathbf{M} = \frac{1}{6}\left(\mathrm{tr}^3\mathbf{M} - 3\mathrm{tr}\mathbf{M}\,\mathrm{tr}\,\mathbf{M}^2 + 2\mathrm{tr}\,\mathbf{M}^3\right), \end{aligned} \tag{2.11}$$

where $\mathrm{tr}\,\mathbf{M}$ is the trace of $\mathbf{M}$. In an orthonormal basis $\{\mathbf{e}_i\}$ $(i = 1,2,3)$ we have: $\mathbf{M} = M_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, $\mathrm{tr}\,\mathbf{M} = M_{ii}$; $(\mathbf{M}^2)_{ij} = (\mathbf{M}\,\mathbf{M})_{ij} = M_{ik}M_{kj}$, etc.

Note that if

$$d = 4\mathrm{II}_M^3 - \mathrm{I}_M^2\mathrm{II}_M^2 + 4\mathrm{I}_M^3\mathrm{III}_M - 18\mathrm{I}_M\mathrm{II}_M\mathrm{III}_M + 27\mathrm{III}_M^2 < 0, \tag{2.12}$$

then M_i in (2.9) are different; for $d = 0$ two of the eigenvalues are equal. In other words, the tensor **M** is then two-dimensional. Finally, for

$$\mathrm{I}_M^2 = 3\mathrm{II}_M \,, \tag{2.13}$$

M is a spherical tensor.

In the case of three different eigenvalues, the eigentensors $\mathbf{i}_j \otimes \mathbf{i}_j$ (no summation over j) can be determined in a unique manner:

$$\mathbf{i}_j \otimes \mathbf{i}_j = \frac{1}{m_j}\left[\mathbf{M}^2 - (\mathrm{I}_M - M_j)\mathbf{M} + \mathrm{III}_M M_j^{-1}\mathbf{I}\right] \text{ (no summation over } j), \tag{2.14}$$

where

$$m_j = 2M_j^2 - \mathrm{I}_M M_j + \mathrm{III}_M M_j^{-1} \,. \tag{2.15}$$

Consequently, the fabric tensor (2.3) satisfying (2.4) can be represented in the following form:

$$\mathbf{H} = H_1 \mathbf{i}_1 \otimes \mathbf{i}_1 + H_2 \mathbf{i}_2 \otimes \mathbf{i}_2 + H_3 \mathbf{i}_3 \otimes \mathbf{i}_3 \,, \tag{2.16}$$

where

$$H_i = \frac{1}{\sqrt{M_i}} \,, \qquad i = 1, 2, 3. \tag{2.17}$$

Cowin [8] concluded that mechanical properties of cancellous bone are independent of the pore size and that the largest and smallest values of the set $\{H_1, H_2, H_3\}$ are associated with the largest and smallest values of Young's moduli, respectively. The fabric tensor may be normalized by the requirement [8, 39]

$$\operatorname{tr}\mathbf{H} = H_1 + H_2 + H_3 = 1. \tag{2.18}$$

An alternative approach to the fabric tensor has been discussed by Zysset and Curnier [42].

An elementary microstructural description is contained in a single scalar property such as relative density, while material anisotropy requires fabric tensors of higher even rank [24]. Kanatani's [24] approach can be applied to a class of materials with strictly positive morphological properties that are radially symmetric. In these situations we can use a scalar-valued orientation distribution function $h(\mathbf{N}) > 0$, where $\mathbf{N} = \mathbf{n} \otimes \mathbf{n}$ is the tensor product of the unit vector $\mathbf{n}$ specifying the orientation. Assuming the function to be square integrable, it can be expanded in a convergent Fourier series:

$$\begin{aligned} h(\mathbf{N}) &= g(\mathbf{N})1 + \mathbf{G} \cdot \mathbf{F}(\mathbf{N}) + \mathbb{G}\colon \mathbb{F}(\mathbf{N}) + \ldots \\ &= g(\mathbf{N})1 + G_{ij}F_{ij}(\mathbf{N}) + \mathbb{G}_{ijkl}\mathbb{F}_{ijkl}(\mathbf{N}) + \ldots \,, \end{aligned} \tag{2.19}$$

where 1, $\mathbf{F}(\mathbf{N})$ and $\mathbb{F}(\mathbf{N})$ are even rank tensorial basis functions and g, $\mathbf{G}$ and $\mathbb{G}$ the corresponding even rank fabric tensors [24]. In bone mechanics we can use an approximation based on a scalar and a symmetric, traceless second rank fabric tensor. Then the first tensorial basis function is

$$\mathbf{F} - \frac{1}{3}\mathbf{I}, \tag{2.20}$$

while the tensorial coefficients are calculated by

$$g = \frac{1}{4\pi}\int_S h(\mathbf{N})\, dS, \qquad \mathbf{G} = \frac{15}{8\pi}\int_S h(\mathbf{N})\mathbf{F}(\mathbf{N})\, dS, \tag{2.21}$$

where S is the surface of the unit sphere. For the particular case of an ellipsoidal distribution function we have

$$h(\mathbf{N}) = \frac{1}{\sqrt{\mathbf{N}\cdot\mathbf{M}}}. \tag{2.22}$$

We observe that since $\mathbf{H}$ is positive definite, therefore its normalization according to (2.18) is admissible. It seems, however, that a natural norm for a second-order tensor is

$$\|\mathbf{H}\| = \sqrt{\operatorname{tr}\mathbf{H}^2}. \tag{2.23}$$

Consequently, more convenient to apply is the structural tensor defined by

$$\overline{\mathbf{H}} = \frac{1}{\|\mathbf{H}\|}\mathbf{H}. \tag{2.24}$$

Structural tensors are not necessarily constructed according to (2.3) or (2.21). A conceptually different approach consists in measuring the pore surfaces in almost the same way as in continuum damage mechanics, and not just the MIN (mean intercept length). Obviously, here we do not discuss the counterparts of the fourth-order tensors describing damage behaviour.

RYCHLEWSKI and ZHANG [31] introduced the following measure of orthotropy degree of $\mathbf{H}$:

$$\delta(\mathbf{H}) = \frac{\sqrt{2}}{2}\frac{(H_1 - H_3)}{\|\mathbf{H}\|}. \tag{2.25}$$

We note that in [31] a general anisotropy measure has also been introduced for tensors, tensor functions and tensor functionals. In the specific case of symmetric second-order tensors this measure reduces to (2.25). From (2.25) we conclude that if $\mathbf{H}$ is an isotropic tensor then $\delta(\mathbf{H}) = 0$. From the results due to TURNER *et al.* [39] it follows that in the case of human proximal tibia, the average eigenvalues of $\mathbf{H}$ normalized according to (2.18) are equal to: $H_1 = 0.429$, $H_2 = 0.292$, $H_3 = 0.278$; then we have $\delta(\mathbf{H}) = 0.185$. TURNER *et al.* [39] report the eigenvalues of $\mathbf{H}$ between 0.178 and 0.585. By using (2.25) we find $\delta(\mathbf{H})_{\max} = \sqrt{2}/2$. Hence we conclude that the morphological property of the bone investigated is not so strongly orthotropic. Closer inspection of the average eigenvalues of $\mathbf{H}$ given in [39] reveals that trabecular bone of the human proximal tibia behaves *approximately* as an transversely isotropic material (since $H_2 \approx H_3$). From Table 1 d of [39, pp. 556] it also follows that the bone investigated is significantly inhomogeneous.

