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STABILITY DERIVATIVES CAST IN THE FRAME OF SUBSONIC UNSTEADY AERODYNAMICS

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On the basis of numerical results and theoretical considerations, a general form of the (unsteady) linear, discretized aerodynamic operator in the Laplace- and in the time-domain, for two- and three-dimensional subsonic flow is proposed. It corresponds to the classical Theodorsen solution for an airfoil in incompressible flow. The model of aerodynamic derivatives uses a polynomial approximation to the transfer functions. There are identified terms, which are neglected in this approach: these are the deficiency function and, in the case of compressible fluid, also the term responsible for the initial pulse. These results clears the limitations and possible improvements of the aerodynamic derivatives model.

1. Introduction

In Flight Dynamics, the aerodynamic loads are usually determined by means of aerodynamic (stability) derivatives. The aerodynamic forces are assumed to be linear functions of generalized coordinates, velocities and their first derivatives (with respect to time). The coefficients in these linear functions are called "aerodynamic derivatives" and are assumed to be constant, independent of parameters which describe the motion. They are usually calculated on the basis of a simple steady, linear aerodynamic model. This approach was first introduced into the engineering practice in 1904 by Bryan [1], and since then is used with great success in the stability analysis of aeroplanes, when the structure may be regarded as rigid and all changes of the state variables are sufficiently slow. The characteristic feature of this model is the assumption, that the aerodynamic forces depend only on instantaneous values of parameters used to describe the motion of the structure.

In the flutter analysis of aircrafts, it is necessary to use more complex aerodynamic models, which take into account the influence of the flow disturbances (determined by the history of motion of the structure) on aerodynamic forces. Although the model of stability derivatives may be regarded as a particular case of the unsteady aerodynamic model, it neglects some factors which may be important if the aerodynamic forces change very rapidly, and also the asymptotic behaviour (for large time after a perturbation) of both aerodynamic models is also different. The aim of this paper is to determine the relations between these models and identify the terms which are neglected in the aerodynamic derivatives model.

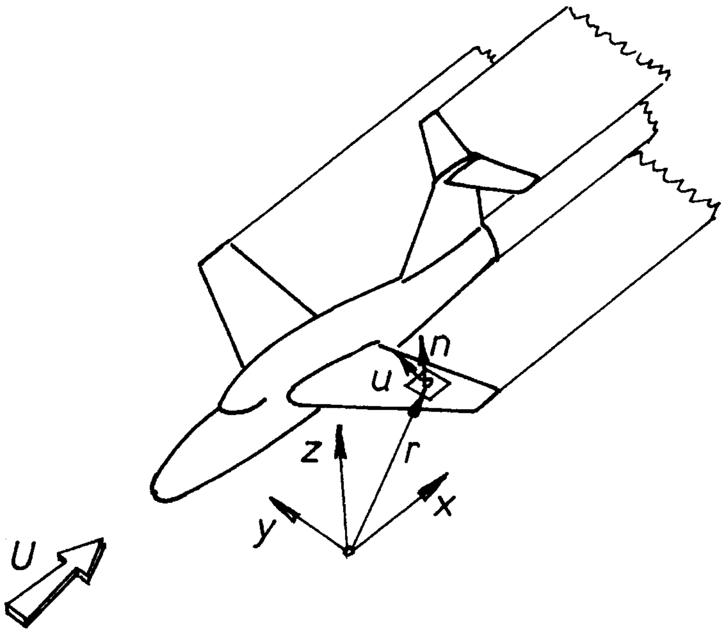

FIG. 1. The aeroplane in rectilinear flight.

Let us consider the problem of determination of aerodynamic forces acting on a flexible aeroplane, undergoing small perturbations from a steady state of flight at constant velocity U (Fig. 1). It is assumed, that the displacements $\mathbf{u}(\mathbf{r},t)$ of the structure, relative to a moving frame of reference x, y, z are small. The n generalized coordinates $q_1, q_2, ..., q_n$ are defined by an expansion of the displacement vector of any point $\mathbf{r}$ of the structure in terms of given modes $\mathbf{\Phi}_i(\mathbf{r})$,

$$\mathbf{u}(\mathbf{r},t) = \sum_{i=1}^{n} \mathbf{\Phi}_i(\mathbf{r})\, q_i(t), \tag{1.1}$$

usually chosen from the set of the natural vibrations modes of the free structure (including the "rigid" modes), which correspond to the lower vibration frequencies. This method of discretization of the structure is commonly used in aeroelasticity, but it may also be used to describe small displacements of a rigid structure. In what follows, all lengths are nondimesionalized with a reference length b (e.g. airfoil semichord), all velocities are nondimensionalized with U as the reference speed, and t is the nondimensional time defined by

$$t = \frac{U \cdot t_{\text{real}}}{b}.$$

Taking into account only the aerodynamic forces caused by pressure distribution on the external surface S of the aircraft body, the generalized forces are defined by the expressions

$$Q_i(t) = \frac{\rho U^2}{2} f_i(t), \tag{1.2}$$

where

$$f_i(t) = \iint_S \mathbf{\Phi}_i(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) c_p(\mathbf{r}, t)\, dS \quad \text{for} \quad i = 1, 2, ..., n,$$

where ρ is the fluid density, $\mathbf{n}(\mathbf{r})$ – outer normal to the body (Fig. 1), and $c_p(\mathbf{r}, t)$ is the pressure coefficient (pressure nondimensionalized by the dynamic pressure $\rho U^2/2$).

The generalized aerodynamic forces are related to the generalized coordinates which describe the motion of the body by means of an aerodynamic operator

$$\{f(t)\} = \mathcal{A}\{q(t)\}, \tag{1.3}$$

where $\{f(t)\}$ and $\{q(t)\}$ are vectors with elements $f_i(t)$ and $q_i(t)$ $(i = 1, 2, ..., n)$.

The aerodynamic derivatives are defined under the assumption, that this operator is linear and has the form

$$\{f(t)\} = [A_0]\{q(t)\} + [A_1]\{\dot{q}(t)\} + [A_2]\{\ddot{q}(t)\}, \tag{1.4}$$

where the superscribed dot indicates the derivative with respect to time. The elements of constant matrices $[A_0]$, $[A_1]$, $[A_2]$ are the aerodynamic (or stability) derivatives. The last term in (1.4) appears only in the case when the fluid is incompressible and the apparent mass effect was taken into account.

The motion of a rigid aircraft can be described by a set of six nonlinear differential equations. In Flight Mechanics [2] this nonlinear system is usually linearized about a prescribed trajectory. The generalized coordinates and velocities are in this case usually defined not on the basis of (1.1), but directly. Also the aerodynamic forces are defined directly, such as lift, drag, sideforce, and moments about body axes. Numerous reference systems are used. The first is the *earth-fixed* reference frame x, y, z. The vehicle position (x and y) and altitude (h) are measured from the origin of this reference system. The *vehicle-carried* vertical axis system (Fig. 2) has its origin at the center of gravity of the aeroplane. This axis system is obtained by a translation of the earth-fixed axis system to the aeroplane center of gravity. The origin of the *body axis system* x_b, y_b, z_b is also the center of gravity. The x_b axis is directed toward the nose of the aeroplane, the y_b axis toward the right wing, and the z_b toward the bottom of the aeroplane. As the generalized coordinates are used three coordinates of the center of gravity $x(t)$, $y(t)$, $h(t)$ and three Euler angles $\phi(t)$, $\theta(t)$, $\psi(t)$ which describe the orientation of body axes with respect to the vehicle-carried axes. The instantaneous velocity field of the rigid body is determined by the velocity of the center of gravity $\mathbf{V}$ and the angular velocity $\mathbf{\Omega}$. The generalized velocities are defined by the column matrix $[V, \alpha, \beta, p, q, r]^T$, where $V = |\mathbf{V}|$, α is the angle of attack, β is the angle of sideslip (Fig. 2), and p, q, r are the roll, pitch, and yaw rates (the projections of $\mathbf{\Omega}$ on the body axes).

