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A New Asymptotic-Tolerance Model of Dynamic Problems for Transversally Graded Cylindrical Shells

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The objects of consideration are thin linearly elastic Kirchhoff-Love-type open circular cylindrical shells having a functionally graded macrostructure and a tolerance-periodic microstructure in circumferential direction. The aim of this note is to formulate and discuss a new non-asymptotic averaged model for the analysis of selected dynamic problems for these shells. The proposed *asymptotic-tolerance model equations* have continuous and slowly varying coefficients depending also on a cell size. An important advantage of this model is that it makes it possible to study micro-dynamics of tolerance-periodic shells independently of their macro-dynamics.

Key words: thin transversally graded shells, tolerance modelling, length-scale effect.

1. Formulation of the problem, starting equations

Thin linearly elastic Kirchhoff-Love-type open circular cylindrical shells with a tolerance-periodic microstructure in circumferential direction are analysed. It means that *on the microscopic level*, the shells under consideration consist of many small elements. These elements, called *cells*, are treated as thin shells. It is assumed that *the adjacent cells are nearly identical, but the distant elements can be very different*. An example of such shell is shown in Fig. 1. At the same time, the shells have constant structure in axial direction. On the microscopic level, the geometrical, elastic and inertial properties of these shells are determined by highly oscillating non-continuous *tolerance-periodic functions* in x. By *tolerance periodic functions* we shall mean functions which in every cell can be approximated by periodic functions in x.

On the other hand, *on the macroscopic level*, the averaged (effective) properties of the shells are described by functions being *smooth* and *slowly varying*

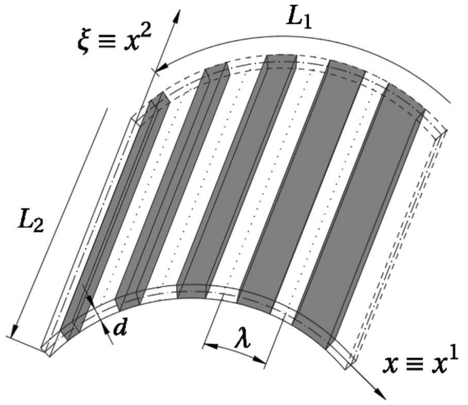

Fig. 1. An example of a shell with a tolerance-periodic microstructure.

along circumferential direction. It means that the tolerance-periodic shells under consideration can be treated as made of *functionally graded materials* (FGM), cf. [2], and called *functionally graded shells*. Moreover, since effective properties of the shells are graded in direction normal to interfaces between constituents, this gradation is referred to as *the transversal gradation*.

The dynamic problems of such shells are described by partial differential equations with highly oscillating, tolerance-periodic and non-continuous coefficients, so these equations are too complicated to apply to investigations of engineering problems. To obtain averaged equations with continuous and slowly varying coefficients, a lot of different approximate modelling methods have been proposed. Periodic and tolerance-periodic structures are usually described using *homogenized models* derived by means of *asymptotic methods*, cf. [1, 2]. Unfortunately, in the models of this kind *the effect of a microstructure size* (called *the length-scale effect*) on the overall shell behaviour is neglected. This effect can be taken into account using *the tolerance averaging technique*, cf. [4, 5]. Some applications of this method to the modelling of mechanical and thermomechanical problems for various periodic and tolerance-periodic structures are shown in many works. The extended list of papers and books on this topic can be found in [3–5].

The aim of this contribution is to formulate *a new averaged combined asymptotic-tolerance model for the analysis of selected dynamic problems for the transversally graded cylindrical shells under consideration. Governing equations of the proposed model have continuous and slowly varying coefficients depending also on a microstructure size* λ. Hence, this model makes it possible to describe the effect of a length scale on the dynamic shell behaviour. The model will be derived applying *the combined asymptotic-tolerance modelling technique* given by Woźniak in [5].