A specific form of the fabric tensors $\mathbf{M}$, $\mathbf{H}$ or $\mathbf{J} = g\mathbf{I} + \mathbf{G}$ is not required for our subsequent developments. The only assumption is that $\mathbf{M}$, $\mathbf{H}$ and $\mathbf{J}$ be positive definite and symmetric second order tensors. In the next sections, for the fabric tensor we use the notation $\mathbf{H}$ for brevity, remembering that this is not necessarily the tensor defined by (2.3).

3. ELASTICITY

For small deformations both compact and cancellous bones exhibit elastic properties, cf. Fig. 2. Below we propose elastic constitutive relationships for the cancellous bone.

As is well known, elastic models in Green's sense derived via energy formulation are insensitive to the loading path and the whole deformation process is reversible. Two equivalent descriptions of the constitutive relationships are possible, namely

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{E}} \qquad \text{or} \qquad \mathbf{E} = \frac{\partial W^*}{\partial \mathbf{T}}, \tag{3.1}$$

where $\mathbf{T}$ is the Cauchy stress tensor and $\mathbf{E}$ is the small strain tensor. The specific elastic energy W and the specific complementary energy W^* are convex scalar-valued functions

$$W = W(\mathbf{E}), \qquad W^* = W^*(\mathbf{T}), \tag{3.2}$$

which have to satisfy the following relations:

$$\begin{gathered} W(\mathbf{0}) = 0, \quad W^*(\mathbf{0}) = 0, \quad \mathbf{T}\cdot\mathbf{E} = \operatorname{tr}\mathbf{T}\mathbf{E} = W + W^*, \\ \forall\,\mathbf{Q} \in S, \quad W(\mathbf{E}) = W\left(\mathbf{Q}\,\mathbf{E}\,\mathbf{Q}^T\right), \quad W^*(\mathbf{T}) = W^*\left(\mathbf{Q}\,\mathbf{T}\,\mathbf{Q}^T\right), \\ \forall\,\widetilde{\mathbf{E}} \in \mathsf{E}^3_s, \quad \widetilde{\mathbf{E}}\cdot\frac{\partial^2 W}{\partial\mathbf{E}\otimes\partial\mathbf{E}}\cdot\widetilde{\mathbf{E}} \geq 0; \quad \forall\,\widetilde{\mathbf{T}} \in \mathsf{E}^3_s, \quad \widetilde{\mathbf{T}}\cdot\frac{\partial^2 W^*}{\partial\mathbf{T}\otimes\partial\mathbf{T}}\cdot\widetilde{\mathbf{T}} \geq 0. \end{gathered} \tag{3.3}$$

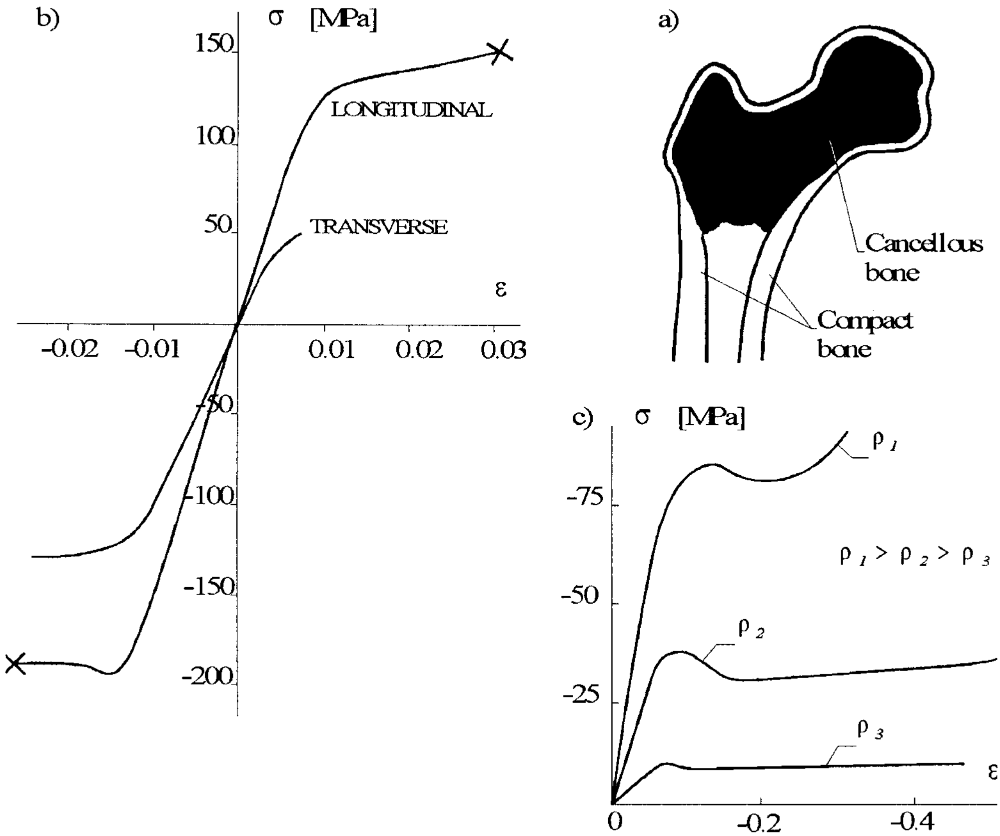

FIG. 2. a. A schematic drawing showing the cancellous (trabecular) bone and the compact (cortical) bone in the head of the human femur. b. Two stress-strain curves for wet compact bone loaded in the longitudinal and transverse directions, after [12]. c. Compressive stress-strain curves for several relative densities ρ_i $(i = 1, 2, 3)$ of wet cancellous bone, after [12].

Here S stands for the anisotropy group of the material considered, $S \subset O(3)$, see (3.5) below. We recall that for isotropic materials $S = O(3)$. Once W is known, the complementary potential is calculated as the Fenchel conjugate:

$$W^*(\mathbf{T}) = \sup\left\{\mathbf{T}\cdot\mathbf{E} - W(\mathbf{E}) \mid \; \mathbf{E} \in \mathsf{E}^3_s\right\}. \tag{3.4}$$

In this case W may be only piecewise regular and $(3.3)_3$ is no longer valid in the whole space. The specific energy is differentiable only once for materials with different properties in tension and compression. Biomaterials like bones are of such a type. The presence of microscopic damage also influences the macroscopic response of bones. We hope to study the problem in the future.