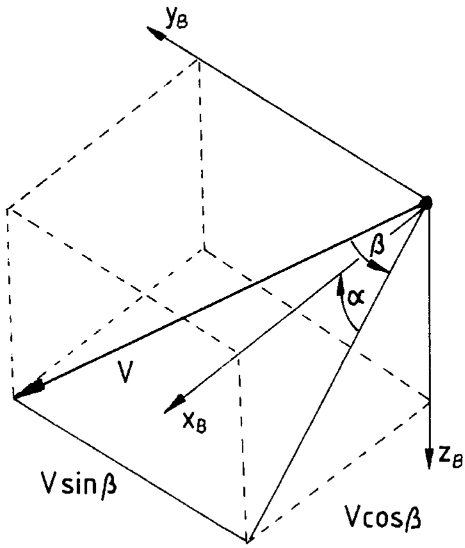

FIG. 2. Coordinates used in Flight Dynamics.

Aerodynamic derivatives defined in Flight Dynamics are based on a linear aerodynamic model, and therefore can be used only in the analysis of a perturbed motion about a given trajectory. From the aerodynamic point of view, they are equivalent to derivatives (1.4), which correspond to a rectilinear trajectory. Therefore the following discussion will be restricted to the model (1.4), where the generalized coordinates and forces are defined in (1.1) and (1.2).

2. THE PHYSICAL BASIS OF A LINEAR UNSTEADY AERODYNAMIC MODEL

The aerodynamic operator relates the aerodynamic forces acting on the body to the motion of the body. The surface of the body constitutes a moving boundary for the flowfield surrounding the structure. To determine the flowfield, an appropriate boundary problem for the governing partial differential equations must be solved, with given initial conditions. It may be assumed, that all disturbances of the flowfield arose by the motion of the body at earlier instants of time. This may be interpreted as a "memory" of the aerodynamic system, which is realised physically by free vortex wakes generated in boundary layers on the body.

The mechanics and modelling of wake generation is shown schematically in Fig. 3. A vibrating airfoil is located in a uniform flow with velocity U. On both sides of the airfoil beginning from the leading edge, grow the boundary layers which, after detaching at the trailing edge, form a free vortex wake. At high Reynolds numbers (typical for aeroplanes are of the order of $\mathrm{Re} = 10^7$ and more), the boundary layers and the vortex wake (at least not far from the airfoil) are relatively thin. In the limit $\mathrm{Re} \to \infty$, the vortex wake transforms in a free surface of discontinuous tangential velocity, and the flow outside may be assumed to be potential. In this simplified model, the viscosity of the fluid plays only a catalytic role and does not appear explicitly in the equations. The pressure across the vortex sheet is continuous and from this boundary condition it follows, that the wake moves with a velocity which is the average of the velocities on both sides of the sheet. The boundary condition on the airfoil takes the same form as in an inviscous flow. In addition, the Kutta-Joukowski condition (which states that the pressure at the trailing edge is continuous) must be satisfied.

The linearization of the aerodynamic model means not only the linearization of the governing equations, but also the linearization of boundary conditions in the wake (including the trailing edge, where it transforms into the Kutta-Joukowski condition), and on the surface of the body. For thin airfoils and low Mach numbers, the linearization about the main, uniform flow with velocity U is possible (Fig. 3). It follows, that the wake is rectilinear and is transported with velocity U. In the context of small perturbations theory, the boundary conditions on the moving surface $h(x,t)$ are stated on a chord parallel to the main flow

$$w(x,t) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t}\,. \tag{2.1}$$

Contrary to the common conviction, the linear model may be often used with success also in the transonic range. However, when on the airfoil surface a shock wave is formed, the linearization should be made about the steady flow over the body. Although the governing field equations are inherently nonlinear, the aerodynamic forces and shock motion are linear in the (sufficiently small) changes of the angle of attack. The range in which this linear behaviour occurs, increases

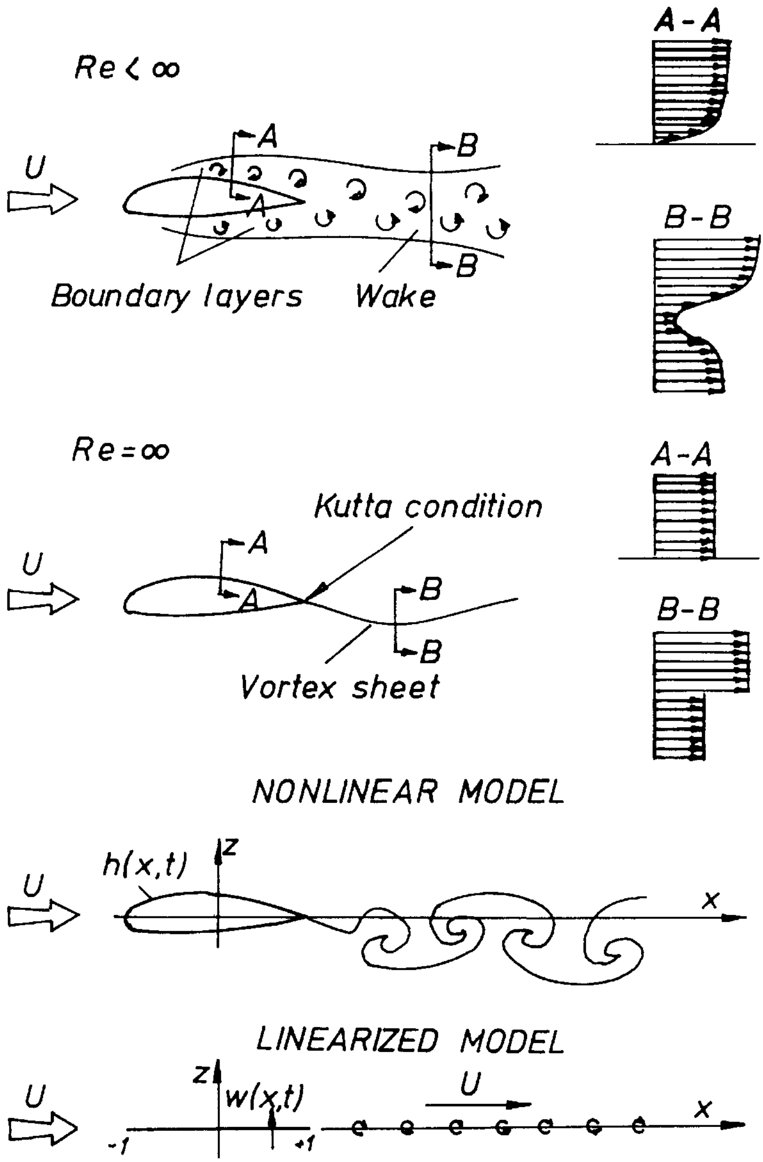

FIG. 3. The linearized unsteady aerodynamic model.

with the frequency coefficient [3]. When the shock reaches the trailing edge, the linear region is again unbounded (when the influence of viscosity is neglected).

The described aerodynamic modelling scheme is not restricted to two-dimensional problems. It enables also the determination of pressure coefficient distribution $c_p(\mathbf{r}, t)$ over a three-dimensional body (Fig. 1) in terms of the normal (to the surface) component of fluid velocity distribution $v_n(\mathbf{r}, \tau)$, fixed by the body's history of motion for $-\infty < \tau \le t$. The whole process of determination of aerodynamic forces consists of the following steps:

$$\{q(\tau)\} \Rightarrow v_n(\mathbf{r},\tau) \Rightarrow c_p(\mathbf{r},t) \Rightarrow \{f(t)\}, \qquad \text{where} \quad -\infty < \tau \le t. \tag{2.2}$$

This scheme describes also the internal structure of the aerodynamic operator $\mathcal{A}$.

