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Research Paper

A Generalized Hypothesis of Elastic Energy Equivalence in Continuum Damage Mechanics

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A new generalized hypothesis of elastic energy equivalence is proposed. The proposed generalized hypothesis is inclusive of all the existing different hypotheses of equivalence in continuum damage mechanics and all are obtained as special cases. Specifically, the hypothesis of elastic strain equivalence and the hypothesis of elastic energy equivalence are obtained as special cases of the generalized hypothesis proposed here. In addition, the generalized hypothesis has some unusual properties when the integer exponent n approaches infinity. In particular, it turns out that the strain energy density function is a vector for even values of the integer exponent. This conclusion is totally unexpected but an attempt is made to explain this result based on geometry.

Key words: damage mechanics, energy equivalence, generalized hypothesis, thermodynamics, vector exponentiation.

Notations

ϕ – scalar cross-sectional damage variable,

A – cross-sectional area in the deformed/damaged configuration,

$\overline{A}$ – cross-sectional area in the effective/undamaged configuration,

σ – Cauchy stress,

$\overline{\sigma}$ – effective Cauchy stress,

ε – elastic strain,

$\overline{\varepsilon}$ – effective elastic strain,

n – integer exponent with values $n = 0, 1, 2, 3, \ldots$,
$\{\sigma\}$ – vector representation of the second-rank stress tensor,
$\{\overline{\sigma}\}$ – vector representation of the second-rank effective stress tensor,
$\{\varepsilon\}$ – vector representation of the second-rank strain tensor,
$\{\overline{\varepsilon}\}$ – vector representation of the second-rank effective strain tensor,
U – elastic strain energy density in the deformed/damaged configuration,
$\overline{U}$ – effective elastic strain energy density (in the effective/undamaged configuration),
$\{x\}$ – arbitrary vector,
α – a scalar,
$[A]$ – a matrix.

1. Introduction

The subject of continuum damage mechanics was pioneered by Kachanov [13] and more fully developed by Lee *et al.* [22], Voyiadjis and Kattan [32, 35–37], Sidoroff[30], Krajcinovic [18], and Kattan and Voyiadjis [15–17].

In 1958 Kachanov [13] pioneered the concept of effective stress and introduced the topic of continuum damage mechanics. This development was followed by Rabotnov [27] and by others later [16, 17, 19, 32, 36, 37, 43]. In the framework of continuum damage mechanics, a scalar damage variable ϕ is introduced that has values in the range $0 \leq \phi \leq 1$. Thus the value of the damage variable is zero when the virgin material is undamaged while the value approaches 1 upon complete rupture. However, practically the damage value cannot exceed 0.3 to 0.4 without violating the concept of a continuum mechanics.

Research on damage mechanics accelerated rapidly in the past few years [5, 7, 9, 14, 15, 20, 22–25, 28, 31, 33, 35]. Another approach to damage characterization is to use an entropy generation rate as a damage metric rather than the damage potential surface [1, 2]. Recently, Voyiadjis and Kattan [43] introduced new concepts in damage mechanics such as damageability and integrity of materials, applied damage mechanics to graphene [39], and formulated a decomposition scheme for the degradation of elastic stiffness in damaged solids [38].

This work consists of two main sections. In Sec. 2, the one-dimensional scalar formulation is presented. In this case, the mathematical derivation is simple and straightforward. However, the situation changes radically when the general three-dimensional tensorial formulation is attempted in Sec. 3. In this case, the mathematical formulation is complicated and resort is made to adopting a new matrix operation in order to reach conclusions. The new matrix operation involves raising a vector to an integer exponent. In this respect, some unexpected results are obtained. For example, it is concluded that (1) the vector evolves into a matrix as the value of the integer exponent approaches infinity provided it is an even number, and (2) the strain energy density function is a vector in four-dimensional space for even values of the integer exponent. An attempt is made to explain these results in terms of the geometry of space.

2. One-dimensional (scalar) formulation

Consider a body (in the form of a cylinder) in the initial undeformed and undamaged configuration. Consider also the configuration of the body that is both deformed and damaged after a set of external agencies act on it (see Fig. 1). Next, consider a fictitious configuration of the body obtained from the damaged configuration by removing all the damage that the body has undergone, i.e. this is the state of the body after it had only deformed without damage (see Fig. 1). Therefore, in defining a damage variable ϕ, its value must vanish in the fictitious configuration.

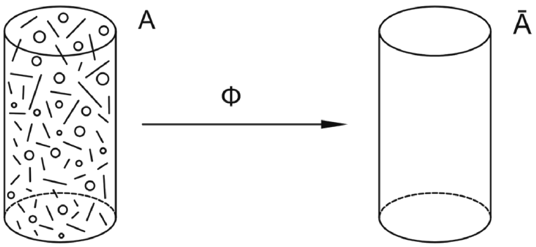

Fig. 1. Damaged and fictitious undamaged configurations.

