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ON DYNAMICS OF THIN PLATES WITH A PERIODIC STRUCTURE

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A new modelling approach to thin elastic Kirchhoff plates with a periodic structure along the midplane based on that given in [7] is shown. The main feature of this model is that it describes the length-scale effect on the plate dynamics, which is neglected in the known asymptotic theories of periodic composite plates. The structural model, which takes into account also the effect of the rotational inertia, and the comparison between this model and the local models without the length-scale effect are presented.

1. Introduction

The formulation of different modelling approaches to composite mechanics of periodic structures is motivated by the fact that the exact analysis of microperiodic heterogeneous materials within solid mechanics can be carried out only for a few special problems. In general, the equations of micromechanics are too complicated to constitute the basis for investigations of most engineering problems. That is why different approximate models of periodic heterogeneous materials and structures are formulated. These models are often called *the macro-models* and investigate the effect of constituents only as averaged properties of a body within the framework of the macromechanics of composite materials. Many macromodelling procedures make it possible to detect the micromechanical behaviour of a composite. We outline below some trends in the formulation of approximate (asymptotic) theories for periodic structures and in particular, for plates with a periodic structure.

The known macromodelling methods can be separated into two groups. We can deal with special and general macromodelling procedures. The first group consists of methods developed for special types of composite materials, e.g. for laminated composites, for fibrous composites, for solids with inclusions; however, the second group consists of general procedures, in which no restrictions are imposed on distribution of constituents within the periodicity cell. These methods have the practical meaning provided that they can be applied to the analysis of special types of composites.

In this paper a new macromodelling procedure is presented, which is related to the non-stationary processes in periodic plates, where the size of the periodicity cell plays a crucial role and cannot be neglected. The effect of the microstructure size on the macrobehaviour of a body is called the length-scale effect. In order to estimate this effect, we will investigate both – *the macromodel with the length-scale effect* and *the local macromodel*, in which this effect is neglected.

The main efforts in creating the new macromodels in dynamics of composites are posed on formulation of special methods. We can mention here the effective stiffness theories for periodically laminated composites introduced in [1].

Among the general methods we can mention those based on the asymptotic homogenization approach (e.g. [3, 8]). The results of macromodelling are determined by equations with constant coefficients (called the effective moduli). In the case of periodic plates, these approximate methods were presented e. g. in [5, 6] (where a "technical" theory of anisotropic plates was presented), in [4] (in which periodic plates were investigated, using two small parameters – thickness of a plate h and the characteristic size of a periodicity unit cell ε), in [9] on thin plates with rapidly varying thickness, in [11] (where periodic plates were investigated in the framework of theories with microlocal parameters), or in [10] (where the homogenized stiffnesses were analysed). In the homogenized models we investigate a certain "substitutional" plate with constant effective stiffnesses and mass densities. Using asymptotic methods, these averaged moduli have to be determined for every periodic structure by obtaining solutions to certain variational problems posed on the periodicity cell. The formulation of macromodels by using the asymptotic homogenization methods is rather complicated from the computational point of view. This is why the asymptotic procedures are restricted to the first approximation. Within this approximation, we obtain local macromodels, which neglect the length-scale effect on the behaviour of a plate. To formulate the length-scale macromodels in the framework of asymptotic homogenization, we have to consider the higher steps in the formal asymptotic procedure, [13].

In many non-stationary processes the length-scale effect on the macrobehaviour of composites cannot be neglected. In this case, we have to use the length-scale models. We are mainly interested in models which are physically reasonable and simple enough to be applied in the analysis of engineering problems.

The macromodels of this kind were applied to selected dynamic problems of periodic structures in papers [14, 15, 17, 2] (on dynamics of periodic plates based on the Reissner – Hencky assumptions), [12] (on dynamics of periodic wavy plates), [7] (on dynamics of thin periodic plates based on the Kirchhoff assumptions), cf. also the monograph [16]. The problem of modelling the thin elastic

Kirchhoff plates having the microperiodic structure in planes parallel to the midplane was started in the contribution [7] and continued here.

The results of macromodelling presented in this paper are called *the structural macrodynamics of periodic plates.* The word "*structural*" is related to the fact that the obtained equations describe the length-scale effect (the effect of the size of the periodicity unit cell) on the dynamic plate behaviour. In this paper we will investigate the linear-elastic plates having the microperiodic structure and satisfying the assumptions of the Kirchhoff plate theory. The theory presented here can describe plates, in which the thickness is a periodic function and whose material and inertia properties are also periodic functions of the Cartesian coordinates parametrizing the plate midplane.

The structural approach shown in this paper is different from the known modelling methods of asymptotic homogenization. Using this new approach we can investigate a certain class of motions of microperiodic plates, but do not estimate effective moduli of these plates.