3. The general form of a linear aerodynamic operator

The aerodynamic operator $\mathcal{A}$ is a correspondence that assigns the generalized aerodynamic forces expressed by the function $\{f(t)\}$ to every history of motion expressed by the generalized coordinates $\{q(\tau)\}$ for $-\infty < \tau \le t$. It is preferable to assume, that $\mathcal{A}$ is an operator in the space of distributions [4]. It has then a simple convolution representation

$$\{f(t)\} = [A(M,t)] * \{q(t)\}, \tag{3.1}$$

where $[A(M,t)]$ is the unit impulse response matrix distribution (called also the hereditary matrix [5]), the (i,j) element of which is the generalized response in the i-th mode due to the pressure $c_p(\mathbf{r},t)$ generated by the motion in the j-th mode, with $q_j(t) = \delta(t)$. The elements of this matrix depend also on the Mach number M. The aerodynamic forces can depend only on the history and not on the future of the motion. It means that the aerodynamic system is causal, and therefore

$$[A(M,t)] = 0 \qquad \text{for} \quad t < 0.$$

The existence of the convolution representation (3.1) of the operator $\mathcal{A}$ results from the assumed properties: single-valuedness, linearity, time-invariance and continuity. The distributional convolution is the generalisation of the integral convolution

$$\{f(t)\} = \int_{-\infty}^{t} [A(M,t)]\,\{q(t-\tau)\}\,d\tau,$$

but in the aerodynamic model, the elements of this matrix contain delta distributions and their derivatives, and the integral may be divergent.

The aerodynamic derivatives model is a particular case of (3.1), when

$$[A(M,t)] = [A_0]\delta(t) + [A_1]\dot{\delta}(t) + [A_2]\ddot{\delta}(t). \tag{3.2}$$

Direct calculation of the elements of $[A(M,t)]$ for arbitrary time may be difficult and in practice, these functions are usually determined only by means of the inversion of Fourier or Laplace transforms. Taking the Laplace transformation of the convolution (3.1) it follows that

$$\left\{\hat{f}(p)\right\} = \left[\hat{A}(M,p)\right]\{\hat{q}(p)\}, \tag{3.3}$$

where p is the Laplace variable, and the circumflex accents $(\hat{})$ denote transforms. The Laplace transform of the hereditary matrix

$$\left[\hat{A}(M,p)\right] = \mathcal{L}[A(M,t)], \qquad [A(M,t)] = \mathcal{L}^{-1}\left[\hat{A}(M,p)\right], \tag{3.4}$$

is called the aerodynamic transfer functions matrix. It plays an important role in the stability analysis of an aircraft. The aerodynamic transfer functions matrix $[A(M,t)]$ is a Laplace transform of a real distribution and is real whenever p is real. Hence

$$\left[\hat{A}(M,p)\right]^* = \left[\hat{A}(M,p^*)\right], \tag{3.5}$$

where the star $(^*)$ denotes complex conjugate values.

The aerodynamic transfer functions are holomorphic functions in the p plane for $\mathrm{Re}\,(p) > 0$, but they are not ordinary Laplace transforms, because they do not fulfil the condition $O(|p|^{-\alpha})$ for $|p| \to \infty$ and $\alpha \geq 1$. The relations (3.4) are valid only under the assumption, that $\mathcal{L}$ is a distributional Laplace transformation [4].

If the aerodynamic forces acting on an aeroplane are calculated by means of stability derivatives (1.4), the transfer functions are polynomials

$$\left[\hat{A}(M,p)\right] = [A_0] + [A_1]p + [A_2]p^2. \tag{3.6}$$

Aerodynamic transfer functions were introduced in 1956 by Etkin [2], and he pointed out their usefulness for calculating dynamic stability derivatives. He proposed a truncated power series expansion of the functions $[A(M,t)]$ as an "unsteady" model of aerodynamic derivatives. Unfortunately, at subsonic flight velocities, the transfer functions possess a branch point in the origin of the (complex) Laplace plane and the accuracy of a polynomial approximation in the vicinity of the origin may be poor. Better results may be achieved by means of an approximation by rational functions and this approach is commonly used in aeroelastic applications.

If the Laplace variable is pure imaginary $p = ik$, then (3.1) determines the steady-state frequency response function, which relates the amplitudes of generalized forces to the amplitudes of generalized coordinates in harmonic motion

$$\left\{\hat{f}(ik)\right\} = \left[\hat{A}(M,ik)\right]\{\hat{q}(ik)\}, \tag{3.7}$$

where k is the frequency coefficient (reduced frequency)

$$k = \frac{\omega b}{U},$$

and

$$\left[\hat{A}(M, ik)\right] = \lim_{p \to ik} \left[\hat{A}(M, p)\right]$$

is the matrix of harmonic transfer functions.

Many methods for determination of the oscillatory aerodynamic loads for harmonic small displacements of the structure were developed. The aerodynamic transfer functions are usually determined by means of an analytic continuation of the elements of matrix $[\hat{A}(M, ik)]$ from the imaginary axis into the whole complex plane. Modern approaches to the approximation are based on the calculation of the values of harmonic transfer functions over a specified range of the frequency coefficients, and the transfer functions are approximated by rational functions which fit best the calculated values. It is then possible to cast the equations of motion of the aeroplane in a linear time-invariant state-space form, although the size of the state vector increases due to the approximation. In addition to the state variables introduced initially to describe the motion of the aeroplane, there appear also augmented state variables that belong to the aerodynamic model. Currently there are three basic formulations used in approximating aerodynamic transfer functions by means of rational functions: least-squares [6], modified matrix-Padé [7] and minimum-state [8]. In the minimum-state formulation, the approximation of aerodynamic transfer functions is taken in the form

$$\left[\hat{A}(M, p)\right] \approx \left([A_0] + [A_1]p + [A_2]p^2\right) + [D]\left([I]p - [R]\right)^{-1}[E]p,$$

and the number of augmented state variables is equal to the range of the (positive definite) matrix $[R]$. The calculation of the constant matrices in the formula is a task of nonlinear programming [9]. After transformation to the time domain, it follows

$$\{f(t)\} = [A_0(M)]\{q(t)\} + [A_1(M)]\{\dot{q}(t)\} + [A_2(M)]\{\ddot{q}(t)\} + [D]\{x(t)\},$$

with a set of additional differential equations for the augmented state variables

$$\{\dot{x}(t)\} = [E]\{\dot{q}(t)\} + [R]\{x(t)\}.$$

The last step in these procedures consists in an analytic continuation of the transfer functions from the imaginary axis into the whole Laplace-plane. This is an ill-posed process and the accuracy of results depends on the analytic properties (singularities) of the approximated transfer functions. There are no poles in the right half of the Laplace-plane, since the transient aerodynamic response is always stable. The analytic continuation into this half-plane is possible without any restrictions. The behaviour of transfer functions in the left half of the p-plane is more complicated. UEDA [10] stated in 1987 that at high subsonic velocities, there exist poles of the transfer functions in the left half of the Laplace-plane. They determine the limits for the approximation by means of rational functions and may significantly influence the aerodynamic forces in the time domain.

Another interesting approach to the approximation of transfer functions was given by STARK [11]. He proposed a simple expression for the lift deficiency function in the time domain and assumed that this function is independent of the deflection mode of the wing. The Laplace transform of his deficiency function possesses a branch point in the origin, which is the only singularity of the approximate transfer functions in the finite part of the Laplace-plane,

$$\left[\hat{A}(M,p)\right] \approx [A_0] + [A_1]p + [A_2]p^2 + ([R_0] + [R_1]p)\, apF_m(ap),$$

where $F_m(p) = \mathcal{L}(1+t)^{-m}$, $a = 5.5$ and $m = 3$. This approach leads to a good approximation in the incompressible case, but for non-zero Mach numbers the results are less satisfactory.