The first damage variable ϕ is usually defined as follows:

$$\phi = \frac{A - \overline{A}}{A}, \tag{2.1}$$

where A is the cross-sectional area in the damaged configuration while $\overline{A}$ is the cross-sectional area in the fictitious configuration with $A > \overline{A}$. It is clear that when a body is undamaged, i.e. when $A = \overline{A}$, then $\phi = 0$.

The stress in the fictitious configuration is called the effective stress and is denoted by $\overline{\sigma}$. The value of the effective stress $\overline{\sigma}$ may be obtained using the equilibrium relation $\overline{\sigma}\overline{A} = \sigma A$ where σ is the stress in the damaged configuration. Therefore, using this relation along with the definition in Eq. (2.1), one obtains:

$$\overline{\sigma} = \frac{\sigma}{1 - \phi}. \tag{2.2}$$

It should be mentioned that the equilibrium condition in the paragraph above reflects a mean-field type of assumption on the stress redistribution (uniform over the resistive section) and therefore appears to be appropriate only in the dilute damage regime, away from the stress-strain peak where cooperate effects dominate and damage localization takes place.

Next, one needs to utilize one of the equivalence hypotheses that are used in the literature of damage mechanics. For this purpose, one introduces the following generalized hypothesis of elastic energy equivalence of order n. The expression for the generalized hypothesis was introduced recently by the authors within the context of the theory of undamageable materials [34, 40–42, 45]. It will be shown that all the equivalence hypotheses of damage mechanics can be obtained as special cases of this generalized hypothesis:

$$\frac{1}{2}\overline{\sigma}\,\overline{\varepsilon}^n = \frac{1}{2}\sigma\varepsilon^n, \tag{2.3}$$

where $n = 0, 1, 2, 3, \ldots$. The classical hypothesis of elastic energy equivalence is obtained as a special case of Eq. (2.3) when $n = 1$. Next one investigates what happens to Eq. (2.3) when $n \to \infty$. To investigate this extreme and hypothetical case, one raises both sides of Eq. (2.3) to the power $1/n$ to obtain:

$$\frac{1}{2}\overline{\sigma}^{1/n}\,\overline{\varepsilon} = \frac{1}{2}\sigma^{1/n}\,\varepsilon. \tag{2.4}$$

Substituting $n \to \infty$ into Eq. (2.4) and noting that $\overline{\sigma}^{1/\infty} \to 1$ and $\sigma^{1/\infty} \to 1$, Eq. (2.4) reduces to:

$$\overline{\varepsilon} = \varepsilon. \tag{2.5}$$

It is noted that the expression given in Eq. (2.5) represents exactly the hypothesis of elastic strain equivalence that is used frequently in the literature. The simple derivation in Eqs. (2.3)–(2.5) establishes the *missing link* in damage mechanics and shows that the hypothesis of elastic strain equivalence is a special case of the hypothesis of elastic energy equivalence of order n when $n \to \infty$. This extreme and special case is hypothetical, is not physically based, and should not be used in practical applications. Table 1 shows a summary of the results thus far.

Table 1. Various hypotheses of equivalence shown as special cases of Eq. (2.3).

n	Equation	Hypothesis title
$n \to 0$	$\overline{\sigma} = \sigma$	Hypothesis of stress equivalence
$n \to 1$	$\frac{1}{2}\overline{\sigma}\,\overline{\varepsilon} = \frac{1}{2}\sigma\varepsilon$	Hypothesis of elastic energy equivalence
$n \to 2$	$\frac{1}{2}\overline{\sigma}\,\overline{\varepsilon}^2 = \frac{1}{2}\sigma\varepsilon^2$	Hypothesis of elastic energy equivalence of order 2
...	...	...
...	...	...
$n \to \infty$	$\overline{\varepsilon} = \varepsilon$	Hypothesis of elastic strain equivalence

3. General three-dimensional (tensorial) formulation

In this section the mathematical equations are systematically and consistently developed for the generalized hypothesis of elastic energy equivalence using tensors. The vector and matrix representation of tensors is utilized. The use of tensors thus implies the applicability of the equations to general three-dimensional states of deformation and damage.