Here

$$S \equiv \{\mathbf{Q} \in 0(3) | \ \mathbf{Q}\,\mathbf{H}\,\mathbf{Q}^T = \mathbf{H}\}. \tag{3.5}$$

If the eigenvalues of the tensor $\mathbf{H}$ are different then a constitutive equation of the type (3.1) has the form

$$\mathbf{T} = \alpha_1\mathbf{I} + \alpha_2\mathbf{H} + \alpha_3\mathbf{H}^2 + 2\alpha_4\mathbf{E} + \alpha_5(\mathbf{E}\,\mathbf{H} + \mathbf{H}\,\mathbf{E}) + \alpha_6(\mathbf{E}\,\mathbf{H}^2 + \mathbf{H}^2\mathbf{E}) + 3\alpha_7\mathbf{E}^2, \tag{3.6}$$

where

$$\alpha_m = \frac{\partial f}{\partial I_m}, \qquad \frac{\partial \alpha_m}{\partial I_n} = \frac{\partial \alpha_n}{\partial I_m}, \qquad m, n = 1, \ldots, 7, \tag{3.7}$$

and, in turn

$$W(\mathbf{E}) = f(I_m(\mathbf{E})) = f\left(\operatorname{tr}\mathbf{E},\ \operatorname{tr}\mathbf{E}\,\mathbf{H},\ \operatorname{tr}\mathbf{E}\,\mathbf{H}^2,\ \operatorname{tr}\mathbf{E}^2,\ \operatorname{tr}\mathbf{E}^2\mathbf{H},\ \operatorname{tr}\mathbf{E}^2\mathbf{H}^2,\ \operatorname{tr}\mathbf{E}^3\right). \tag{3.8}$$

For an inhomogeneous material W and W^* depend explicity on $\mathbf{x} \in \Omega$ since $\mathbf{H}$ depends on x, where $\overline{\Omega}$ denotes the closure of a domain occupied by the body considered in its undeformed configuration.

We observe that the above physically nonlinear constitutive relationships are not identical with the equations proposed in the papers [5, 8, 9, 32, 39, 42]. According to our approach, the structural tensor $\mathbf{H}$ is not an argument of the function (3.2) Consequently, the material functions α_m appearing in (3.6) do not depend explicitly on three invariants of $\mathbf{H}$. In Eq. (3.6) the tensor $\mathbf{H}$ describes only the microstructure of the material. Experimental data justify the assumption of small elastic deformations for bones, cf. [25]. Those deformations are of the order of 1%. The tensor $\mathbf{H}$ could be treated as an argument of the function (3.2) provided that elastic deformations would lead to a significant change of this tensor. Change of $\mathbf{H}$ with time would require application of viscoelastic or elasto-viscoplastic constitutive relationships completed with an evolution equation for this tensor. Such an approach would enable us to describe quantitatively the phenomenon of bone adaptation, which is out of scope of the present contribution. The reader is referred to [7, 8, 11] for more details on adaptation of bones to the loading. As we already know, bone is an inhomogeneous material. It means that both the coefficients α_m $(m = 1, ..., 7)$ and the tensor $\mathbf{H}$ depend on the point in the bone. Consequently, they depend in an explicit manner on, for instance, the density of bone at this point.

Linearization of Eq. (3.6) with respect to $\mathbf{E}$ leads to the equation with the following functions α_m:

$$\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{12} & a_{22} & a_{23}\\ a_{13} & a_{23} & a_{33}\end{bmatrix}\begin{bmatrix}\operatorname{tr}\mathbf{E}\\ \operatorname{tr}\mathbf{E}\mathbf{H}\\ \operatorname{tr}\mathbf{E}\mathbf{H}^2\end{bmatrix}, \tag{3.9}$$

$$\alpha_4 = a_{44}, \qquad \alpha_5 = a_{55}, \qquad \alpha_6 = a_{66}, \qquad \alpha_7 = 0,$$

where $a_{ij} = a_{ji}$ $(i,j = 1,2,3)$, a_{44}, a_{55} and a_{66} are coefficients.

A matrix form of Hooke's law for orthotropic materials is the following:

$$\mathbf{T}_{6\times 1} = \mathbf{C}_{6\times 6}\mathbf{E}_{6\times 1}, \tag{3.10}$$

where

$$\mathbf{C}_{6\times 6} = \begin{bmatrix}\mathbf{A}_{3\times 3} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_{3\times 3}\end{bmatrix} = \begin{bmatrix} e_1 & f_3 & f_2 & 0 & 0 & 0\\ f_3 & e_2 & f_1 & 0 & 0 & 0\\ f_2 & f_1 & e_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 2g_3 & 0 & 0\\ 0 & 0 & 0 & 0 & 2g_2 & 0\\ 0 & 0 & 0 & 0 & 0 & 2g_1\end{bmatrix},$$

$$\mathbf{T}_{6\times 1} = \begin{bmatrix} T_{11}\\ T_{22}\\ T_{33}\\ \sqrt{2}T_{12}\\ \sqrt{2}T_{23}\\ \sqrt{2}T_{13}\end{bmatrix}, \qquad \mathbf{E}_{6\times 1} = \begin{bmatrix} E_{11}\\ E_{22}\\ E_{33}\\ \sqrt{2}E_{12}\\ \sqrt{2}E_{23}\\ \sqrt{2}E_{13}\end{bmatrix}. \tag{3.11}$$

The formulas (3.10) and (3.11) mean that classical Hooke's law: $\mathbf{T} = \mathbb{C}\cdot\mathbf{E}$ is written in the normalized basis

$$\mathbf{J}_K, \qquad K = 1,...,6,$$

where

$$\mathbf{J}_1 = \mathbf{e}_1\otimes\mathbf{e}_1, \qquad \mathbf{J}_2 = \mathbf{e}_2\otimes\mathbf{e}_2, \qquad \mathbf{J}_3 = \mathbf{e}_3\otimes\mathbf{e}_3,$$

$$\mathbf{J}_4 = \frac{1}{\sqrt{2}}(\mathbf{e}_2\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_2), \qquad \mathbf{J}_5 = \frac{1}{\sqrt{2}}(\mathbf{e}_1\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_1),$$

$$\mathbf{J}_6 = \frac{1}{\sqrt{2}}(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1).$$

The matrix $\mathbf{C}_{6\times 6}$ is obviously the representation of the tensor $\mathbb{C}$ in the basis $\mathbf{J}_K\otimes\mathbf{J}_L$, $K,L = 1,...,6$.