4. THE AERODYNAMIC TRANSFER FUNCTIONS

The principal part in the calculation of aerodynamic forces acting on the body (2.2) is the determination of the relation between the distribution of pressure coefficient $c_p(\mathbf{r},t)$ on the body, and the normal (to the surface) component of fluid velocity distribution $v_n(\mathbf{r},\tau)$ (for $-\infty < \tau \leq t$). In the linear model described in Sec. 2, the external (to the body and wake) flow is potential. It makes possible the use of the boundary integral equations method to formulate a direct relation between the given boundary conditions and the distribution of velocity potential over the body, without determination of the whole flowfield. The Kutta - Joukowski condition relate the vorticity in the wake to the velocities at the trailing edge at earlier instants of time. The last step in the procedure is the determination of pressure distribution by means of the Cauchy - Lagrange integral. The (integral) equations are usually derived by the assumption of oscillatory motion, when the vorticity distribution in the wake is described by a simple (sinusoidal) function. All methods derived for the harmonic motion may be used also to calculate the Laplace transforms, by means of an (exact) analytic continuation $ik \to p$ (in the equations) from the imaginary axis into the whole Laplace plane.

Generally, the whole process of the determination of Laplace transforms of the pressure distribution may be described as the solution of an equation

$$\hat{v}_n(\mathbf{r}, p) = \mathcal{K}\hat{c}_p(\mathbf{r}, p), \tag{4.1}$$

where the operator $\mathcal{K}$ is determined by the method used. A typical example of (4.1) is the lifting-surface equation.

The solution of (4.1) is a difficult numerical problem. An important exception is the exact solution for a harmonically oscillating airfoil in an incompressible flow [12]. DENGLER and LUKE [13] have extended this solution to the whole Laplace plane by an analytic continuation, but it gave rise in the past to arguments on validity of the results in the left-hand half-plane of the Laplace variable. This problem was later resolved by MILNE [14], EDWARDS [15] and others.

For an airfoil with chord $2b$ (Fig. 2), the Theodorsen solution may be written in the form

$$\begin{aligned} \Delta\hat{c}_p(x,p) = {} & \frac{4}{\pi}\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,\frac{\widehat{w}(\xi,p)}{\xi - x}\,d\xi + \frac{4}{\pi}p\int_{-1}^{1}\Lambda(x,\xi)\widehat{w}(\xi,p)\,d\xi \\ & + \frac{4}{\pi}\left(C(-ip)-1\right)\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,\widehat{w}(\xi,p)\,d\xi, \end{aligned} \tag{4.2}$$

where $\Delta\hat{c}_p(x,p)$ denotes the transform of the difference of pressure coefficients between the upper and lower sides of the airfoil,

$$\Lambda(x,\xi) = \frac{1}{2}\ln\left|\frac{1 - x\xi + \sqrt{1-x^2}\,\sqrt{1-\xi^2}}{1 - x\xi - \sqrt{1-x^2}\,\sqrt{1-\xi^2}}\right|,$$

and

$$C(-ip) = \frac{K_1(p)}{K_0(p) + K_1(p)} \tag{4.3}$$

is the Theodorsen function ($K_0(p)$ and $K_1(p)$ are modified Bessel functions). This function was originally defined for harmonic motion, when $p = ik$, and the argument of (4.3) determines the analytic continuation into the whole Laplace plane.

The first term in the integral in (4.2) determines the steady solution. The second term (proportional to p) is the result of the apparent mass effect, and the last term (proportional to $C(-ip) - 1$) expresses the influence of the vortex wake.

When the generalized coordinates are defined by (1.1), then on the basis of (4.2), (2.1) and (1.2), the aerodynamic transfer functions matrix for an airfoil in incompressible flow ($M = 0$) may be put in the form

$$\left[\hat{A}(0,p)\right] = [A_0] + [A_1]p + [A_2]p^2 + ([A_3] + [A_4]p)\,(C(-ip) - 1), \tag{4.4}$$

where the elements of the constant matrices $[A_0] - [A_4]$ depend only on the definition of the generalized coordinates (1.1).

The model of aerodynamic derivatives may be derived from (4.4) by the assumption, that the last term (proportional to $C(-ip) - 1$) is equal to zero. This term is responsible for the behaviour of the aerodynamic transfer functions in the vicinity of the origin $p = 0$.

The Theodorsen function $C(-ip)$ is holomorphic in the complex plane p, cut along the negative real semi-axis [16],

$$C(-ip) = 1+p\Big(\ln\frac{p}{2}+\gamma\Big)-p^2\Big(\ln\frac{p}{2}+\gamma\Big)^2+O\Big(p^3(\ln p)^3\Big) \qquad \text{for } p \to 0. \tag{4.5}$$

and

$$C(-ip) \approx 1 - \frac{1}{2}\left(1 - \frac{1}{4p} + \frac{1}{8p^2} - \frac{1}{64p^3} + \ldots\right) \qquad \text{for } |p| \to \infty. \tag{4.6}$$

The approximation of the function $C(-ip)$ by a polynomial is not convenient because of the logarithmic branch-point in the origin (4.5). Better results were obtained by numerous approximations which use rational functions, e.g. by the approximation of R.T. JONES [17]

$$C(-ip) \approx 1 - \frac{b_1 p}{p - a_1} - \frac{b_2 p}{p - a_2},$$

where $b_1 = 0.165$, $b_2 = 0.335$ and $a_1 = -0.0445$, $a_2 = -0.3$. An important contribution to the problem of approximation of the Theodorsen function is the result of DESMARAIS [18]. He has established a continued fraction representation, which may be truncated, to obtain approximations of any desired accuracy, by means of rational functions with poles on the negative real axis. The poles become infinitely dense as the order of the approximation is increased. It follows, that the simulation of the branch point by means of rational functions is possible, but in practice may be troublesome. The maximum percentage error based on exact values for the R.T. Jones approximation in harmonic motion ($p = ik$) is equal to 8.5 per cent for $\operatorname{Re}(C)$ and -13.5 per cent for $\operatorname{Im}(C)$.

In compressible flow $M \neq 0$ and for more complex structures, the calculation of $\hat{c}_p(\mathbf{r}, p)$ on the basis of $\hat{v}_n(\mathbf{r}, p)$ requires numerical methods. The discretization of the equation (4.1) leads to a system of algebraic equations

$$(4.7) \qquad \underset{N\times 1}{\{\hat{w}(p)\}} = \underset{N\times N}{\left[\hat{K}(M,p)\right]} \underset{N\times 1}{\{\hat{c}_p(p)\}},$$

where the matrices are finite-dimensional approximations

$$\mathcal{K} \Rightarrow \left[\hat{K}(M,p)\right], \qquad \hat{c}_p(\mathbf{r},p) \Rightarrow \{\hat{c}_p(p)\} \qquad \text{and} \qquad \hat{v}_p(p) \Rightarrow \{\hat{w}(p)\},$$

determined by the choice of the used discretization method. An illustrative example of the transformation of $\mathcal{K}$ in the case of an airfoil is given in the Appendix.

The matrix $[\hat{K}(M,p)]$ is called the aerodynamic influence coefficients matrix. It describes the aerodynamic system and, contrary to the transfer matrix, does not depend on the choice of generalized coordinates (1.1) used to describe the motion of the structure.

The use of the same discretization to the substantial derivative (2.1) and to the definition of generalized aerodynamic forces (1.2), leads to the following expressions

$$(4.8) \qquad \underset{N\times 1}{\{\hat{w}(p)\}} = \underset{N\times n}{([D_1] + p[D_2])} \underset{n\times 1}{\{\hat{q}(p)\}}$$

and

$$(4.9) \qquad \underset{n\times 1}{\left\{\hat{f}(p)\right\}} = \underset{n\times N}{[S]} \underset{N\times 1}{\{\hat{c}_p(p)\}}.$$

Finally, the transfer functions may be expressed by the formula

$$(4.10) \qquad \underset{n\times n}{\left\{\hat{A}(M,p)\right\}} = \underset{n\times N}{[S]} \underset{N\times N}{\left[\hat{K}(M,p)\right]^{-1}} \underset{N\times n}{([D_0] + p[D_1])},$$

where N is the number of aerodynamic elements used by the discretization of the aerodynamic model, and n denotes the number of generalized coordinates (used to describe the motion of the structure). Usually $N \gg n$.