The general thesis of this research is that in many dynamic problems for periodic plates the length-scale effects cannot be neglected. The obtained equations involve terms which depend on the size of the periodicity cell, and show the effect of the plate microstructure parameter on the plate behaviour. Here, three structural models are presented. The first of them is the general structural model, which takes account also of the effect of the rotational inertia. The other models are the structural models with some simplifications. Moreover, from the proposed structural (refined) theory of periodic plates, by scaling down the microstructure parameter, a certain special effective stiffness theory can be derived. Models obtained in the framework of this theory will be called the local models. Comparing these theories the microperiodic aspect of this problem can be investigated only within the framework of the structural theory.

2. Preliminaries

Throughout the paper subscripts α, β, ... $(i, j, ...)$ run over 1, 2 (over 1, 2, 3) and indices A, B, ... run over 1, ..., N. Summation convention holds for all the aforementioned indices.

Let $0x_1x_2x_3$ be the orthogonal Cartesian coordinate system in the physical space. Setting $\mathbf{x} \equiv (x_1, x_2)$ and $z \equiv x_3$, we assume that the region of underformed plate is defined by $\Omega := \{(\mathbf{x}, z) : -h(\mathbf{x})/2 < z < h(\mathbf{x})/2,\ \mathbf{x} \in \Pi\}$, where Π is the region of midplane and $h(\mathbf{x})$ is the plate thickness at a point $\mathbf{x} \in \Pi$, cf. Fig. 1. We shall denote by $\Delta := (0, l_1) \times (0, l_2)$ the periodicity unit cell on $0x_1x_2$ plane, where l_1, l_2 are length dimensions sufficiently small compared to L_Π, which is the

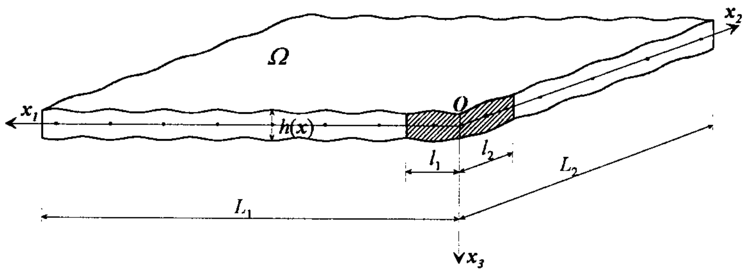

FIG. 1. Example of the microperiodic plate.

minimum characteristic length dimension of Π. The size of the cell is described by *the microstructure length parameter* l (defined by $l := \sqrt{l_1^2 + l_2^2}$, where $l \ll L_\Pi$, $\lambda_l \equiv l/L_\Pi$ and $\lambda_l \ll 1$). We assume that $h(\mathbf{x})$ is a Δ-periodic function of $\mathbf{x}$ and all material and inertial properties of the plate are also Δ-periodic functions of $\mathbf{x}$ and even functions of z. For an arbitrary integrable Δ-periodic function $f(\cdot)$ we define

$$\langle f \rangle := (l_1 l_2)^{-1} \int_\Delta f(\mathbf{x})\, da,$$

where $\langle f \rangle$ is the averaged (constant) value of f. We also define t as the time coordinate.

Our considerations will be based on the Kirchhoff plate theory assumptions and carried out within the framework of the linear elasticity theory. The analysis of these plates was presented in the paper [7]. Below, we will quote the general formulations of the theory under consideration.

Let u_i, e_{ij}, s_{ij} stand for the displacements, strains and stresses. Denoting by a_{ijkl} components of the elastic moduli tensor and assuming that $z = \text{const}$ are material symmetry planes ($a_{3\alpha\beta\gamma} = 0$, $a_{333\gamma} = 0$), we shall define $c_{\alpha\beta\gamma\delta} := a_{\alpha\beta\gamma\delta} - a_{\alpha\beta33}a_{\gamma\delta33}(a_{3333})^{-1}$.

2.1. The Kirchhoff plate equations

The direct description of linear-elastic thin periodic plates will be governed by:

(i) *the strain-displacement equations*

$$e_{ij} = u_{(i,j)}\,; \tag{2.1}$$

(ii) *the stress-strain relations in the form*

$$s_{\alpha\beta} = c_{\alpha\beta\gamma\delta} e_{\gamma\delta}\,, \tag{2.2}$$

with the plane stress assumption $s_{33} = 0$;

(iii) *the kinematic relations*

$$u_\alpha(\mathbf{x}, z, t) = -zw_{,\alpha}(\mathbf{x}, t), \qquad u_3(\mathbf{x}, z, t) = w(\mathbf{x}, t), \tag{2.3}$$

where $w(\mathbf{x}, t)$ are the displacements of points of the midplane Π; and by