Nine elasticity constants e_i, f_i, g_i $(i = 1, 2, 3)$ depend on the constants a_{ij}, a_{44}, a_{55}, a_{66} and the eigenvalues of $\mathbf{H}$ in the following manner:

$$
\begin{aligned}
& e_i = a_{11}+2a_{44}+2H_i(a_{12}+a_{55})+H_i^2[a_{22}+2(a_{13}+a_{66})]+2a_{23}H_i^3+a_{33}H_i^4\,,\\
& f_i = a_{11} + a_{12}(H_j + H_k) + H_jH_k[a_{22} + a_{23}(H_j + H_k) + a_{33}H_jH_k]\\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + a_{13}(H_j^2 + H_k^2),\\
& g_i = a_{44} + \frac{1}{2}a_{55}(H_j + H_k) + \frac{1}{2}a_{66}(H_j^2 + H_k^2),\\
& (i, j, k) = (1, 2, 3);\ (2, 3, 1);\ (3, 1, 2).
\end{aligned}
\tag{3.12}
$$

We observe that the constants a_{11} and a_{44} are not associated with the eigenvalues of $\mathbf{H}$. For $\mathbf{H} = \mathbf{0}$, a_{11} and a_{44} are the so-called Lamé's constants of an isotropic material. It can easily be verified that if two of the eigenvalues of $\mathbf{H}$ are equal then the matrix specified by $(3.11)_1$ contains five independent constants (the transverse isotropy). Further, if all of the eigenvalues coincide then only two constants are independent (the isotropy). In case of orthotropy, six different eigenvalues of the matrix $(3.11)_1$ define six Kelvin's moduli, cf. [1, 28–31]. The remaining three nondimensional constants, the so-called stiffness distributors, determine the tensorial basis in which the matrix $(3.11)_1$ is diagonal. Kelvin's moduli are obviously the invariants of the stiffness tensor in the Hooke law. In the paper [31] it has been shown how to define the measure of the degree of orthotropy of a material provided that the spectral decomposition of the stiffness tensor is available.

If the principal axes of orthotropy are known, determination of Kelvin's moduli is easy since they have the following form:

$$
\lambda_i = A_i\,, \qquad \lambda_{i+3} = 2g_i\,, \qquad i = 1, 2, 3. \tag{3.13}
$$

Here A_i $(i = 1, 2, 3)$ are the eigenvalues of the matrix $\mathbf{A}_{3\times3}$ and g_i are Kirchhoff's moduli. To determine the ordered eigenvalues of this matrix, we apply the formulas (2.9)–(2.11), where $\mathbf{M}$ is to be replaced by $\mathbf{A}_{3\times3}$. The ellipticity condition $(3.3)_3$ reduces then to simple inequalities: $\lambda_K \geq 0$, $K = 1, ..., 6$. Next, to find the stiffness distributors, a standard procedure of linear algebra is used. More precisely, from eigenversors of the matrix $\mathbf{A}_{3\times3}$ an orthogonal matrix $\mathbf{R}_{3\times3}$ is constructed. The diagonal form of $\mathbf{A}_{3\times3}$ is then given by

$$
\mathbf{R}_{3\times3}\mathbf{A}_{3\times3}\mathbf{R}_{3\times3}^T = \mathrm{diag}[A_1, A_2, A_3]. \tag{3.14}
$$

By using the invariant, cf. [14]

$$
\cos\phi = \frac{1}{2}(\mathrm{tr}\,\mathbf{R}_{3\times3} - \det\mathbf{R}_{3\times3}), \qquad \phi \in (0, \pi), \tag{3.15}
$$

one can represent the elements of $\mathbf{R}_{3\times 3}$ in the following form:

$$R_{ij} = (\det \mathbf{R}_{3\times 3})[r_i r_j + (\cos\phi)(\delta_{ij} - r_i r_j) - (\sin\phi)\epsilon_{ijk} r_k], \tag{3.16}$$

where

$$r_i = \frac{\epsilon_{ijk} R_{jk}}{2\sin\phi}. \tag{3.17}$$

Here ϵ_{ijk} are components of Ricci's permutation symbol: $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$, the remaining components being equal to zero. Note that r_i are components of the unit vector of the rotation axis by angle ϕ in the basis $\mathbf{J}_i$ $(i = 1, 2, 3)$. These are not rotations or rotations with reflexions of the basis $\mathbf{e}_i$. For instance, when spherical coordinates are used, then r_i can be expressed as functions of two angular parameters, say γ and η:

$$\begin{aligned} \mathbf{r}^T_{3\times 1} &= \frac{1}{2\sin\phi}[R_{23} - R_{32},\ R_{31} - R_{13},\ R_{12} - R_{21}] \\ &= [\sin\gamma\cos\eta,\ \sin\gamma\sin\eta,\ \cos\eta], \end{aligned} \tag{3.18}$$

where $\gamma \in (0, \pi)$, $\eta \in (0, 2\pi)$. Consequently, apart from six Kelvin's moduli the angles ϕ, γ and η uniquely determine the elastic moduli of an orthotropic material. The above procedure has been applied to the determination of the angles ϕ, γ and η. We observe that eigenversors are not determined in a unique manner: if $\mathbf{y}$ is an eigenversor associated with an engenvalue λ then $-\mathbf{y}$ is also an eigenversor. Spectral decomposition of the matrix $\mathbf{A}_{3\times 3}$ (and consequently, of the tensor $\mathbb{C}$) is, however, unique since matrix representations of eigentensors of the form:

$$\begin{aligned} \mathbf{P}^{(1)}_{3\times 3} &= \mathbf{R}^T_{3\times 3}\mathrm{diag}[1, 0, 0]\mathbf{R}_{3\times 3}, \\ \mathbf{P}^{(2)}_{3\times 3} &= \mathbf{R}^T_{3\times 3}\mathrm{diag}[0, 1, 0]\mathbf{R}_{3\times 3}, \\ \mathbf{P}^{(3)}_{3\times 3} &= \mathbf{R}^T_{3\times 3}\mathrm{diag}[0, 0, 1]\mathbf{R}_{3\times 3}, \end{aligned} \tag{3.19}$$

associated with ordered eigenvalues are uniquely determined. The eigentensors with matrix representations (3.19) in the basis $\mathbf{J}_i \otimes \mathbf{J}_j$ $(i, j = 1, 2, 3)$ can be determined directly from the formulae (2.14) and (2.15), provided that $\mathbf{M}$ and its invariants are replaced by $A_{ij}\mathbf{J}_i \otimes \mathbf{J}_j$ and its invariants, where A_{ij} are elements of the matrix $\mathbf{A}_{3\times 3}$.

We observe that stiffness distributors are not necessarily given by the angles ϕ, γ and η. Three independent nondimensional parameters, which uniquely determine the representation of eigentensors (3.19), are likewise acceptable. For instance, Euler's angles are possible candidates for such parameters.

In case of orthotropic materials, the spectral decomposition of $\mathbb{C}$ is given by

$$\mathbb{C} = \lambda_1 \mathbb{P} + \ldots + \lambda_6 \mathbb{P}_6 , \tag{3.20}$$

where

$$\mathbb{P}_i P^{(i)}_{kl} \mathbf{J}_k \otimes \mathbf{J}_l , \qquad \mathbb{P}_{i+3} = \mathbf{J}_{i+3} \otimes \mathbf{J}_{i+3} , \qquad i,k,l = 1,2,3 \quad \text{(no summation over } i). \tag{3.21}$$

Here $P^{(i)}_{kl}$ are elements of the matrix $\mathbf{P}^{(i)}_{3\times 3}$ and are defined by (3.19).