The "static" aerodynamic derivatives may be obtained by the assumption

$$\left\{\hat{K}(M,p)\right\} \approx \left[\hat{K}(M,0)\right] = \text{const},$$

and the steady approximations to stability derivatives have the form

$$(4.11) \qquad [A_0] = [S]\left[\hat{K}(M,0)\right][D_0], \qquad [A_1] = [S]\left[\hat{K}(M,0)\right][D_1], \qquad [A_2] = 0.$$

The matrix $[\hat{K}(M,0)]$ is in the practice calculated on the basis of any aerodynamic method for the steady flow (e.g. the strip, or "vortex-lattice" method).

The methods described in handbooks of Flight Mechanics are usually equivalent to (4.11). The only exception are the derivatives with respect to the rates of angle of attack $\dot{\alpha}$ and slideship $\dot{\beta}$, which describe the influence of the wake and can be determined only on the basis of an unsteady flow model. Usually, this nonstationary effect is included in the aerodynamic derivatives model approximately, on the basis of steady aerodynamics. This approach is based on a hypothesis that the deflection angle of the flow near the tailplane is caused by the lift change on the wing at the moment, when appropriate elements of the wake shed of the wing. In other words, the deflection of the stream is delayed in comparison with the steady value by the time, which is needed for the wake to move the distance between wing and tailplane.

The direct application of unsteady aerodynamics to calculate these derivatives by approximation of the transfer function with a polynomial (3.6) is also difficult [19], because the derivatives of $[\hat{A}(M,p)]$ are singular at $p = 0$.

The elements of the aerodynamic influence coefficients matrix $[\hat{K}(M,p)]$ are regular, holomorphic functions in the complex p-plane, cut along the negative real axis. This is not true for the elements of the transfer functions $[\hat{A}(M,p)]$ which may have poles in the singular points of this matrix, where

$$\det\left(\left[\hat{K}(M,p)\right]\right) = 0. \tag{4.12}$$

It was stated in [20], that for each Mach number in the range $0 < M < 1$, there exist a large, probably infinite set of latent roots of (4.12). In Fig. 4 and Fig. 5 are shown some roots, calculated on the basis of a lifting-surface model for a rectangular wing with aspect ratio 3, at Mach numbers $M = 0.8$ and $M = 0.95$. The distribution of latent roots vary with the Mach number, and in high subsonic flow, many roots are close to the imaginary axis. A typical latent root loci are shown in Fig. 6. The outer ends of the curves correspond to the Mach number $M = 0.5$, and the inner ends to $M = 0.9$. It is possible to formulate a hypothesis that for $M \to 1$, all roots move to the origin and may significantly influence the behaviour of transfer functions at high subsonic Mach numbers. On the other hand, for $M \to 0$, all roots move to infinity and in the incompressible case ($M = 0$), there are no roots in the finite part of the Laplace plane. The last conclusion agrees with the exact solution for an airfoil (4.2).

In the vicinity of the root $p = p_k$, the inverse matrix $[\hat{K}(M,p)]^{-1}$ may be approximated by

$$\left[\hat{K}(M,p)\right]^{-1} = \frac{\{u_k\}\{v_k\}^T}{p - p_k} + \text{regular function of } p, \tag{4.13}$$

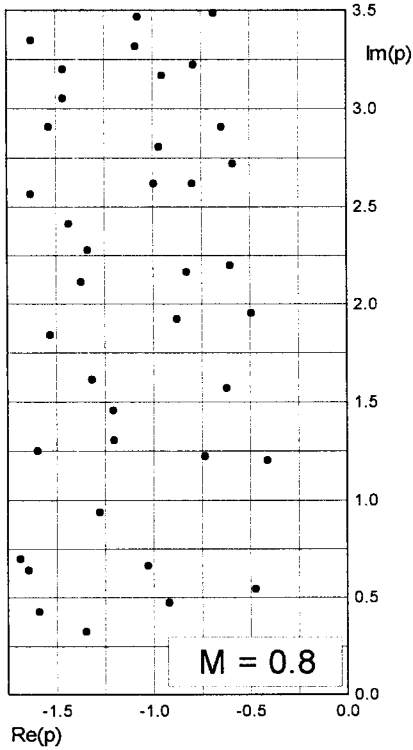

FIG. 4. Latent roots for an aspect-ratio-three rectangular wing.

where $\{u_k\}$ and $\{v_k\}$ are the right and left latent vectors which are solutions of homogeneous equations

$$\{v_k\}^T \left[\hat{K}(M, p_k)\right] = 0 \qquad \text{and} \qquad \left[\hat{K}(M, p_k)\right] \{u_k\} = 0,$$

normalised in such a way, that

$$\{v_k\}^T \frac{\partial}{\partial p} \left[\hat{K}(M, p)\right]\Big|_{p=p_k} \{u_k\} = 1. \tag{4.14}$$

The proximity of many poles to the imaginary axis and to the origin at high subsonic Mach numbers, may significantly disturb not only a polynomial (in the model of aerodynamic derivatives) approximation in this region, but also an approximation of the transfer functions by rational functions with poles on the real axis (in aeroelastic applications).

In subsonic flow, the aerodynamic influence coefficients have a logarithmic branch point. It results from the assumption, that the length of the wake may grow to infinity. Although this assumption is in contradiction with the experience, it is necessary in a model to be consistent with the governing equations. On the

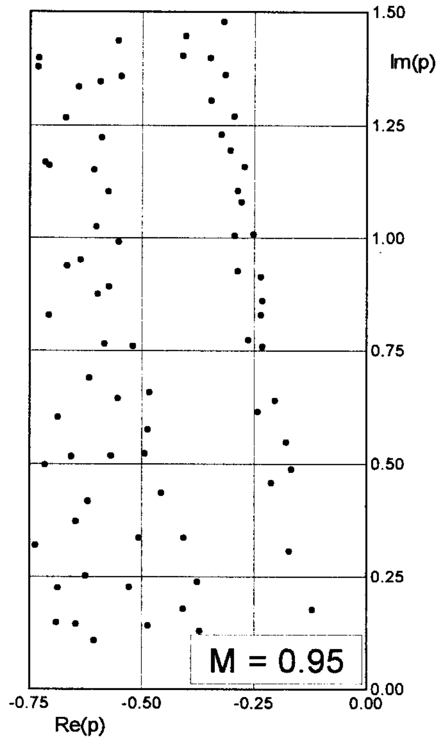

FIG. 5. Latent roots for an aspect-ratio-three rectangular wing.

basis of (4.13) it is possible to write

$$\left[\hat{K}(M,p)\right]^{-1} = \left[\hat{K}(M,0)\right]^{-1} + \sum_k \frac{p}{p_k}\,\frac{\{u_k\}\{v_k\}^T}{p-p_k} + \left[\hat{G}(M,p)\right], \tag{4.15}$$

where $[\hat{K}(M,0)]^{-1}$ is the steady solution, $[\hat{G}(M,p)]$ has no poles and possesses a branch-point in the origin. On the basis of the Mittag-Leffler theorem [21], the second term determines a meromorphic function if the series

$$\sum_k \{u_k\}\{v_k\}^T/p_k^2 \tag{4.16}$$

is convergent. All poles appear in conjugate pairs, and a sum of two consecutive terms in (4.16) is always real. Numerical results have shown, that the contribution of a pole to the values of transfer functions (4.15) does not decrease with increasing distance of the origin. It is a result of the normalisation (4.15) of the latent vectors, because

$$\left\|\{u_k\}\{v_k\}^T\right\| \geq \frac{1}{\left\|\dfrac{\partial}{\partial p}\left[\hat{K}(M,p)\right]\Big|_{p=p_k}\right\|}.$$

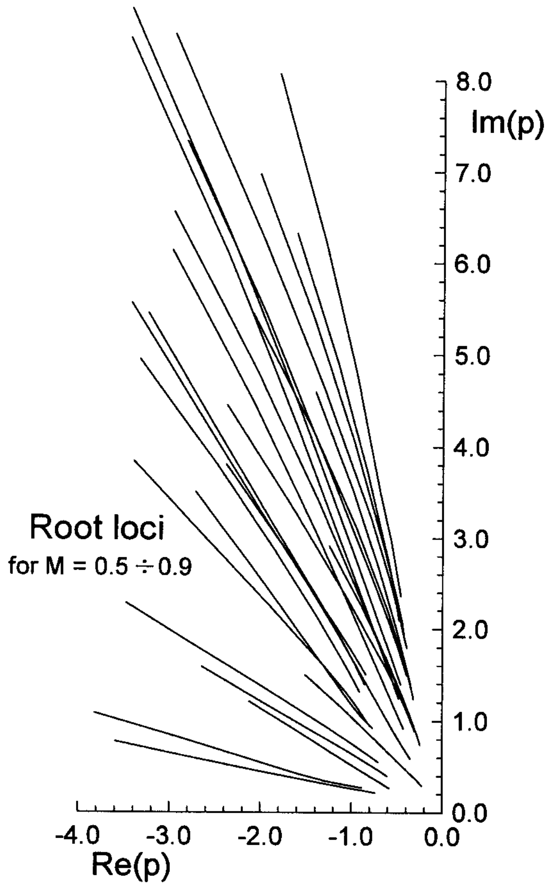

FIG. 6. Latent root loci for an aspect-ratio-three rectangular wing.