The formulation starts with the generalized form of the elastic energy density of order n. One then proceeds to derive systematically an expression for the generalized elastic constitutive relation for these types of materials. One starts with the following general expression:

$$U = \frac{1}{2}\{\sigma\}^{\mathrm{T}}\{\varepsilon\}^{n}. \tag{3.1}$$

Next, a new method of vector exponentiation is proposed in. This method is proposed in order to account for the strain vector that raised to an integer exponent in Eq. (3.1). Therefore, a new operation in matrix algebra is proposed. The operation deals with raising a vector to an integer exponent. A vector $\{x\}$ is considered to be raised to the integer exponent n. Before proceeding with the new operation of vector exponentiation, one first considers the following two well known vector operations [3, 4, 6, 8, 10–12, 21, 26, 29, 46]:

The *dot product* $\{x\}^{\mathrm{T}}\{x\}$ (also know as the *inner product*, and the *scalar product*) is given by the following equation for a vector $\{x\} = [x_1\ x_2\ x_3]^{\mathrm{T}}$, where the superscript "T" indicates the transpose operation:

$$\{x\}^{\mathrm{T}}\{x\} = [x_1\ x_2\ x_3]\left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right\} = x_1^2 + x_2^2 + x_3^2 = \alpha, \tag{3.2}$$

where α is a scalar. This product always produces a scalar. Note that this dot product lowers the vector from a one-dimensional vector to a zero-dimensional scalar.

The *dyadic product* $\{x\}^{\mathrm{T}}\{x\}$ (also known as the *outer product*, *matrix product* or the *tensor product*) is given by the following equation for the same vector $\{x\}$ considered above:

$$\{x\}^{\mathrm{T}}\{x\} = \left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right\}[x_1\ x_2\ x_3] = \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2^2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3^2 \end{bmatrix} = [A], \tag{3.3}$$

where $[A]$ is a square matrix or a tensor represented by a matrix. This product always produces a matrix or a tensor that may be represented by a matrix. Note that this dyadic product raises the vector from a one-dimensional vector to a two-dimensional matrix.

There is a third vector product that is used in the literature called the *cross product* or the *vector product* but this product does not play a role in the new proposed operation of vector exponentiation. The cross product is not discussed here. The interested reader may refer to [3, 4, 6, 8, 10–12, 21, 26, 29, 46] for details about this product.

Next one proceeds to define the operation of raising the vector $\{x\}$ to the integer exponent n. In the next discussion one will have to consider the cases when n takes the values 1, 2, 3, 4, and 5 in detail to try to arrive at a general conclusion and to observe the pattern that is emerging. As will be seen in the next few equations, the operation of vector exponentiation is transformative if certain conditions are satisfied by the integer exponent n.

$\underline{n=1}$: There is only one outcome as follows:

$$\{x\}^1 = \{x\}. \tag{3.4}$$

In this case the result is a vector and indeed it is obviously the same vector. So raising a vector to the exponent 1 does not change the vector and is similar to raising a scalar or a matrix to the exponent 1. The strain energy density U in this case becomes:

$$U = \frac{1}{2}\{\sigma\}^{\mathrm{T}}\{\varepsilon\}^1 = \frac{1}{2}\{\sigma\}^{\mathrm{T}}\{\varepsilon\}. \tag{3.5}$$

Clearly from the above expression, the strain energy density is a scalar for $n = 1$.

$\underline{n=2}$: There are two possible outcomes as follows:

$$\{x\}^2 = \begin{cases} \{x\}^{\mathrm{T}}\{x\} = \alpha, \\ \{x\}\{x\}^{\mathrm{T}} = [A]. \end{cases} \tag{3.6}$$

Thus is clear that raising a vector to the exponent 2 has two possible outcomes. The first outcome produces a scalar α while the second outcome produces a matrix $[A]$. The probabilities of each outcome are listed below with the symbol P denoting probability:

$$P(\text{scalar},\ n=2) = \frac{1}{2}, \tag{3.7}$$

$$P(\text{matrix},\ n=2) = \frac{1}{2}. \tag{3.8}$$

In this case, the strain energy density U becomes:

$$U = \begin{cases} \dfrac{1}{2}\{\sigma\}^{\mathrm{T}}\alpha = \text{vector}, \\[2ex] \dfrac{1}{2}\{\sigma\}^{\mathrm{T}}[A] = \text{vector}. \end{cases} \tag{3.9}$$

Clearly from the above expression, the strain energy density is a vector for $n = 2$.

$\underline{n = 3}$: There are two possible outcomes but both produce a vector as follows:

$$\{x\}^3 = \begin{cases} \{x\}2\,\{x\} = \alpha\,\{x\}, \\ \{x\}^2\,\{x\} = [A]\,\{x\}. \end{cases} \tag{3.10}$$

Clearly raising a vector to the exponent 3 always produces a vector although two possible different vectors can be produced.