As far as practical applications are concerned, an inverse of Eq. (3.10) is necessary ($\mathbf{e} = \mathbb{C}^{-1}\colon \mathbf{T}$, $\mathbb{C}^{-1} = \lambda_1^{-1}\mathbb{P}_1 + \ldots \lambda_6^{-1}\mathbb{P}_6$):

$$\mathbf{E}_{6\times 1} = \mathbf{C}^{-1}_{6\times 6} \mathbf{T}_{6\times 1} , \tag{3.22}$$

where

$$\mathbf{C}^{-1}_{6\times 6} = \begin{bmatrix} p_1 & r_3 & r_2 & 0 & 0 & 0 \\ r_3 & p_2 & r_1 & 0 & 0 & 0 \\ r_2 & r_1 & p_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}s_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}s_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}s_1 \end{bmatrix} \tag{3.23}$$

and, in turn

$$\begin{aligned} & dp_i = e_j e_k - f_i^2 , \qquad dr_i = f_j f_k - e_i f_i \quad \text{(no summation over } i), \\ & s_i = \frac{1}{g_i} , \qquad d = e_1 e_2 e_3 + 2 f_1 f_2 f_3 - e_1 f_1^2 - e_2 f_2^2 - e_3 f_3^2, \\ & (i,j,k) = (1,2,3);\ (2,3,1);\ (3,1,2). \end{aligned} \tag{3.24}$$

The constants p_i, r_i, s_i ($i = 1,2,3$) can be determined from standard tests performed on an orthotropic material.

Obviously, in order to obtain (3.22) we can directly apply $(3.1)_2$, thus arriving at

$$\mathbf{E} = \beta_1 \mathbf{I} + \beta_2 \mathbf{H} + \beta_3 \mathbf{H}^2 + 2\beta_4 \mathbf{T} + \beta_5 (\mathbf{T}\,\mathbf{H} + \mathbf{H}\,\mathbf{T}) + \beta_6 (\mathbf{T}\,\mathbf{H}^2 + \mathbf{H}^2 \mathbf{T}) + 3\beta_7 \mathbf{T}^2, \tag{3.25}$$

where

$$\beta_m = \frac{\partial g}{\partial J_m} , \qquad \frac{\partial \beta_m}{\partial J_n} = \frac{\partial \beta_n}{\partial J_m} , \qquad m,n = 1,\ldots,7, \tag{3.26}$$

and, in turn

$$W^*(\mathbf{T}) = g(J_m(\mathbf{T})) = g\left(\operatorname{tr}\mathbf{T},\ \operatorname{tr}\mathbf{T}\mathbf{H},\ \operatorname{tr}\mathbf{T}\mathbf{H}^2,\ \operatorname{tr}\mathbf{T}^2,\ \operatorname{tr}\mathbf{T}^2\mathbf{H},\ \operatorname{tr}\mathbf{T}^2\mathbf{H}^2,\ \operatorname{tr}\mathbf{T}^3\right). \tag{3.27}$$

Linearization of Eq. (3.25) under T leads to the equation with the following functions β_m:

$$\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13}\\ b_{12} & b_{22} & b_{23}\\ b_{13} & b_{23} & b_{33}\end{bmatrix} \begin{bmatrix}\operatorname{tr}\mathbf{T}\\ \operatorname{tr}\mathbf{T}\mathbf{H}\\ \operatorname{tr}\mathbf{T}\mathbf{H}^2\end{bmatrix}, \tag{3.28}$$
$$\beta_4 = b_{44}, \qquad \beta_5 = b_{55}, \qquad \beta_6 = b_{66}, \qquad \beta_7 = 0,$$

where $b_{ij} = b_{ji}$ $(i,j = 1,2,3)$, b_{44}, b_{55} and b_{66} are coefficients. The constitutive relationship (3.25) combined with (3.28) is more convenient for experimental verification since tests are usually carried out for a given loading. Relations between the coefficients (3.28) and the classical orthotropic constants p_i, r_i and s_i are analogous to (3.12). In the procedure just outlined we do not explicitly exploit the fact that (3.25) is inverse to (3.6). The coefficients p_i, r_i and s_i have a clear mechanical interpretation. We observe that a search for a simple relation, for instance between Young's moduli and the eigenvalues of **H** is not justified, cf. [37]. In fact, Eqs. (3.12), (3.24), (3.29) – (3.31) imply the relation between the classical coefficients of an orthotropic material and the tensor **H**.

Following Hayes' paper [16], the following relations can be established:

(i) generalized Young's moduli for an arbitrary direction **n**

$$\frac{1}{E(\mathbf{n})} = p_1 n_1^4 + p_2 n_2^4 + p_3 n_3^4 + 2(r_1 + 2s_3)n_2^2 n_3^2 + 2(r_2 + 2s_2)n_1^2 n_3^2 + 2(r_3 + 2s_1)n_1^2 n_2^2, \tag{3.29}$$

where **n** is an arbitrary versor with the components n_i;

(ii) generalized Poisson's ratios for an arbitrary plane (for a pair of orthogonal directions **n**, **m**)

$$\begin{aligned}-\frac{\nu(\mathbf{m},\mathbf{n})}{E(\mathbf{n})} = -\frac{\nu(\mathbf{n},\mathbf{m})}{E(\mathbf{m})} &= p_1 m_1^2 n_1^2 + p_2 m_2^2 n_2^2 + p_3 m_3^2 n_3^2 \\ &+ r_1(m_2^2 n_3^2 + m_3^2 n_2^2) + r_2(m_1^2 n_3^2 + m_3^2 n_1^2) + r_3(m_1^2 n_2^2 + m_2^2 n_1^2) \\ &+ 4s_1 m_1 m_2 n_1 n_2 + 4s_2 m_1 m_3 n_1 n_3 + 4s_3 m_2 m_3 n_2 n_3,\end{aligned} \tag{3.30}$$

where **m** is a versor with the components m_i;

(iii) generalized Kirchhoff moduli for an arbitrary plane (for a pair of orthogonal directions **n**, **m**)

$$(3.31)\qquad \frac{1}{G(\mathbf{m},\mathbf{n})} = \frac{1}{G(\mathbf{n},\mathbf{m})} = 4\Big[p_1 n_1^2 m_1^2 + p_2 n_2^2 m_2^2 + p_3 n_3^2 m_3^2 + 2r_1 m_2 m_3 n_2 n_3 + 2r_2 m_1 m_3 n_1 n_3 + 2r_3 m_1 m_2 n_1 n_2 + s_1(n_2 m_3 + m_2 n_3)^2 + s_2(n_1 m_3 + m_1 n_3)^2 + s_3(n_1 m_2 + m_1 n_2)^2\Big].$$

In Table 1 are presented averaged experimental data of the so-called technical elastic constants, which were obtained by an ultrasonic method, cf. [38].

Table 1. Technical constants (ultrasonic technique, after [38]).