Also poles which are far from the considered region can significantly influence the values of transfer functions. This is the reason of difficulties in the approximation of transfer functions in the compressible case.

If the limit of (4.16) exist also in incompressible flow

$$\lim_{M\to 0} \sum_k \{u_k\}\{v_k\}^T/p_k^2 = -[M_A], \tag{4.17}$$

then the formula (4.15) is valid also in this case, and may be written in the form

$$\left[\hat{K}(0,p)\right]^{-1} = \left[\hat{K}(0,0)\right]^{-1} + [M_A]p + \left[\hat{G}(0,p)\right], \tag{4.18}$$

where $[M_A]$ is the apparent mass matrix.

The expression (4.17) explains the behaviour of the aerodynamic forces, when the Mach number tends to the limit $M \to 0$. With decreasing Mach number, all poles move to infinity, but with increasing intensities. In the limit, there is only one pole in infinity.

The formula (4.15) determines completely the behaviour of the transforms in the whole complex plane of the Laplace variable. It consists of terms which correspond to the terms in the exact Theodorsen solution (4.2) for an airfoil in incompressible flow, but can be applied also to arbitrary 3-dimensional structures, when a discretization is necessary to obtain the numerical results. In the usual methods used to calculate the aerodynamic derivatives, only the first term is taken into account. In incompressible flow ($M = 0$) also the second term may be used. At high subsonic velocities, when the poles are located in the vicinity of the origin, the contribution of this term may be significant and it is difficult to approximate it with a polynomial. Also the approximation of the second term (with the branch point) is difficult. These are the main limitations in the use of aerodynamic derivatives.

It was assumed that the series (4.16) is convergent and that the limit (4.17) exists. The proof of this statements may be difficult, because until now, all information about the distribution of the latent roots of (4.1) were obtained only in numerical calculations.

5. The aerodynamic forces in the time domain

The solution of the equation (4.7) which describes a (discretized) aerodynamic model, may be put in the form

$$(5.1)\qquad \underset{N\times 1}{\{\hat{c}_p(p)\}} = \underset{N\times N}{\left[\hat{K}(M,p)\right]^{-1}} \underset{N\times 1}{\{\hat{w}(p)\}} = \underset{N\times N}{\left[\hat{H}(M,p)\right]} p \underset{N\times 1}{\{\hat{w}(p)\}},$$

where

$$(5.2)\qquad \left[\hat{H}(M,p)\right] = \frac{1}{p}\left[\hat{K}(M,p)\right]^{-1}.$$

The inverse Laplace transform $\mathcal{L}^{-1}$ applied to (5.1) gives the relation

$$(5.3)\qquad \{c_p(t)\} = [H(M,t)] * \{\dot{w}(t)\},$$

where the indicial functions $[H(M,t)] = \mathcal{L}^{-1}[\hat{H}(M,p)]$ are the responses to a unit step change in the (discretized) boundary conditions (normal component of fluid velocity).

The steady solution follows from the final value theorem [4] for the Laplace transforms

$$(5.4)\qquad [H(M,\infty)] = \lim_{t\to\infty} [H(M,t)] = \lim_{p\to 0} p\left[\hat{H}(M,p)\right] = \lim_{p\to 0}\left[\hat{K}(M,p)\right]^{-1} = \left[\hat{K}(M,0)\right]^{-1}.$$

The asymptotic behaviour (for $t\to\infty$) of indicial functions may be determined by the expansion of the aerodynamic influence coefficients (which is different in the two- and three-dimensional cases)

$$(5.5)\qquad \left[\hat{K}(M,p)\right]^{-1} - \left[\hat{K}(M,0)\right]^{-1} = \begin{cases} O(p\ln p) & \text{2-dim} \\ O(p^2\ln p) & \text{3-dim} \end{cases} \quad \text{for} \quad p\to 0,$$

hence

$$(5.6)\qquad [H(M,t)] - [H(M,\infty)] = \begin{cases} O(t^{-1}) & \text{2-dim} \\ O(t^{-2}) & \text{3-dim} \end{cases} \quad \text{for} \quad t\to\infty.$$

From the initial value theorem for Laplace transforms it follows that in the compressible case, a limit should exist

$$(5.7)\qquad \lim_{p\to\infty}\left[\hat{K}(M,p)\right]^{-1} = \lim_{t\to 0^+}[H(M,t)] = [D],$$

which may be determined directly on the basis of the piston theory

$$(5.8)\qquad c_p(\mathbf{r},0^+) = \frac{4}{M}\,\frac{v_n(\mathbf{r},0^+)}{U}\,.$$

The further evolution of the pressure distribution is not simple. An acoustic wave propagates from the leading edge downstream with the velocity $a_\infty + U$ (where a_∞ is the sound velocity). At the same time, from the trailing edge propagates upstream (in subsonic flow) another wave with velocity $a_\infty - U$. In the time interval, until the first wave reaches the trailing edge, and the second wave – the leading edge, the pressure distribution on the surface changes very rapidly. For an airfoil (with the chord $2b$) there are two characteristic times $t_1 = 2M/(1+M)$ and $t_2 = 2M/(1-M)$. The pressure distribution changes for $t > t_2$ (in subsonic flow) are already mild, and in the supersonic flow, the pressure reaches the steady limit at $t = t_1$.

The pressure distribution on a profile (cross-section of the wing) is at the discretization usually approximated by a truncated series of functions which possess appropriate singularities on the leading and trailing edges. The pressure distribution in the piston theory (5.8) follows the boundary condition and has no singularities. Therefore it is not possible to cast the piston theory in the discretization scheme exactly, and the exact result should be projected on the finite-dimensional space used at the discretization to approximate the pressure distribution [20].

In the incompressible flow ($M = 0$) the limit (5.6) does not exist, but

$$\lim_{p\to\infty}\left(\frac{1}{p}\left[\hat{K}(0,p)\right]^{-1}\right) = \lim_{t\to 0+}\left([H(0,t)] * 1_+(t)\right) = [M_A]. \tag{5.9}$$

In the subsonic range, the poles of the function (4.15) are responsible for the starting pulse, which in the time domain is described by the expression

$$\mathcal{L}^{-1}\left(\frac{1}{p}\sum_k \frac{p}{p_k}\frac{\{u_k\}\{v_k\}^T}{p-p_k}\right) = \begin{cases} [M_A]\delta(t) & \text{for} \quad M = 0, \\ \sum\limits_k \dfrac{\{u_k\}\{v_k\}^T}{p_k} e^{p_k t} & \text{for} \quad M \neq 0. \end{cases}$$