In this case, the strain energy density U becomes:

$$U = \begin{cases} \dfrac{1}{2}\,\{\sigma\}^{\mathrm{T}}\,\alpha\,\{\varepsilon\} = \text{scalar}, \\[2ex] \dfrac{1}{2}\,\{\sigma\}^{T}\,[A]\,\{\varepsilon\} = \text{scalar}. \end{cases} \tag{3.11}$$

Clearly from the above expressions, the strain energy density is a scalar for $n = 3$.

$\underline{n = 4}$: There are four possible outcomes as follows:

$$\{x\}^4 = \begin{cases} \{x\}^2\,\{x\}^2 = \alpha\cdot\alpha = \alpha^2, \\ \{x\}^2\,\{x\}^2 = \alpha\cdot[A] = \alpha\,[A], \\ \{x\}^2\,\{x\}^2 = [A]\cdot\alpha = \alpha\,[A], \\ \{x\}^2\,\{x\}^2 = [A]\cdot[A] = [A]^2\,. \end{cases} \tag{3.12}$$

Thus it is clear that raising a vector to the exponent 4 has four possible outcomes. The first outcome produces a scalar α^2 while the three other outcomes produce matrices, namely $\alpha\,[A]$ (twice) and $[A]^2$. The probabilities of each outcome are listed below with the symbol P denoting probability:

$$P(\text{scalar},\ n = 4) \ = \ \frac{1}{2}, \tag{3.13}$$

$$P(\text{matrix},\ n = 4) \ = \ \frac{3}{4}. \tag{3.14}$$

In this case, the strain energy density U becomes:

$$U = \begin{cases} \frac{1}{2}\{\sigma\}^{\mathrm{T}}\alpha^2 = \text{vector}, \\ \frac{1}{2}\{\sigma\}^{\mathrm{T}}\alpha\,[A] = \text{vector}, \\ \frac{1}{2}\{\sigma\}^{\mathrm{T}}\alpha\,[A] = \text{vector}, \\ \frac{1}{2}\{\sigma\}^{\mathrm{T}}[A]^2 = \text{vector}. \end{cases} \tag{3.15}$$

Clearly from the above expressions, the strain energy density is a vector for $n = 4$.

$\underline{n = 5}$: There are four possible outcomes but all produce a vector as follows:

$$\{x\}^5 = \begin{cases} \{x\}^2\{x\}^2\{x\} = \alpha\cdot\alpha\cdot\{x\} = \alpha^2\{x\}, \\ \{x\}^2\{x\}^2\{x\} = \alpha\cdot[A]\cdot\{x\} = \alpha\,[A]\{x\}, \\ \{x\}^2\{x\}^2\{x\} = [A]\cdot\alpha\cdot\{x\} = \alpha\,[A]\{x\}, \\ \{x\}^2\{x\}^2\{x\} = [A]\cdot[A]\cdot\{x\} = [A]^2\{x\}. \end{cases} \tag{3.16}$$

Clearly raising a vector to the exponent 5 always produces a vector although four possible different vectors can be produced.

In this case, the strain energy density U becomes:

$$U = \begin{cases} \frac{1}{2}\{\sigma\}^{\mathrm{T}}\alpha^2\{\varepsilon\} = \text{scalar}, \\ \frac{1}{2}\{\sigma\}^{\mathrm{T}}\alpha\,[A]\{\varepsilon\} = \text{scalar}, \\ \frac{1}{2}\{\sigma\}^{\mathrm{T}}\alpha\,[A]\{\varepsilon\} = \text{scalar}, \\ \frac{1}{2}\{\sigma\}^{\mathrm{T}}[A]^2\{\varepsilon\} = \text{scalar}. \end{cases} \tag{3.17}$$

Clearly from the above expressions, the strain energy density is a scalar for $n = 5$.

$\underline{n=6}$: There are eight possible outcomes as follows:

$$\{x\}^6 = \begin{cases} \{x\}^4\{x\}^2 = \alpha^2 \cdot \alpha = \alpha^3, \\ \{x\}^4\{x\}^2 = \alpha^2 \cdot [A] = \alpha^2 [A], \\ \{x\}^4\{x\}^2 = \alpha [A] \cdot \alpha = \alpha^2 [A], \\ \{x\}^4\{x\}^2 = \alpha [A] \cdot [A] = \alpha [A]^2, \\ \{x\}^4\{x\}^2 = \alpha [A] \cdot \alpha = \alpha^2 [A], \\ \{x\}^4\{x\}^2 = \alpha [A] \cdot [A] = \alpha [A]^2, \\ \{x\}^4\{x\}^2 = [A]^2 \cdot \alpha = \alpha [A]^2, \\ \{x\}^4\{x\}^2 = [A]^2 \cdot [A] = [A]^3. \end{cases} \tag{3.18}$$