Technical constants (average)	human femoral cortical bone	human cancellous bone (proximal tibia)
E_1	11.7 (1.6) [GPa]	237 (63) [MPa]
E_2	13.2 (1.8) [GPa]	309 (93) [MPa]
E_3	19.8 (2.4) [GPa]	823 (337) [MPa]
G_{12}	4.53 (0.37) [GPa]	73 (38) [MPa]
G_{13}	5.61 (0.4) [GPa]	112 (48) [MPa]
G_{23}	6.23 (0.48) [GPa]	134 (49) [MPa]
ν_{12}	0.375 (0.095)	0.169 (0.304)
ν_{21}	0.416 (0.118)	0.209 (0.209)
ν_{23}	0.237 (0.083)	0.063 (0.217)
ν_{32}	0.346 (0.096)	0.245 (0.626)
ν_{13}	0.374 (0.108)	0.423 (0.356)
ν_{31}	0.234 (0.088)	0.145 (0.123)

Average technical constants for 60 specimens of human femoral cortical bone, where the 1-direction is radial, the 2-direction is circumferential and the 3-direction is longitudinal. Average technical constants for 9 specimens of human cancellous bone from the proximal tibia, where the 1-direction is anterior-posterior, the 2-direction is medial-lateral and the 3-direction is longitudinal. The numbers in parentheses are the standard deviations.

From Eqs. (3.29) – (3.31) the off-axis technical elastic constants in a plane of an orthotropic, linearly elastic material can be represented as a function of off-axis angle by the following equations, cf. Figs. 3–5

$$(3.32)\qquad \begin{aligned} \frac{1}{E(\varphi)} &= \frac{1}{E_1}\cos^4\varphi + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)\sin^2\varphi\cos^2\varphi + \frac{1}{E_2}\sin^4\varphi, \\ \frac{1}{G(\varphi)} &= 2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^2\varphi\cos^2\varphi + \frac{1}{G_{12}}(\sin^4\varphi + \cos^4\varphi), \\ \nu(\varphi) &= E(\varphi)\left[\frac{\nu_{12}}{E_1}(\sin^4\varphi + \cos^4\varphi) - \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}}\right)\sin^2\varphi\cos^2\varphi\right], \end{aligned}$$

where φ is referred to the 1-direction.

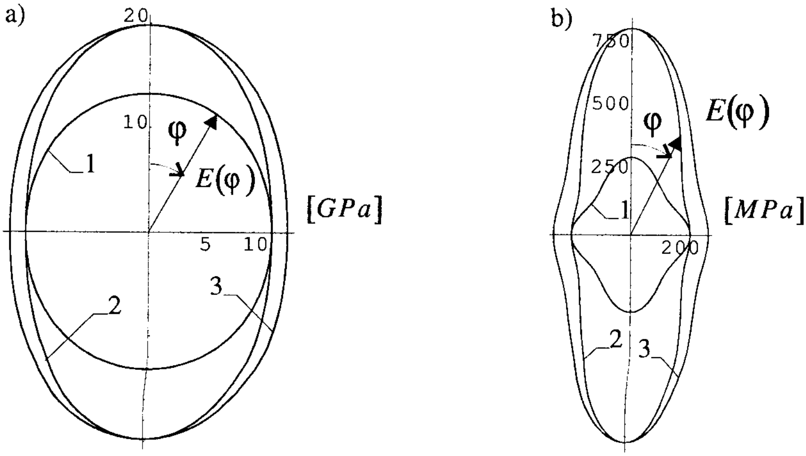

FIG. 3. The off-axis Young's moduli as a function of off-axis angle Eq. $(3.32)_1$; 1-plane 1-2, 2-plane 1-3, 3-plane 2-3; a) human femoral cortical bone, b) human cancellous bone from the proximal tibia.

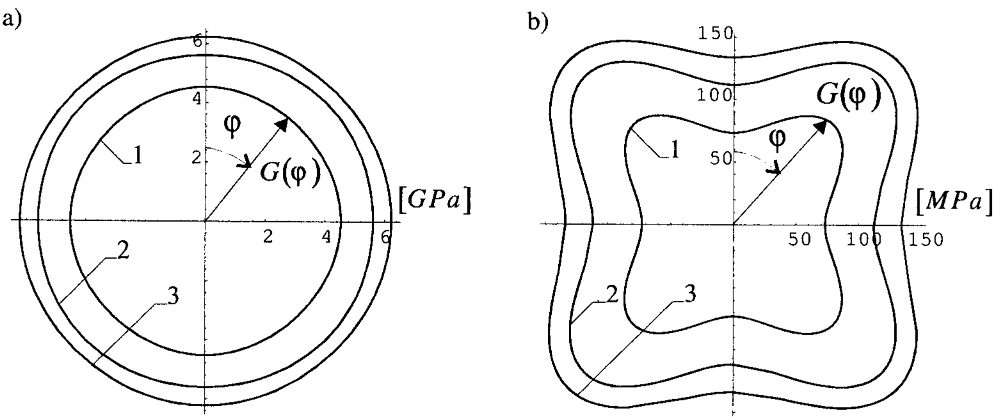

FIG. 4. The off-axis shear moduli Eq.$(3.32)_2$; 1-plane 1-2, 2-plane 1-3, 3-plane 2-3; a) human femoral cortical bone, b) human cancellous bone from the proximal tibia.

Similar relationships hold true for the 1-3 plane and 2-3 plane. From Table 1 and Figs. 3, 4 and 5 it follows that the human femoral cortical bone may be treated approximately as a transversely isotropic material. On the other hand, such an approximation would not be justified for the cancellous bone. Comparing the standard deviations we conclude that the cancellous bone is considerably more inhomogeneous than the cortical bone. In our opinion the tests performed by TURNER *et al.* [39] should additionally be completed by the determination of eigenvectors of **H**. The direction of orthotropy would then be determined more precisely. We note that the data provided in Table 1 have been obtained under the assumption that the directions of orthotropy of all samples are the same.

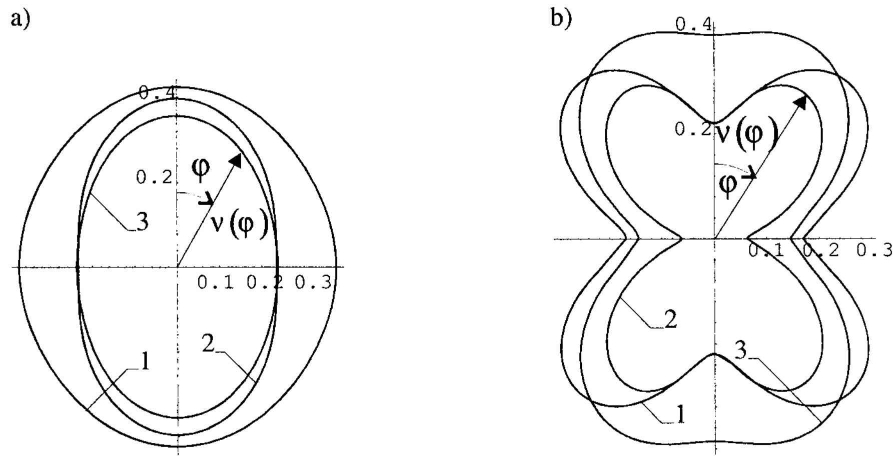

FIG. 5. The off-axis Poisson's ratios Eq. $(3.32)_3$; 1-plane 1-2, 2-plane 1-3, 3-plane 2-3; a) human femoral cortical bone, b) human cancellous bone from the proximal tibia.