Finally, the indicial functions matrix in (5.3) has the following structure

$$[H(M,t)] = \left[\hat{K}(M,0)\right] + \begin{cases} [M_A]\delta(t) + [C_D(0,t)] & \text{for } M = 0, \\ \sum\limits_k \dfrac{\{u_k\}\{v_k\}^T}{p_k} e^{p_k t} + [C_D(M,t)] & \text{for } M \neq 0, \end{cases} \tag{5.10}$$

where

$$[C_D(M,t)] = \mathcal{L}^{-1}\left(\frac{1}{p}\left[\hat{G}(M,p)\right]\right) \tag{5.11}$$

is the deficiency function, which determines the difference between the indicial function $[H(M,t)]$ and its steady limit $[H(M,\infty)]$. On the basis of (5.6) its asymptotic behaviour is given by

$$[C_D(M,t)] = \begin{cases} O(t^{-1}) & \text{2-dim} \\ O(t^{-2}) & \text{3-dim} \end{cases} \quad \text{for} \quad t \to \infty.$$

This asymptotic behaviour of the deficiency functions is responsible for the logarithmic branch point in $[\hat{G}(M,p)]$ and consequently, also in the aerodynamic transfer functions $[\hat{A}(M,p)]$. STARK [11] proposed a very simple method to approximate the deficiency function

$$[C_D(M,t)] = \sum_{k=k_1}^{k_2} [B_k]\left(\frac{a}{a+t}\right)^k,$$

where the constants k_1, k_2, a and $[B_k]$ should be chosen. For an airfoil in incompressible flow the choice $k_1 = k_2 = 1$, $a = 4$, corresponds to the Garrick approximation of the Wagner function [17]. For a rectangular wing, very good results were obtained (but only for $M = 0$), with the parameters $k_1 = k_2 = 3$, $a = 5.5$. In the aerodynamic derivatives model, the deficiency function is completely neglected, and all disturbances decay exponentially.

The second term in (5.10) determines the initial pulse after a stepwise change of the boundary conditions. For $M = 0$ it is concentrated at $t = 0$ and it is easy to take it into account in the calculations. For $M > 0$ it is distributed in time and in the aerodynamic derivatives model, as well as in the STARK model [11], it is neglected. The approximation of the starting pulse is the main difficulty in numerical calculations in the case of compressible fluid flow.

The indicial functions (5.10) may be compared with the known, exact results for an airfoil. The inverse Laplace transform $\mathcal{L}^{-1}$ applied to the Theodorsen solution (5.1) gives the relation

$$\Delta c_p(x,t) = \frac{4}{\pi}\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\frac{w(\xi,t)}{\xi - x}d\xi + \frac{4}{\pi}\int_{-1}^{1}\Lambda(x,\xi)\dot{w}(\xi,t)\,d\xi + \frac{4}{\pi}(\phi(t) - 1_+(t)) * \left(\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\dot{w}(\xi,t)\,d\xi\right), \tag{5.12}$$

where

$$\phi(t) = \mathcal{L}^{-1}\left(\frac{C(-ip)}{p}\right) \tag{5.13}$$

is the Wagner function [17].

For a step change in the angle of attack: $w(x,t) = -\alpha 1_+(t)$, $\dot{w}(x,t) = -\alpha\delta(t)$ and

$$\frac{\Delta c_p(x,t)}{\alpha} = -4\sqrt{\frac{1-x}{1+x}} - 4\sqrt{1-x^2}\,\delta(t) - 4\sqrt{\frac{1-x}{1+x}}(\phi(t) - 1). \tag{5.14}$$

The three terms in (5.14) correspond to the appropriate terms in (5.10). The time dependence of the deficiency function is in this case expressed by a multiplier $(\phi(t) - 1)$.

LOMAX *et al.* [22] derived exact formulas to calculate the pressure distribution on an airfoil in the first stage after a step change in the angle of attack α. In Fig. 7, the results at several times, t, for a representative subsonic Mach number $M = 0.8$ are reproduced. For $t = 0^+$, the result is given by the piston theory.

The pressure distributions may be integrated along the chord ($2b$) to obtain the total force on the airfoil. In Fig. 8 are shown representative results for the

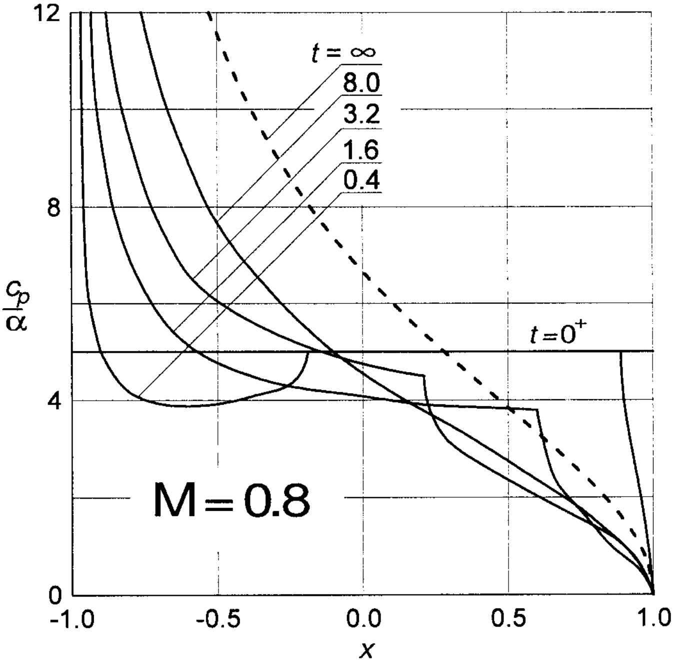

FIG. 7. Chordwise lifting pressure distributions on an airfoil. After LOMAX [23].

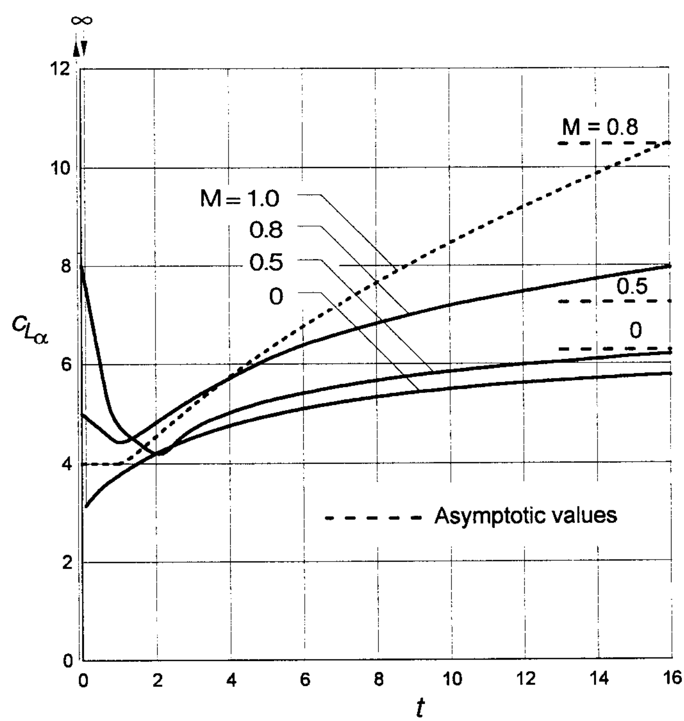

FIG. 8. Time history of lift curve slope. After LOMAX [23].

time history of lift curve slope, defined by

$$c_{L_\alpha} = \frac{L}{\frac{\rho U^2}{2}(2b)\alpha},$$

where L is the lift on the airfoil.

6. Conclusions

The aerodynamic transfer functions are holomorphic functions in the complex plane cut along the negative real semi-axis and have (in the case of subsonic flow) a branch-point at the origin and a set of poles in the left half-plane. The branch-point (caused by the infinite length of the wake) is responsible for the asymptotic behaviour of aerodynamic forces for large time. The poles are responsible for the initial pulse after a step change of upwash distribution on the surface of the body. With decreasing Mach number, all poles tend to infinity and for incompressible flow, there is only one pole at infinity (responsible for the apparent mass effect).

The aerodynamic model of stability derivatives uses a polynomial approximation to the transfer functions, and it neglects the influence of the vortex wake on aerodynamic forces. In the compressible case, the lack of poles makes impossible a correct prediction of the forces for a discontinuous change of the boundary conditions (e.g. after a step change of control surface deflection or in a sharp gust). This conclusion remains true also when the stability derivatives have been calculated on the basis of an unsteady model (but with a polynomial approximation to the transfer functions). The use of the Bryan's model must be limited to the cases, when it may be assumed, that the aerodynamic influence coefficients are almost constant.