(iv) *the virtual work principle*

$$\int_{\Pi}\int_{-h/2}^{h/2} \varrho \ddot{u}_i \delta u_i \, dz \, da + \int_{\Pi}\int_{-h/2}^{h/2} (s_{\alpha\beta}\delta e_{\alpha\beta} + 2s_{\alpha 3}\delta e_{\alpha 3}) \, dz \, da \tag{2.4}$$
$$= \int_{\Pi} \left[p^+ \delta u_3 \left(\mathbf{x}, \frac{h}{2}\right) + p^- \delta u_3 \left(\mathbf{x}, -\frac{h}{2}\right) \right] da + \int_{\Pi}\int_{-h/2}^{h/2} \varrho \delta u_3 \, dz \, da,$$

which has to hold for every virtual displacement δu_i. These displacements are defined in the form $\delta u_\alpha = -z\delta w_{,\alpha}(\mathbf{x})$, $\delta u_3 = \delta w(\mathbf{x})$ and $\delta u_i = 0$ on the boundary $\partial\Pi$ of the midplane. Functions $c_{\alpha\beta\gamma\delta}$, ϱ, h, in the general case, are arbitrary regular Δ-periodic functions of $\mathbf{x}$.

Because our considerations are based on the above relations, we will assume, that periodicity cells $\Delta(\mathbf{x})$ ($\Delta(\mathbf{x}) = \mathbf{x} + \Delta$, $\Delta(\mathbf{x}) \subset \Pi$) have the form of thin plates. Using the known modelling procedures, from Eqs. (2.1)÷(2.4) we shall obtain the partial differential equation of the fourth order for $w(\mathbf{x}, t)$. This is the known Kirchhoff plate theory equation involving $w(\mathbf{x}, t)$. However, for the microperiodic plates the equation includes highly oscillating Δ-periodic coefficients and that is why it does not constitute a proper analytical tool for a computational analysis of special problems. On the other hand, different homogenization macro-modelling approaches (the effective stiffness plate theories), governed by different equations with constant coefficients, do not describe the length-scale effect on the dynamic plate behaviour. In order to retain this effect we propose below the new approach which will be called the refined theory (structural model) of microperiodic thin plates, cf. [7].

3. Modelling approach

The foundations of the refined theory describing the dynamic behaviour of microperiodic composite plates will be based on certain heuristic hypotheses, which were presented in papers [14, 17]. In order to formulate these hypotheses, we will recall two auxiliary concepts, introduced in [14, 15, 16] and [7].

The first is the concept of a *macrofunction*. Function F defined on Π (which can also depend on the time coordinate t), related to the microstructural length parameter l and to a certain accuracy parameter ε_F, will be called *macrofunction* if for every $x', x'' \in \Pi$ the condition $\|x' - x''\| < l \Rightarrow |F(x') - F(x'')| < \varepsilon_F$ is satisfied. If F is a differentiable function and conditions of this form also hold for all derivatives of F, then F is called a *regular macrofunction.*

Let $f(\cdot)$ be a Δ-periodic continuous function and let $F(\cdot)$ stand for a continuous macrofunction defined on Π. Because we want to investigate certain micromotions, which are possible in the frame of the unit cell Δ, the form of the cell depends on the class of these analysed micromotions. Hence, the cell Δ can be assumed as one, two or a few repeated elements of the plate under consideration. Let us approximate the region Π by a sum $\operatorname{int}\bigcup\overline{\Delta}(\mathbf{x})$, $\mathbf{x} \in \Lambda$, of mutually disjoint cells $\Delta(\mathbf{x})$, where Λ is a lattice of points on Π such that $\Delta(\mathbf{x}) \subset \Pi$ for every $\mathbf{x} \in \Lambda$. Under this assumption we obtain that

$$\begin{aligned} \int_{\Pi} f(\mathbf{x})F(\mathbf{x})\,da &= \sum_{\mathbf{x}\in\Lambda}\ \int_{\Delta(\mathbf{x})} f(\mathbf{x}')F(\mathbf{x}')\,da' + \mathcal{O}(\lambda_l), \\ \int_{\Delta(\mathbf{x})} f(\mathbf{x}')F(\mathbf{x}')\,da' &= \int_{\Delta(\mathbf{x})} f(\mathbf{x}')F(\mathbf{x})\,da' + \mathcal{O}(\varepsilon_F), \\ \sum_{\mathbf{x}\in\Lambda}\ \int_{\Delta(\mathbf{x})} f(\mathbf{x}')F(\mathbf{x}')\,da' &= \int_{\Pi} \langle f\rangle F(\mathbf{x})\,da + \mathcal{O}(\lambda_l) + \mathcal{O}(\varepsilon_F). \end{aligned} \tag{3.1}$$

We will use these formulae in the modelling procedure leading to a refined theory of the composite plates under consideration.