Thus it is clear that raising a vector to the exponent 6 has eight possible outcomes. The first outcome produces a scalar α^3 while the three other outcomes produce matrices, namely $\alpha^2 [A]$ (three times), $\alpha [A]^2$ (three times), and $[A]^3$. The probabilities of each outcome are listed below with the symbol P denoting probability:

$$P(\text{scalar},\ n=6) = \frac{1}{8}, \tag{3.19}$$

$$P(\text{matrix},\ n=6) = \frac{7}{8}. \tag{3.20}$$

In this case, the strain energy density U becomes:

$$U = \begin{cases} \frac{1}{2}\{\sigma\}^{\mathrm{T}} \alpha^3 = \text{vector}, \\ \frac{1}{2}\{\sigma\}^{\mathrm{T}} \alpha^2 [A] = \text{vector}, \\ \frac{1}{2}\{\sigma\}^{\mathrm{T}} \alpha^2 [A] = \text{vector}, \\ \frac{1}{2}\{\sigma\}^{\mathrm{T}} \alpha [A]^2 = \text{vector}, \\ \frac{1}{2}\{\sigma\}^{\mathrm{T}} \alpha^2 [A] = \text{vector}, \\ \frac{1}{2}\{\sigma\}^{\mathrm{T}} \alpha [A]^2 = \text{vector}, \\ \frac{1}{2}\{\sigma\}^{\mathrm{T}} \alpha [A]^2 = \text{vector}, \\ \frac{1}{2}\{\sigma\}^{\mathrm{T}} [A]^3 = \text{vector}. \end{cases} \tag{3.21}$$

Clearly from the above expressions, the strain energy density is a vector for $n = 6$.

$\underline{n = 7}$: There are eight possible outcomes but all eight produce vectors as follows:

$$\{x\}^7 = \begin{cases} \{x\}^4 \{x\}^2 \{x\} = \alpha^2 \cdot \alpha \cdot \{x\} = \alpha^3 \{x\}, \\ \{x\}^4 \{x\}^2 \{x\} = \alpha^2 \cdot [A] \cdot \{x\} = \alpha^2 [A] \{x\}, \\ \{x\}^4 \{x\}^2 \{x\} = \alpha [A] \cdot \alpha \cdot \{x\} = \alpha^2 [A] \{x\}, \\ \{x\}^4 \{x\}^2 \{x\} = \alpha [A] \cdot [A] \cdot \{x\} = \alpha [A]^2 \{x\}, \\ \{x\}^4 \{x\}^2 \{x\} = \alpha [A] \cdot \alpha \cdot \{x\} = \alpha^2 [A] \{x\}, \\ \{x\}^4 \{x\}^2 \{x\} = \alpha [A] \cdot [A] \cdot \{x\} = \alpha [A]^2 \{x\}, \\ \{x\}^4 \{x\}^2 \{x\} = [A]^2 \cdot \alpha \cdot \{x\} = \alpha [A]^2 \{x\}, \\ \{x\}^4 \{x\}^2 \{x\} = [A]^2 \cdot [A] \cdot \{x\} = [A]^3 \{x\}. \end{cases} \tag{3.22}$$

Clearly raising a vector to the exponent 7 always produces a vector although eight possible different vectors can be produced.

In this case, the strain energy density U becomes:

$$U = \begin{cases} \frac{1}{2} \{\sigma\}^{\mathrm{T}} \alpha^3 \{\varepsilon\} = \text{scalar}, \\ \frac{1}{2} \{\sigma\}^{\mathrm{T}} \alpha^2 [A] \{\varepsilon\} = \text{scalar}, \\ \frac{1}{2} \{\sigma\}^{\mathrm{T}} \alpha^2 [A] \{\varepsilon\} = \text{scalar}, \\ \frac{1}{2} \{\sigma\}^{\mathrm{T}} \alpha [A]^2 \{\varepsilon\} = \text{scalar}, \\ \frac{1}{2} \{\sigma\}^{\mathrm{T}} \alpha^2 [A] \{\varepsilon\} = \text{scalar}, \\ \frac{1}{2} \{\sigma\}^{\mathrm{T}} \alpha [A]^2 \{\varepsilon\} = \text{scalar}, \\ \frac{1}{2} \{\sigma\}^{\mathrm{T}} \alpha [A]^2 \{\varepsilon\} = \text{scalar}, \\ \frac{1}{2} \{\sigma\}^{\mathrm{T}} [A]^3 \{\varepsilon\} = \text{scalar}. \end{cases} \tag{3.23}$$

Clearly from the above expressions, the strain energy density is a scalar for $n = 7$.