Table 2 contains Kelvin's moduli and parameters determining stiffness distributors for the analyzed cortical and cancellous bones, and it confirms our earlier assertions.

Table 2. Kelvin's moduli and parameters determining stiffness distributors for the analyzed cortical and cancellous bones.

	human femoral cortical bone	human cancellous bone (proximal tibia)
λ_1	46.239 [GPa]	1190.75 [MPa]
λ_2	13.-15 [GPa]	324.80 [MPa]
λ_3	8.828 [GPa]	210.43 [MPa]
λ_4	11.22 [GPa]	224 [MPa]
λ_5	12.46 [GPa]	268 [MPa]
λ_6	9.06 [GPa]	146 [MPa]
ϕ	0.470π	0.399π
γ	0.477π	0.416π
η	0.267π	1.484π
$\det \mathbf{R}_{3\times 3}$	1	-1

In case of transverse isotropy, one has four independent Kelvin's moduli and only one angular parameter defining the stiffness distributor, cf. [1, 28–30].

Closer inspection of Tables 1 and 2, account being taken of the standard deviations given in round brackets in Table 1, leads to conclusion that the cortical bone may be regarded as a transversely isotropic material since the appropriately underlined values of Kelvin's moduli are practically identical while the angles ϕ and γ only insignificantly differ from $\pi/2$.

The spectral decomposition of stiffness tensors of cortical and cancellous bones are obtained by applying the appropriate formulae derived earlier and the data provided in Table 2.

4. Elastic-perfectly plastic model

Let us denote by $\dot{\mathbf{E}}_e$, $\dot{\mathbf{E}}_p$ the elastic and plastic part of the strain rate tensor. As usual, we assume that

$$\dot{\mathbf{E}} = \dot{\mathbf{E}}_e + \dot{\mathbf{E}}_p \tag{4.1}$$

and construct the constitutive relationships for elastic perfectly-plastic materials. The elastic behaviour is described by $\dot{\mathbf{E}}_{6\times1} = \mathbf{C}^{-1}_{6\times6}\dot{\mathbf{T}}_{6\times1}$. General form of the yield function is assumed in the form

$$G(\mathbf{T}) = F(J_m(\mathbf{T})) = F(\operatorname{tr}\mathbf{T},\ \operatorname{tr}\mathbf{T}\mathbf{H},\ \operatorname{tr}\mathbf{T}\mathbf{H}^2,\ \operatorname{tr}\mathbf{T}^2,\ \operatorname{tr}\mathbf{T}^2\mathbf{H},\ \operatorname{tr}\mathbf{T}^2\mathbf{H}^2,\ \operatorname{tr}\mathbf{T}^3) \tag{4.2}$$

while the yield condition is given by

$$G(\mathbf{T}) - 1 = 0. \tag{4.3}$$

The associated flow rule assumes the form

$$\dot{\mathbf{E}}_p = \lambda\frac{\partial G}{\partial \mathbf{T}}, \qquad \lambda \geq 0. \tag{4.4}$$

Since the cancellous bone reveals different plastic behaviour in tension and compression (cf. [7, 12, 22]), therefore we propose to assume Hoffman's criterion [18], cf. also [4, 20, 33–35]. Written in an invariant form, this criterion is expressed by [19]

$$c_1(K_2 - K_3)^2 + c_2(K_3 - K_1)^2 + c_3(K_1 - K_3)^2 + 2c_4K_6 + 2c_5K_5 + 2c_6K_4 + c_7K_1 + c_8K_2 + c_9K_3 - 1 = 0, \tag{4.5}$$

where

$$\begin{aligned}
c_1 &= \frac{1}{2}\left(\frac{1}{Y_{t2}Y_{c2}} + \frac{1}{Y_{t3}Y_{c3}} - \frac{1}{Y_{t1}Y_{c1}}\right),\\
c_2 &= \frac{1}{2}\left(\frac{1}{Y_{t3}Y_{c3}} + \frac{1}{Y_{t1}Y_{c1}} - \frac{1}{Y_{t2}Y_{c2}}\right),\\
c_3 &= \frac{1}{2}\left(\frac{1}{Y_{t1}Y_{c1}} + \frac{1}{Y_{t2}Y_{c2}} - \frac{1}{Y_{t3}Y_{c3}}\right),\\
2c_4 &= \frac{1}{k_{23}^2}, \qquad 2c_5 = \frac{1}{k_{13}^2}, \qquad 2c_6 = \frac{1}{k_{12}^2},\\
c_7 &= \frac{Y_{c1} - Y_{t1}}{Y_{c1}Y_{t1}}, \qquad c_8 = \frac{Y_{c2} - Y_{t2}}{Y_{c2}Y_{t2}}, \qquad c_9 = \frac{Y_{c3} - Y_{t3}}{Y_{c3}Y_{t3}}.
\end{aligned} \tag{4.6}$$

Here Y_{ci}, Y_{ti} and k_{ij} are the yield limit in compression and tension in the directions of orthotropy and the yield limit in shear in the principal planes of orthotropy, respectively. The invariants K_p ($p = 1, ..., 6$) are given by

$$\begin{aligned} K_1 &= \operatorname{tr}\mathbf{M}_1\mathbf{T}, \qquad K_2 = \operatorname{tr}\mathbf{M}_2\mathbf{T}, \qquad K_3 = \operatorname{tr}\mathbf{M}_3\mathbf{T}, \\ K_4 &= \frac{1}{2}\Big[(\operatorname{tr}\mathbf{M}_3\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_1\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_2\mathbf{T})^2 - \operatorname{tr}\mathbf{M}_1\mathbf{T}^2 \\ &\qquad + \operatorname{tr}\mathbf{M}_1\mathbf{T}^2 + \operatorname{tr}\mathbf{M}_2\mathbf{T}^2\Big], \\ K_5 &= \frac{1}{2}\Big[(\operatorname{tr}\mathbf{M}_2\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_1\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_3\mathbf{T})^2 - \operatorname{tr}\mathbf{M}_2\mathbf{T}^2 \\ &\qquad + \operatorname{tr}\mathbf{M}_1\mathbf{T}^2 + \operatorname{tr}\mathbf{M}_3\mathbf{T}^2\Big], \\ K_6 &= \frac{1}{2}\Big[(\operatorname{tr}\mathbf{M}_1\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_2\mathbf{T})^2 - (\operatorname{tr}\mathbf{M}_3\mathbf{T})^2 - \operatorname{tr}\mathbf{M}_1\mathbf{T}^2 \\ &\qquad + \operatorname{tr}\mathbf{M}_2\mathbf{T}^2 + \operatorname{tr}\mathbf{M}_3\mathbf{T}^2\Big]. \end{aligned} \tag{4.7}$$