The second auxiliary concept is *the microshape function system.* It is a system of N linear-independent Δ-periodic functions

$$g^A = g^A(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2) \in \mathbf{R}^2, \quad A = 1, 2, ..., N,$$

which satisfies the following conditions:

(i) g^A are continuous together with their first and second order derivatives;

(ii) $g^A(\mathbf{x}) \in \mathcal{O}(l^2)$, $g^A_{,\alpha}(\mathbf{x}) \in \mathcal{O}(l)$, $g^A_{,\alpha\beta}(\mathbf{x}) \in \mathcal{O}(1)$ for every $\mathbf{x} \in \mathbf{R}^2$;

(iii) and $\langle \mu g^A\rangle = 0$, $\langle g^A{}_{,\alpha}\rangle = 0$.

Generally speaking, every linear combination of microshape functions in an arbitrary but fixed periodicity cell $\Delta(\mathbf{x})$ ($\Delta(\mathbf{x}) = \mathbf{x} + \Delta$, $\Delta(\mathbf{x}) \subset \Pi$) has to describe disturbances of the plate deflections $w(\mathbf{x}', t)$, $\mathbf{x}' \in \Delta(\mathbf{x})$, caused by the periodic plate microstructure. Hence, the choice of these functions depends on the problem under consideration (the form of the periodicity cell, for instance – for a symmetrical cell it is necessary to define a certain symmetrical microshape function, the class of micromotions, which we want to investigate) and the accuracy of modelling. As a simple example of these functions we can assume N

functions of the form $l^2 \sin(n\pi x_1/l_1)\sin(m\pi x_2/l_2)$, where n, m are positive integers. For cells with the complicated form we have to take more functions g^A.

The formulation of the refined 2D-theory for the microperiodic thin plates will be based on Eqs. (2.1)÷(2.4) and on the following hypotheses (which were presented in [7]):

• MACRO-KINEMATIC HYPOTHESIS (MKH)

This hypothesis is based on the assumption that the midplane plate deflections can be given by

$$w(\mathbf{x},t) = W(\mathbf{x},t) + g^A(\mathbf{x})V^A(\mathbf{x},t), \tag{3.2}$$

where g^A are postulated *a priori* microshape functions and W, V^A are arbitrary linear independent macrofunctions. The first term in Eq. (3.2) describes the effect of the plate macrostructure and the second term describes disturbances in the plate deflections caused by the microperiodic structure of the plate. The form of MKH is based on the fact, that an arbitrary function, in this case – a function describing the effect of the microperiodic structure on plate deflections, can be presented by the Fourier series. This series can be approximated by the sum of the first N terms $g^A(\mathbf{x})V^A(\mathbf{x},t)$, $A = 1,...,N$, $N \geq 1$, where N has to be specified in every problem under consideration. Hence, we do not obtain the functions g^A as solutions to certain local problems posed on the periodicity cell, what is made in the known asymptotic homogenization approaches (cf. [4, 9]). These functions are certain coefficients of the Fourier series and their form depends on the class of micromotions, which will be analysed. In most cases every g^A can represent a certain form of free vibrations inside the postulated *a priori* cell $\Delta(\mathbf{x})$. Hence, the choice of their form is related to the assumed unit cell Δ, which can be defined as one, two or a few repeated elements of the plate under consideration. Moreover, these micromotions must be referred to the mass centre of the cell, hence the condition $\langle \mu g^A \rangle = 0$ assumed for functions g^A. Functions V^A can be counted as some amplitudes of disturbances caused by the microstructure of the plate. Macrofunctions W, V^A represent the new unknown kinematic fields of the refined theory of thin microperiodic plates. They are called macrodeflections and inhomogeneity correctors, respectively.

• VIRTUAL WORK HYPOTHESIS (VWH)

The principle of virtual work (2.4) is assumed to hold for every virtual displacement field satisfying the conditions

$$\delta w(\mathbf{x}) = \delta W(\mathbf{x}) + g^A(\mathbf{x})\delta V^A(\mathbf{x}), \tag{3.3}$$

where δW, δV^A are arbitrary regular and linear independent macrofunctions.

• Macro-Modelling Hypothesis (MMH)

In calculations of integrals over Π in the principle of virtual work (2.4) combined together with Eqs. (2.1)÷(2.3) and (3.2)÷(3.3), terms of order $\mathcal{O}(\lambda_l)$, $\mathcal{O}(\varepsilon_F)$ will be neglected, where F run over $W(\cdot,t$, $W_{,\alpha\beta}(\cdot,t)$, $\ddot{W}(\cdot,t)$, $\ddot{W}_{,\alpha}(\cdot,t)$, $V^A(\cdot,t)$, $V^A_{,\alpha}(\cdot,t)$, $\ddot{V}^A(\cdot,t)$, ... This assumption will be applied to the formulae of the form (3.1).