$\underline{n = 8}$: There are 16 possible outcomes that will not be listed. However, it is noted that only one outcome produces a scalar while the other 15 outcomes produce matrices. Thus the probabilities for this case are as follows:

$$P(\text{scalar},\ n = 8) = \frac{1}{16}, \tag{3.24}$$

$$P(\text{matrix},\ n = 8) = \frac{15}{16}. \tag{3.25}$$

In this case, the strain energy density U turns out to be a vector.

$\underline{n = 9}$: There are 16 possible outcomes but all 16 produce vectors although different vectors can come out of this operation. In this case, the strain energy density U turns out to be a scalar.

$\underline{n = 10}$: There are 32 possible outcomes that will not be listed. However, it is noted that only one outcome produces a scalar while the other 31 outcomes produce matrices. Thus the probabilities for this case are as follows:

$$P(\text{scalar},\ n = 10) = \frac{1}{32}, \tag{3.26}$$

$$P(\text{matrix},\ n = 10) = \frac{31}{32}. \tag{3.27}$$

In this case, the strain energy density U turns out to be a vector.

The above results are summarized in Table 2 (when is an odd integer) and Table 3 (when is an even integer) as follows.

Table 2. Summary: n is an odd integer.

n	1	3	5	7	9	∞
$(n+1)/2$	1	2	3	4	5	
P(vector) decimal	1	1	1	1	1	1
U	scalar	scalar	scalar	scalar	scalar	scalar

Table 3. Summary: n is an even integer.

n	2	4	6	8	10	∞
$n/2$	1	2	3	4	5	∞
$\frac{1}{2^{n/2}}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	0
P(matrix) decimal	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	$\frac{31}{32}$	1
P(matrix) percentage	50%	75%	87.5%	93.75%	96.875%	100%
U	vector	vector	vector	vector	vector	vector

In general and based on the patterns emerging from the above results, and Tables 2 and 3, one arrives at the following conclusions:

1. When the integer exponent n is odd, the vector exponentiation always produces a vector. The nature of the vector never changes under this operation and for this range of values for the exponent. The results of this case are summarized in Table 2.
2. When the integer exponent n is even, the vector exponentiation changes the nature of the vector. This is very interesting as the vector can devolve into a scalar or evolve into a matrix. The nature of the vector is changed under this operation and for the values of exponent specified. The results are summarized in Table 3. It is shown that for low even values of the integer exponent, e.g. $n = 2$, the probability of the vector becoming a scalar is 0.5 and the probability of the vector becoming a matrix is also 0.5. However, as the even values of the exponent become larger, the probability of the vector becoming a scalar diminishes considerably while the probability of the vector becoming a matrix increases exponentially. In this case the probability of the vector becoming a matrix is given by the following expression;

$$P\,(\text{matrix},\ n\ \text{even}) = 1 - \frac{1}{2^{n/2}}, \tag{3.28}$$

$$\lim_{n\to\infty}\left(1 - \frac{1}{2^{n/2}}\right) = 1. \tag{3.29}$$

   It is noted from the above expression that as $n \to \infty$, the probability becomes 100% and the vector is guaranteed to become a matrix. Thus at infinity, the vector undergoes a transformation – it changes from a one-dimensional vector into a two-dimensional matrix. Thus the vector evolves and crosses dimensions. This transformation of the vector happens only when the integer exponent has an even value and approaches infinity.
3. The generalized strain energy density U turns out to be a scalar (as expected) but only for odd values of the integer exponent n. The generalized strain energy density U becomes a vector (unexpected) for even values of the integer exponent n. A possible explanation is that the generalized strain energy density function postulated here operates in a four-dimensional space. In the 4D space, the strain energy density function is a vector when n is even. This is possibly interpreted that as the 4D energy density vector intersects 3D space, a point is projected that represents the scalar strain energy density in the three-dimensional space. Thus the scalar strain energy density that is used in continuum and solid mechanics can be defined as the intersection of the 4D energy density vector with the 3D space (check Figs. 2 and 3). But this interpretation applies when n is even. The other case when n is odd does not require this interpretation as the strain energy density is scalar in both 3D and 4D space.