Finite-dimensional representations of the aerodynamic operator are derived in the Laplace domain (4.15) and in the time domain (5.10), which enable a clear look inside its structure in the case of subsonic flow. They are convenient to make a qualitative estimate of the accuracy of diverse methods of approximation to the unsteady aerodynamic forces. However, it should be mentioned that although these representations seem to be consistent with all known properties of unsteady aerodynamic forces, they were derived mainly on the basis of numerical results for a lifting surface, and until now, they can't be treated as an exact model. The convergence of the series (4.16) and the existence of the limit (4.17) need a proof, and it seems to be a very difficult task.

Appendix. Discretization of the integral equation relating the unknown pressure distribution to the known upwash on an airfoil in subsonic flow

As an example of the general procedure, a discretization process for the aerodynamic operator of an airfoil in subsonic flow will be described. This procedure is, at the same time, also the main part of the lifting-lines method to solve the lifting-surface equation [24]. The integral equation expressing the boundary-value problem of an oscillating airfoil in subsonic flow was first given by Possio in 1938. After generalisation to the Laplace domain, the Possio equation may be written in the form

$$\hat{w}(x,p) = \int_{-1}^{1} \hat{K}(M,x,\xi,p)\Delta\hat{c}_p(\xi,p)\,d\xi \tag{A.1}$$

where the lengths are referred to the airfoil semichord b, U is the main velocity, $\hat{w}(x,p)$ is the Laplace transform of the upwash (normal velocity, positive up, related to U), and $\Delta\hat{c}_p(x,p)$ is the transform of the pressure coefficient difference between the upper and lower side of the airfoil. The kernel function may be expressed in the form [15]

$$\hat{K}(M,x,\xi,p) = \frac{p}{4\pi\beta}e^{p(x-\xi)}\left\{e^{p(x-\xi)/\beta^2}\left[M\frac{|x-\xi|}{x-\xi}K_1\left(\frac{pM}{\beta^2}|x-\xi|\right) + K_0\left(\frac{pM}{\beta^2}|x-\xi|\right)\right] - \beta\ln\left(\frac{1+\beta}{M}\right) - p\int_0^{x-\xi} e^{p\lambda/\beta^2}K_0\left(\frac{pM}{\beta^2}|\lambda|\right)d\lambda\right\}, \tag{A.2}$$

where $\beta = \sqrt{1-M^2}$.

The approximate solution of the Possio equation is usually sought in the form of a truncated series

$$\Delta\hat{c}_p(\xi,p) = \sqrt{\frac{1-\xi}{1+\xi}}\sum_{k=1}^{N} a_k P_{k-1}(\xi), \tag{A.3}$$

where $P_{k-1}(\xi)$ are Jacobi polynomials defined by the recurrence formula

$$P_{k-1}(\xi) - 2\xi P_k(\xi) + P_{k+1}(\xi) = 0 \quad \text{for} \quad k = 1,2,\ldots \quad \text{and} \quad P_0(\xi) = 1, \quad P_1(\xi) = 2\xi + 1. \tag{A.4}$$

These polynomials fulfil the orthogonality condition

$$\int_{-1}^{1}\sqrt{\frac{1-\xi}{1+\xi}}P_i(\xi)P_j(\xi)\,d\xi = \begin{cases} 0 & \text{for} \quad i \neq j, \\ \pi & \text{for} \quad i = j. \end{cases} \tag{A.5}$$

After substitution of (A.3) in (A.1), the coefficients a_k may be calculated from a system of algebraic equations obtained by the collocation method. Another useful method to solve the Possio equation is the Galerkin scheme. The equation (A.1) is multiplied by the functions

$$\sqrt{\frac{1+x}{1-x}}Q_{k-1}(x)$$

in turn for $k = 1, 2, ..., n$, and then integrated along the chord in the interval $-1 \leq x \leq 1$. Functions $Q_k(x) = P_k(-x)$ are polynomials orthogonal on the interval $-1 \leq x \leq 1$ with the weighting function $\sqrt{1+x}/\sqrt{1-x}$. As a result, a set of algebraic equations is obtained

$$\{v(p)\} = [H(M,p)]\{a(p)\}, \tag{A.6}$$

where

$$\{v(p)\} = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\{Q(x)\}\widehat{w}(x,p)\, dx \tag{A.7}$$

and

$$[H(M,p)] = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \int_{-1}^{1} \sqrt{\frac{1-\xi}{1+\xi}} \hat{K}(M,x,\xi,p)\{Q(x)\}\{P(\xi)\}^T\, d\xi\, dx. \tag{A.8}$$

The chordwise integrals can be calculated numerically according to the m-node Gauss-Jacobi quadrature

$$\int_{-1}^{1} \sqrt{\frac{1-\xi}{1+\xi}}\varphi(\xi)\, d\xi = \sum_{k=1}^{m} w_k \varphi(\xi_k),$$

where

$$\xi_k = \cos\left(\frac{2\pi k}{2m+1}\right) \qquad \text{and} \qquad w_k = (1-\xi_k)\frac{2\pi}{2m+1}. \tag{A.9}$$

The same scheme may be applied to the integrals

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\varphi(x)\, dx = \int_{-1}^{1} \sqrt{\frac{1-\xi}{1+\xi}}\varphi(-\xi)\, d\xi = \sum_{k=1}^{m} w_k \varphi(x_k),$$

where

$$x_k = -\xi_k = -\cos(2\pi k/(2m+1)). \tag{A.10}$$

The additional assumption $m = N$ leads to the lifting-lines method. The equations may then be written in the form

$$\{v(p)\} = [P][W][K(M,p)][W][P]^T\{a(p)\}, \tag{A.11}$$

where $[P]$ and $[\hat{K}(M,p)]$ are square matrices of order N with the elements

$$[P]_{ij} = P_{j-1}(\xi_i), \quad \left[\hat{K}(M,p)\right]_{ij} = \hat{K}(M,x_i,\xi_j,p), \quad \text{where} \quad x_i = -\xi_i\,. \tag{A.12}$$

The matrix $[W]$ is a diagonal matrix composed of Gauss–Jacobi quadrature weights (A.9), and the orthogonality condition may be expressed also in the matrix form

$$[P][W][P]^T = [I]. \tag{A.13}$$

Introducing new vectors to describe the upwash distribution

$$\{\widehat{w}(p)\} = ([P][W])^{-1}\{v(p)\} = \frac{1}{\pi}[P]^T\{v(p)\}, \tag{A.14}$$

and the pressure distribution

$$\{\hat{c}_p(p)\} = [W][P]^T\{a(p)\}, \tag{A.15}$$

the equations (A.9) may be simplified to the form

$$\{\widehat{w}(p)\} = \left[\hat{K}(M,p)\right]\{\hat{c}_p(p)\}\,. \tag{A.16}$$

This system of algebraic equations corresponds to the system (4.7) considered in the paper. For an arbitrary upwash distribution $\widehat{w}(x,p)$, the vector $\{\widehat{w}(p)\}$ may by calculated from the expressions (A.7) and (A.14). When the function is a polynomial in x of degree less than $2N-1$, then the elements of the vector are equal

$$\{\widehat{w}(p)\}_k = \hat{w}(x_k,p), \qquad k = 1,2,...,N,$$

where x_k are defined in (A.10). In the case of piecewise linear functions, the integrals (A.7) may be calculated exactly [24].

The solution of the system (A.16) determines the pressure distribution (A.3), because

$$\{a(p)\} = \left([W][P]^T\right)^{-1}\{\hat{c}_p(p)\} = \frac{1}{\pi}[P]\{\hat{c}_p(p)\} \tag{A.17}$$

It is possible to correct the solution and to take into account the logarithmic singularities in the pressure distribution at upwash discontinuities [24]. The projection of the piston theory results on the finite-dimensional space used at the discretization (A.3) was described in [20].

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