We outline now the general line of the modelling procedure in the analysis of the dynamic plate behaviour. This procedure leading to the refined theory of the thin microperiodic plates has to be carried out by means of:

(i) combining together the principle of virtual work (2.4) and Eqs. (2.1)÷(2.3), (3.2)÷(3.3);

(ii) introducing fields averaged over the thickness of the plate;

(iii) averaging the Δ-periodic functions in the obtained equations.

Using the three modelling hypotheses (MKH, VWH, MMH), from the Kirchhoff theory relations (2.1)÷(2.4) by applying the divergence theorem as well as the du Bois - Reymond lemma, we obtain the system of differential equations for the kinematic fields W, V^A, what will be presented in the next section.

In order to write down the governing equations of the refined macrotheory of microperiodic plates, we shall introduce the following notations for the Δ-periodic functions

$$\mu := \int_{-h/2}^{h/2} \varrho\, dz, \qquad j := \int_{-h/2}^{h/2} \varrho z^2 \varrho\, dz, \qquad d_{\alpha\beta\gamma\delta} := \int_{-h/2}^{h/2} z^2 c_{\alpha\beta\gamma\delta}\, dz. \tag{3.4}$$

After some calculations we obtain

$$\begin{gathered}
\int_\Pi \Big[\langle d_{\alpha\beta\gamma\delta}\rangle W_{,\alpha\beta\gamma\delta} + \langle d_{\alpha\beta\gamma\delta} g^B_{,\gamma\delta}\rangle V^B_{,\alpha\beta} + \langle\mu\rangle \ddot{W} + \langle \mu g^B\rangle \ddot{V}^B \\
- \langle j\rangle \ddot{W}_{,\alpha\alpha} - \langle j g^B_{,\alpha}\rangle \ddot{V}^B_{,\alpha} - (p + b\langle\mu\rangle)\Big] \delta W\, da = 0, \\
\int_\Pi \Big[\langle d_{\alpha\beta\gamma\delta} g^A_{,\alpha\beta}\rangle W_{,\gamma\delta} + \langle d_{\alpha\beta\gamma\delta} g^A_{,\gamma\delta} g^B_{,\alpha\beta}\rangle V^B + \langle\mu g^A\rangle \ddot{W} + \langle\mu g^A g^B\rangle \ddot{V}^B \\
+ \langle j g^A_{,\alpha}\rangle \ddot{W}_{,\alpha} + \langle j g^A_{,\alpha} g^B_{,\alpha}\rangle \ddot{V}^B - b\langle\mu g^A\rangle\Big] \delta V^A\, da = 0
\end{gathered} \tag{3.5}$$

Using the above formulae, the governing equations of the refined theory of microperiodic plates can be derived. These equations will be presented in the subsequent section.

4. Governing equations of the refined theory (structural models)

The macromodelling procedure based on the assumptions formulated above yields a system of equations for macrodeflections W and inhomogeneity correctors V^A representing what are called the structural models.

4.1. The general structural model

Introducing averaged values of Δ-periodic functions occurring in (3.5) by

$$D_{\alpha\beta\gamma\delta} \equiv \langle d_{\alpha\beta\gamma\delta} \rangle, \qquad D^A_{\alpha\beta} \equiv \langle d_{\alpha\beta\gamma\delta} g^A_{,\gamma\delta} \rangle, \qquad D^{AB} \equiv \langle d_{\alpha\beta\gamma\delta} g^A_{,\gamma\delta} g^B_{,\alpha\beta} \rangle, \tag{4.1}$$

using the condition $\langle \mu g^A \rangle = 0$ (from assumptions for functions g^A) and after some manipulations we obtain the following system of equations with constant coefficients:

$$\begin{aligned} D_{\alpha\beta\gamma\delta} W_{,\alpha\beta\gamma\delta} + D^B_{\alpha\beta} V^B_{,\alpha\beta} + \langle \mu \rangle \ddot{W} - \langle j \rangle \ddot{W}_{,\alpha\alpha} - \underline{\langle j g^B_{,\alpha} \rangle} \ddot{V}^B_{,\alpha} &= p + b\langle \mu \rangle, \\ D^A_{\alpha\beta} W_{,\alpha\beta} + D^{AB} V^B + \underline{\langle \mu g^A g^B \rangle} \ddot{V}^B + \underline{\langle j g^A_{,\alpha} \rangle} \ddot{W}_{,\alpha} + \underline{\langle j g^A_{,\alpha} g^B_{,\alpha} \rangle} \ddot{V}^B &= 0. \end{aligned} \tag{4.2}$$

We have arrived at the system of $N+1$ differential equations with constant coefficients. Equations (4.2) can be used as a basis for computational analysis of the thin microperiodic plates. The values of underlined terms in Eqs. (4.2) depend on the microstructure size (on the microstructure length parameter l) and describe the length-scale effect on the plate behaviour. Moreover, the terms with the Δ-periodic function j describe the effect of rotational inertia.