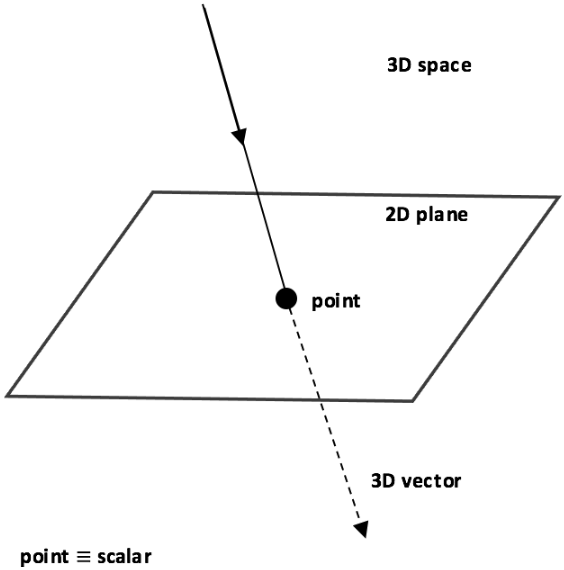

Fig. 2. A schematic diagram of a 3D vector intersecting a 2D plane.

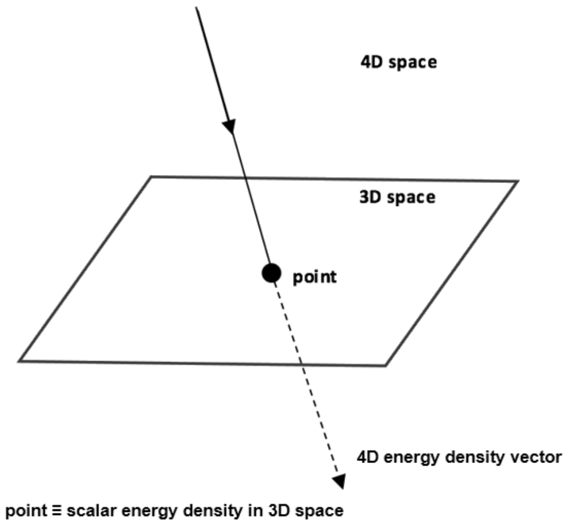

Fig. 3. A schematic diagram showing a 4D energy density vector intersecting a 3D space.

An attempt is made to try to explain the unexpected result obtained for the strain energy density when the integer exponent n is even. By analogy, looking at Fig. 2, one finds that the intersection of a three-dimensional vector (3D vector) with the two-dimensional plane (3D plane) is a point that represents a scalar. Similarly, looking at Fig. 3, one finds that the intersection of a four-dimensional strain energy density vector (4D energy density vector) with the three-dimensional space (3D space) is a point that represents the scalar strain energy density. Thus, the usual scalar strain energy density may be defined as the intersection of the 4D energy density vector with the 3D space. But this is possible only for even values of the integer exponent. For odd values of the exponent, the scalar energy density is a scalar in both the 3D space and 4D space. Several questions arise as a result of this: (1) what is exactly the integer exponent n? What is its physical significance? What is the role that n plays in the nature of the strain energy density function? Why is it that the nature of strain energy density changes depending on whether the integer exponent n is odd or even? All these questions need to be answered in future research on this topic.

Finally it is shown how the strain energy density approaches zero as the value of n approaches infinity. Based on the results for the values of n from 1 to 7 shown above, it can be shown that for any odd value of n, there are only four distinct expressions that the strain energy density U can take (check equations (3.17) and (3.23)). They are listed below for odd n values:

$$U = \begin{cases} \dfrac{1}{2}\{\sigma\}^{\mathrm{T}}\alpha^{\frac{n-1}{2}}\{\varepsilon\} = \text{scalar}, \\[2ex] \dfrac{1}{2}\{\sigma\}^{\mathrm{T}}\alpha^{\frac{n-3}{2}}[A]\{\varepsilon\} = \text{scalar}, \\[2ex] \dfrac{1}{2}\{\sigma\}^{\mathrm{T}}\alpha\,[A]^{\frac{n-3}{2}}\{\varepsilon\} = \text{scalar}, \\[2ex] \dfrac{1}{2}\{\sigma\}^{\mathrm{T}}[A]^{\frac{n-1}{2}}\{\varepsilon\} = \text{scalar}. \end{cases} \tag{3.30}$$

The other four possible expressions for the strain energy density for even values of n are listed below:

$$U = \begin{cases} \dfrac{1}{2}\{\sigma\}^{\mathrm{T}}\alpha^{\frac{n}{2}} = \text{vector}, \\[2ex] \dfrac{1}{2}\{\sigma\}^{\mathrm{T}}\alpha^{\frac{n-2}{2}}[A] = \text{vector}, \\[2ex] \dfrac{1}{2}\{\sigma\}^{\mathrm{T}}\alpha\,[A]^{\frac{n-2}{2}} = \text{vector}, \\[2ex] \dfrac{1}{2}\{\sigma\}^{\mathrm{T}}[A]^{\frac{n}{2}} = \text{vector}. \end{cases} \tag{3.31}$$