The tensors $\mathbf{M}_j = \mathbf{i}_j \otimes \mathbf{i}_j$ (no summation over j) are the eigentensors of $\mathbf{H}$, cf. (2.14). By using the following relation

$$\begin{bmatrix} \operatorname{tr}\mathbf{T}^\alpha \\ \operatorname{tr}\mathbf{H}\,\mathbf{T}^\alpha \\ \operatorname{tr}\mathbf{H}^2\mathbf{T}^\alpha \end{bmatrix} = \mathbf{h}_{3\times 3} \begin{bmatrix} \operatorname{tr}\mathbf{M}_1\mathbf{T}^\alpha \\ \operatorname{tr}\mathbf{M}_2\mathbf{T}^\alpha \\ \operatorname{tr}\mathbf{M}_3\mathbf{T}^\alpha \end{bmatrix}, \qquad \alpha = 1, 2, \tag{4.8}$$

where

$$\mathbf{h}_{3\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ H_1 & H_2 & H_3 \\ H_1^2 & H_2^2 & H_3^2 \end{bmatrix}, \tag{4.9}$$

the criterion (4.5) can be written in the form (4.3).

If

$$\det(\mathbf{h}_{3\times 3}) = (H_2 - H_1)(H_3 - H_1)(H_3 - H_2) \neq 0, \tag{4.10}$$

or the eigenvalues of $\mathbf{H}$ are different then, by using the inverse matrix

$$\mathbf{h}_{3\times 3}^{-1} = \frac{1}{\det(\mathbf{h}_{3\times 3})} \begin{bmatrix} H_2H_3^2 - H_3H_2^2 & H_2^2 - H_3^2 & H_3 - H_2 \\ H_3H_1^2 - H_1H_3^2 & H_3^2 - H_1^2 & H_1 - H_3 \\ H_1H_2^2 - H_2H_1^2 & H_1^2 - H_2^2 & H_2 - H_1 \end{bmatrix}, \tag{4.11}$$

we establish (4.5) as claimed.

For $Y_{ci} = Y_{ti}$ the criterion (4.5) reduces to HILL'S criterion [17], which has also been applied in the bone mechanics, cf. [3, 27].

The canonical form of Hoffman's criterion has been derived in [20], where an alternative interpretation of the coefficients c_i has also been provided. Moreover, on the basis of available data for the compact bone, the applicability of the criterion has been verified.

By using Eq. (4.5) and the transformation formula of tensor components under orthogonal transformations, one can readily derive the formulae for the determination of a sample strength in case of compression and tension, in the direction defined by an angle ϕ, in each of the principal orthotropy planes. Particularly, in the case of the orthotropy plane 1-2 this formula is given by

$$\sigma_{\varphi} = \frac{-B_{\varphi} \pm \sqrt{\Delta_{\varphi}}}{2A_{\varphi}}, \tag{4.12}$$

where

$$\begin{aligned} &\Delta_{\varphi} = (B_{\varphi})^2 + 4A_{\varphi}, \qquad B_{\varphi} = C_7 + (C_8 - C_7)\sin^2\varphi, \\ &A_{\varphi} = C_2 + C_3 + (-4C_3 - 2C_2 + 2C_6)\sin^2\varphi \\ &\qquad\qquad + (4C_3 + C_1 + C_2 - 2C_6)\sin^4\varphi. \end{aligned} \tag{4.13}$$

Here the sign + (−) refers to tension (compression).

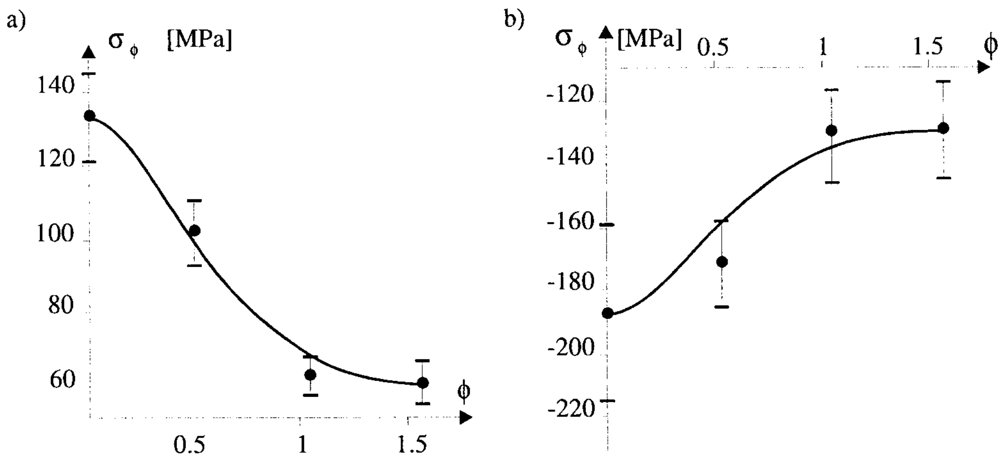

FIG. 6. Strength limit of the human femoral cortical bone in: a) tension, b) compression. The angle φ is taken with respect to the long axis of the bone. • – the experimental data (jointly with error range), after Reilly's thesis (1974), reproduced after [5].

Figure 6 depicts the relation (4.12) for a human cortical bone. As we already know, this bone may be treated as a transversely isotropic material. The constants C_1, C_2, C_5, C_7 and C_8 have been determined by exploiting the experimental data presented in the paper [4]. Table 3 summarizes the data necessary for the calculation, which uses Eqs. (4.12) and (4.13), for this type of bone.

Table 3. Data for the determination of the strength in tension and compression in case of the human femoral cortical bone.

[MPa]	$Y_{t1} = 132$	$Y_{c1} = 187$	$Y_t = 58$	$Y_c = 132$	$k = 67$
[MPa^{-2}]	$C_1 = 1.104 \cdot 10^{-4}$	$C_2 = C_3$ $= 2.026 \cdot 10^{-5}$	$C_7 = 2.23 \cdot 10^{-3}$	$C_8 = C_9$ $= 9.67 \cdot 10^{-3}$	$2C_5 = 2C_6$ $= 2.228 \cdot 10^{-4}$

5. Final remarks

The present paper confirms the usefulness of the concept of fabric tensor in bone mechanics. The available experimental data reveal that bones are anisotropic, in general orthotropic, materials. Their properties depend on the location, i.e. they are inhomogeneous materials. Our consideration have deliberately been confined to elastic and elastic-perfectly plastic modelling of bones. In fact, bone is a porous material with a very complicated hierarchical structure. For instance, one can treat the bone as consisting of piezoelectric skeleton filled with a conductive biofluid (Telega and Wojnar, in preparation). It seems, however, that a specific bone model assumed depends on the problem investigated. We believe that in contact problems of orthopaedic biomechanics, the anisotropic models studied in this paper can provide reliable information on stress distribution in joints after arthroplasty. More precisely, to model the stress distribution in a human joint after arthroplasty, one has to take into account the properties of the bone in the vicinity of a prosthesis. The problem is still discussed in the relevant literature.

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