Equations (4.2) can be written in the alternative form. To this end, we introduce the following functions depending on $\mathbf{x} = (x_1, x_2)$, $\mathbf{x} \in \Pi$, and the time t

$$\begin{aligned} M_{\alpha\beta} &= D_{\alpha\beta\gamma\delta} W_{,\gamma\delta} + D^B_{\alpha\beta} V^B, \\ M^A &= D^{AB} V^B + D^A_{\alpha\beta} W_{,\alpha\beta}\,. \end{aligned} \tag{4.3}$$

In this way, from Eqs. (4.2) we obtain the following equations:

$$\begin{aligned} M_{\alpha\beta,\alpha\beta} + \langle \mu \rangle \ddot{W} - \langle j \rangle \ddot{W}_{,\alpha\alpha} - \underline{\langle j g^B_{,\alpha} \rangle} \ddot{V}^B_{,\alpha} &= p + b\langle \mu \rangle, \\ M^A + \underline{\langle \mu g^A g^B \rangle} \ddot{V}^B + \underline{\langle j g^A_{,\alpha} \rangle} \ddot{W}_{,\alpha} + \underline{\langle j g^A_{,\alpha} g^B_{,\alpha} \rangle} \ddot{V}^B &= 0. \end{aligned} \tag{4.4}$$

Equations (4.3) are called *the constitutive equations* and Eqs. (4.4) are *the equations of motion.*

4.2. The structural model without rotational inertia terms

Neglecting in Eqs. (2.3) the rotational inertia terms (involving j), we obtain the governing equations in the form

$$
\begin{aligned}
D_{\alpha\beta\gamma\delta}W_{,\alpha\beta\gamma\delta} + D^{B}_{\alpha\beta}V^{B}_{,\alpha\beta} + \langle\mu\rangle\ddot{W} &= p + b\langle\mu\rangle,\\
D^{A}_{\alpha\beta}W_{,\alpha\beta} + D^{AB}V^{B} + \underline{\langle\mu g^{A}g^{B}\rangle}\ddot{V}^{B} &= 0.
\end{aligned}
\tag{4.5}
$$

This is also the system of $N+1$ differential equations, with the new basic unknowns W, V^A. The underlined terms describe the length-scale effect on the dynamic plate behaviour.

4.3. The model without terms of order $\mathcal{O}(l^4)$

Neglecting in Eqs. (2.3) terms $\langle\mu g^{A}g^{B}\rangle \in \mathcal{O}(l^4)$, we obtain the following system of equations:

$$
\begin{aligned}
D_{\alpha\beta\gamma\delta}W_{,\alpha\beta\gamma\delta} + D^{B}_{\alpha\beta}V^{B}_{,\alpha\beta} + \langle\mu\rangle\ddot{W} - \langle j\rangle\ddot{W}_{,\alpha\alpha} - \underline{\langle j g^{B}_{,\alpha}\rangle}\ddot{V}^{B}_{,\alpha} &= p + b\langle\mu\rangle,\\
D^{A}_{\alpha\beta}W_{,\alpha\beta} + D^{AB}V^{B} + \underline{\langle j g^{A}_{,\alpha}\rangle}\ddot{W}_{,\alpha} + \underline{\langle j g^{A}_{,\alpha}g^{B}_{,\alpha}\rangle}\ddot{V}^{B} &= 0.
\end{aligned}
\tag{4.6}
$$

The underlined terms in the above system describe the length-scale effect on the dynamic plate behaviour.

Let us observe that in the above equations (4.2), (4.5) and (4.6), the basic unknowns are the macrodeflections W and inhomogeneity correctors V^A, $A = 1, \ldots, N$.

At the end of this section let us consider a thin plate strip made of a homogeneous isotropic material and having the l-periodic thickness along the x_1-axis (the periodicity unit cell is defined in the form $\Delta_1 := (0, l)$). The thickness of the plate h is assumed for $x = x_1 \in \Delta_1 = (0, l)$ in the form

$$
h(x) = \begin{cases} h_1 & \text{if } x \in ((1-\lambda)l/2, (1+\lambda)l/2),\\ h_2 & \text{if } x \in [0, (1-\lambda)l/2] \cup [(1+\lambda)l/2, l],\end{cases}
\tag{4.7}
$$

where l is the microstructure parameter, λ is a real number from $[0, 1]$, which defines relation between the length size of the cell part having the thickness h_1 and the length size of the cell l, which is the microstructure parameter. For the sake of simplicity we assume only one microshape function $g = g^1 = l^2[\cos(2\pi x/l) + c]$, where c is a constant derived from the condition $\langle \mu g \rangle = 0$. After some manipulations we obtain from Eqs. (4.1) the following relations:

$$
\begin{aligned}
D_{1111} &= \frac{E}{12(1-\nu^2)}[\lambda h_1^3 + (1-\lambda)h_2^3], \\
D_{11}^1 &= \frac{\pi E}{3(1-\nu^2)} \sin(\pi\lambda)(h_1^3 - h_2^3), \\
D^{11} &= \frac{2\pi^3 E}{3(1-\nu^2)}[(h_1^3 - h_2^3)[\pi\lambda + \sin(\pi\lambda)\cos(\pi\lambda)] + \pi h_2^3],
\end{aligned} \tag{4.8}
$$

where E is the constant Young modulus and ν is the constant Poisson ratio. The above averaged values can be viewed as certain "effective stiffnesses" of the plate under consideration. This example will be continued in the next paper [18].

5. Governing equations of the local models

Local models of dynamics of thin microperiodic plates can be derived from Eqs. (2.1)÷(2.4) by the asymptotic homogenization approach in which the plate microstructure is scaled down. In this way, by setting $l \searrow 0$, we arrive at the asymptotic approximation of the refined theory in which we neglect the underlined terms in Eqs. (4.2). From Eqs. $(4.2)_2$ we obtain for V^A the system of linear algebraic equations

$$
D^{AB}V^A = -D^B_{\gamma\delta}W_{,\gamma\delta}\,.
$$

It can be shown that the $N \times N$ matrix D^{AB} is non-singular. To this end we can use the expression of the strain energy:

$$
\mathcal{E} = \int_{\Pi}\int_{-h/2}^{h/2} \frac{1}{2} c_{\alpha\beta\gamma\delta} e_{\alpha\beta} e_{\gamma\delta}\, dz\, da\,.
$$

Using expressions (2.1), (2.3), (3.2), (3.1), the operator of an averaging and MMH we can expand the above formula of $\mathcal{E}$ to the form:

$$
\mathcal{E} \cong \frac{1}{2}\int_{\Pi} \Big(D_{\alpha\beta\gamma\delta}W_{,\alpha\beta}W_{,\gamma\delta} + D^A_{\alpha\beta}W_{,\gamma\delta}V^A + D^B_{\gamma\delta}W_{,\alpha\beta}V^B + D^{AB}V^AV^B \Big)\, da\,.
$$

Because the energy is positive definite, we can confirm that the matrix D^{AB} is non-singular. Denoting by E^{AB} elements of the inverse matrix of D^{AB}, we can eliminate inhomogeneity correctors from the governing equations by means of

$$
V^A = -E^{AB}D^B_{\gamma\delta}W_{,\gamma\delta}\,. \tag{5.1}
$$

Let us denote

$$B_{\alpha\beta\gamma\delta} \equiv D_{\alpha\beta\gamma\delta} - D^A_{\alpha\beta} E^{AB} D^B_{\gamma\delta}, \tag{5.2}$$

and define by $B_{\alpha\beta\gamma\delta}$ the coefficients, which can be called "effective stiffnesses" of the microperiodic plate in the framework of local models under consideration.

Using Eq. (3.1) to eliminate functions V^A from Eqs. (4.2) and Eq. (5.2), we arrive at the following equations of local models with or without the rotational inertia terms.

5.1. The general local model

If we neglect in Eqs. (2.3) the underlined terms, then we arrive at

$$B_{\alpha\beta\gamma\delta} W_{,\alpha\beta\gamma\delta} + \langle\mu\rangle \ddot{W} - \langle j\rangle \ddot{W}_{,\alpha\alpha} = p + b\langle\mu\rangle, \tag{5.3}$$

where $B_{\alpha\beta\gamma\delta}$ are the coefficients defined by Eq. (5.2) and the term with the function j denotes the effect of the rotational inertia.

5.2. The local model without rotational inertia terms

Neglecting in Eqs. (4.2) also the terms with the Δ-periodic function j, we obtain the one governing equation in the form

$$B_{\alpha\beta\gamma\delta} W_{,\alpha\beta\gamma\delta} + \langle\mu\rangle \ddot{W} = p + b\langle\mu\rangle. \tag{5.4}$$

Let us observe that for a homogeneous plate with a constant thickness we obtain from Eq. (5.4) the governing equation of the classical Kirchhoff plate theory.

Models of thin microperiodic plates governed by Eq. (5.3) or by Eq. (5.4) will be called the local models of dynamics. The coefficients (5.2), considered as certain "effective stiffnesses" in the framework of dynamics local models, are defined for the class of disturbances described by the postulated *a priori* microshape function system. The formulae of the effective stiffnesses can be obtained using different asymptotic approximations ([2, 3 or 11]). All the effective stiffness models (local models) neglect the length-scale effect of the size of the unit cell Δ on the macrobehaviour of the plate. These models are independent of the microstructure length l.