Based on the various expressions of Eqs. (3.30) and (3.31) shown above, and considering the fact that both the values of α and the norm (and elements) of $[A]$

are less than 1 (they involve the strain which is usually between 0 and 1), one can write the following limits when n is odd:

$$\lim_{n\to\infty} \alpha^{\frac{n-1}{2}} = 0, \tag{3.32}$$

$$\lim_{n\to\infty} \alpha^{\frac{n-3}{2}} = 0, \tag{3.33}$$

$$\lim_{n\to\infty} [A]^{\frac{n-1}{2}} = [0], \tag{3.34}$$

$$\lim_{n\to\infty} [A]^{\frac{n-3}{2}} = [0]. \tag{3.35}$$

Similarly, one can write the following limits when n is even:

$$\lim_{n\to\infty} \alpha^{\frac{n}{2}} = 0, \tag{3.36}$$

$$\lim_{n\to\infty} \alpha^{\frac{n-2}{2}} = 0, \tag{3.37}$$

$$\lim_{n\to\infty} [A]^{\frac{n}{2}} = [0], \tag{3.38}$$

$$\lim_{n\to\infty} [A]^{\frac{n-2}{2}} = [0]. \tag{3.39}$$

Based on the above eight limits (which exhaust all the possibilities), one concludes that the strain energy density approaches zero as the value of n approaches infinity. Thus, the final result can be written as follows:

$$\lim_{n\to\infty} U = 0. \tag{3.40}$$

Thus, the material that exists when $n \to \infty$ will have zero elastic strain energy density. This material has been called an undamageable material in the previous work of the authors [34, 40–42, 45].

4. Thermodynamic formulation with internal variables

In this section, we use the thermodynamic theory of Rice [28]. Consider a material sample of size V which is measured in an unloaded reference state and at a reference temperature T_0. Let σ (or ε), T, and ς be the thermodynamic state variables of constrained equilibrium states of the material sample, where T is the temperature and ς is a set of internal state variables that include the damage variable.

Let η be the specific free energy and ψ its Legendre transform where $\eta = \eta\left(\varepsilon, T, \varsigma\right)$ and $\psi = \psi\left(\sigma, T, f\right) = \varepsilon\frac{\partial \eta}{\partial \varepsilon} - \eta$. Let θ be the specific energy and f

be a set of conjugate thermodynamic forces to ς. Then, we have the following relations:

$$\sigma = \sigma(\varepsilon, T, \varsigma) = \frac{\partial \eta(\varepsilon, T, \varsigma)}{\partial \varepsilon}, \tag{4.1}$$

$$\varepsilon = \varepsilon(\sigma, T, \varsigma) = \frac{\partial \psi(\sigma, T, \varsigma)}{\partial \sigma}, \tag{4.2}$$

$$\theta = \theta(\sigma, T, \varsigma) = \frac{\partial \psi(\sigma, T, \varsigma)}{\partial T}. \tag{4.3}$$

The thermodynamic conjugate forces f are then given by:

$$f = V\frac{\partial \psi}{\partial \varsigma} = -V\frac{\partial \eta}{\partial \varsigma}, \tag{4.4}$$

where $f = f(\sigma, T, \varsigma)$ or $f = f(\varepsilon, T, f)$. From Eq. (4.2), we obtain the following relation for the increment of strains:

$$d\varepsilon = \frac{\partial^2 \psi}{\partial \sigma^2} d\sigma + \frac{\partial^2 \psi}{\partial \sigma\, \partial T} dT + \frac{\partial^2 \psi}{\partial \sigma \partial \varsigma} d\varsigma. \tag{4.5}$$

Finally, the flow potential $Q = Q(f, T, \varsigma)$ is given by:

$$Q = \frac{1}{V}\int_0^f \dot{\varsigma}\, df, \tag{4.6}$$

where $dQ = \frac{1}{V}\dot{\varsigma} df$.

5. Conclusion

A generalized form of the strain energy density function is proposed where the strain is raised to an integer exponent. The purpose of this introduction of this extreme form of the strain energy density function is to develop the theory of undamageable materials. It is seen that all the different hypotheses of equivalence of continuum damage mechanics can be obtained as special cases of the generalized hypothesis of equivalence proposed here. In addition, it is concluded that the strain energy density is a scalar function in both the three-dimensional and four-dimensional spaces, but only when n is an odd integer. The other case when n is an even integer is more complicated and needs a radical interpretation. When n is even, it is concluded that the strain energy density is a vector in four-dimensional space. The cross-section of the 4D energy density vector with the usual 3D space represents the usual scalar strain energy density which is represented geometrically by a point.

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