For an isotropic homogeneous plate band with the l-periodic thickness, which was considered at the end of the Sec. 4, we obtain from Eq. (5.2) the relation for the "effective stiffness" in the form

$$B_{1111} = D_{1111} - (D^1_{11})^2 (D^{11})^{-1},$$

where D_{1111}, D^1_{11}, D^{11} are defined by Eqs. (4.8). This example will be continued in the next paper [18].

The governing equations of three structural models of thin microperiodic plates presented in the Sec. 4 (Eqs. (4.2), (4.5) and (4.6)) and also the equations of local models from this section (Eq. (5.3) and (5.4)), will be used to investigate the length-scale and rotational inertia effects in the second part of this paper [18].

6. Conclusions

The relations of the Kirchhoff plate theory (Eqs. (2.1)÷(2.4)) lead to well known differential equation of the fourth order for the deflection function $w(\mathbf{x},t)$, $\mathbf{x} = (x_1, x_2) \in \Pi$, $t \in (t_0, t_f)$. However, for microperiodic plates we obtain the differential equation with highly oscillating Δ-periodic coefficients ($\Delta := (0,l_1) \times (0,l_2)$ is the periodicity unit cell on the midplane Π). From the computational point of view, this equation is too complicated to constitute the basis for investigations of engineering problems. That is why different approximate models for periodic plates have been proposed. These theories are mostly based on the asymptotic procedures, which scale the microstructure of the plate down and neglect the length-scale effect on the dynamic macrobehaviour of the plate. In order to take this effect into account, we have proposed the new refined theory of thin microperiodic plates. In our considerations we make use of the Kirchhoff relations (Eqs. (2.1)÷(2.4)) and three additional modelling hypotheses (MKH, VWH, MMH). Formulating these hypotheses we apply the concept of a regular macrofunction and the system of microshape functions. Finally we obtain a system of differential equations with constant coefficients for macrodeflections $W(\mathbf{x},t)$ and inhomogeneity correctors $V^A(\mathbf{x},t)$, $\mathbf{x} = (x_1,x_2)$, $A = 1,...,N$. This approach is typical for the refined (structural) macrodynamics of microperiodic bodies, which was presented in papers [14, 15, 17 and 16], and is different than the known modelling procedures based on the theory of asymptotic homogenization. Below, we will summarize the equations obtained in Secs. 4. and 5.

• The general differential equations of the structural model are Eqs. (4.2). This is the system of $N+1$ differential equations with constant coefficients, which consists of one partial differential equation of the fourth order for macrodeflections W, and N ordinary differential equations of the second order for inhomogeneity correctors V^A, $A = 1,...,N$. The underlined terms describe the length-scale effect (the effect of the microstructure size length parameter $l = \sqrt{l_1^2 + l_2^2}$) on the dynamic plate behaviour. Moreover, in these equations we can see the rotational inertia terms (the terms with the Δ-periodic function j).

• However, the system of $N+1$ differential equations (4.5) describe the structural model which neglects the terms with the Δ-periodic function j. These equations take into account only the length-scale effect on the dynamic plate behaviour.

• The equations (4.6) are obtained in order to consider the effect of the rotational inertia. In this way we have neglected the terms of the order $\mathcal{O}(l^3)$ and higher. These equations constitute the system of $N+1$ differential equations in macrodeflections and inhomogeneity correctors.

• In the framework of the asymptotic approximation approach we have only one partial differential equation with constant coefficients (5.3) for one unknown function – macrodeflections W. We obtain this equation by neglecting the effect of the microstructure size on the dynamic plate behaviour. We neglect in Eqs. (4.2) the terms of the order $\mathcal{O}(l)$, $\mathcal{O}(l^2)$, $\mathcal{O}(l^3)$, $\mathcal{O}(l^4)$. Thus from $(4.2)_2$ we obtain the system of N linear algebraic equations for inhomogeneity correctors V^A. We can eliminate them by substituting into $(4.2)_1$ relations of V^A. The coefficients $B_{\alpha\beta\gamma\delta}$, which are found in the equation (5.3), are called the "effective stiffnesses" and we calculate them from Eq. (5.2).

• However, the equation (5.4) is the special form of the local model. This equation does not contain the term with the function j, which describes the effect of the rotational inertia on the dynamic plate behaviour. It has been neglected in it the underlined terms of Eqs. (4.2).

Hence, we can observe that the local models do not describe the length-scale effect on the dynamic behaviour of plates under consideration. To investigate this effect for thin plates with a periodic structure, we have to use the structural models (determined by Eqs. (4.2) or (4.5) or also (4.6)), which take into consideration the effect of the microstructure length parameter l. A detailed discussion of these models will be given in a subsequent paper, [18].

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