We assume that x^1 and x^2 are coordinates parametrizing the shell midsurface M in circumferential and axial directions, respectively. We denote $x \equiv x^1 \in \Omega \equiv (0,\ L_1)$ and $\xi \equiv x^2 \in \Xi \equiv (0,\ L_2)$, where L_1, L_2 are length dimensions of M, cf. Fig. 1. Let $O\,\overline{x}^1\overline{x}^2\overline{x}^3$ stand for a Cartesian orthogonal coordinate system in the physical space R^3 and denote $\overline{\mathbf{x}} \equiv (\overline{x}^1,\ \overline{x}^2,\ \overline{x}^3)$. A cylindrical shell midsurface M is given by $M \equiv \left\{\overline{\mathbf{x}} \in R^3 \colon \overline{\mathbf{x}} = \overline{\mathbf{r}}\left(x^1,\ x^2\right),\ \left(x^1,\ x^2\right) \in \Omega \times \Xi\right\}$, where $\overline{\mathbf{r}}(\cdot)$ is the smooth function such that $\partial\overline{\mathbf{r}}/\partial x^1 \cdot \partial\overline{\mathbf{r}}/\partial x^2 = 0$, $\partial\overline{\mathbf{r}}/\partial x^1 \cdot \partial\overline{\mathbf{r}}/\partial x^1 = 1$, $\partial\overline{\mathbf{r}}/\partial x^2 \cdot \partial\overline{\mathbf{r}}/\partial x^2 = 1$. It means that on M we have introduced *the orthonormal parametrization*. Sub- and superscripts α, β, ... run over 1, 2 and are related to x^1, x^2, summation convention holds. Partial differentiation related to x^α is represented by ∂_α. Moreover, it is denoted $\partial_{\alpha\ldots\delta} \equiv \partial_\alpha\ldots\partial_\delta$. Let $a^{\alpha\beta}$ stand for the midsurface first metric tensor. Under orthonormal parametrization $a^{\alpha\beta}$ is the unit tensor. The time coordinate is denoted by $t \in \mathrm{I} \equiv [t_0,\ t_1]$. Let $d(x)$ and r stand for the shell thickness and the midsurface curvature radius, respectively.

The basic cell Δ is defined by: $\Delta \equiv [-\lambda/2,\ \lambda/2]$, where λ is a cell length dimension in $x \equiv x^1$-direction. *The microstructure length parameter* λ satisfies conditions: $\lambda/d_{\max} \gg 1$, $\lambda/r \ll 1$, and $\lambda/L_1 \ll 1$.

Denote by $u_\alpha = u_\alpha(x,\ \xi,\ t)$, $w = w(x,\ \xi,\ t)$, $(x,\ \xi,\ t) \in \Omega \times \Xi \times \mathrm{I}$, the shell displacements in directions tangent and normal to M, respectively. Elastic properties of the shells are described by shell stiffness tensors $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$. Let $\mu(x)$ stand for a shell mass density per midsurface unit area. The external forces will be neglected.

It is assumed that the behaviour of the shell under consideration is described by the action functional determined by Lagrange function L being a highly oscillating function with respect to x and having the well-known form

$$(1.1)\quad L = -\frac{1}{2}\left(D^{\alpha\beta\gamma\delta}\partial_\beta u_\alpha \partial_\delta u_\gamma + \frac{2}{r}D^{\alpha\beta 11} w \partial_\beta u_\alpha \right. \\ \left. + \frac{1}{r^2}D^{1111}ww + B^{\alpha\beta\gamma\delta}\partial_{\alpha\beta}w\partial_{\gamma\delta}w - \mu a^{\alpha\beta}\dot{u}_\alpha\dot{u}_\beta - \mu\dot{w}^2\right).$$

Applying the principle of stationary action we arrive at the system of Euler-Lagrange equations, which can be written in explicit form as

$$(1.2)\quad \begin{aligned} \partial_\beta(D^{\alpha\beta\gamma\delta}\partial_\delta u_\gamma) + r^{-1}\partial_\beta(D^{\alpha\beta 11}w) - \mu a^{\alpha\beta}\ddot{u}_\beta = 0, \\ r^{-1}D^{\alpha\beta 11}\partial_\beta u_\alpha + \partial_{\alpha\beta}(B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w) + r^{-2}D^{1111}w + \mu\ddot{w} = 0. \end{aligned}$$

It can be observed that Eqs. (1.2) coincide with the well-known governing equations of Kirchhoff-Love theory of thin elastic shells. For periodic shells, coefficients $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $\mu(x)$ of (1.1) and (1.2) are highly oscillating, non-continuous and tolerance-periodic functions in x. Applying the combined asymptotic-tolerance modelling technique (cf. [5]) to lagrangian (1.1), we will derive the averaged model equations with continuous and slowly varying coefficients depending also on a cell size.

2. Modelling procedure, equations of combined model

The combined modelling technique used to starting lagrangian (1.1) is realized in two steps. The first step is based on *the consistent asymptotic averaging* of lagrangian (1.1) under *the consistent asymptotic decomposition* of fields u_α, w, in $\Delta(x) \times \Xi \times \mathrm{I}$

$$
\begin{aligned}
u_{\varepsilon\alpha}(x,z,\xi,t) &\equiv u_\alpha(x,z/\varepsilon,\xi,t) = u^0_\alpha(z,\xi,t) + \varepsilon \widetilde{h}_\varepsilon(x,z) U_\alpha(z,\xi,t), \\
w_\varepsilon(x,z,\xi,t) &\equiv w(x,z/\varepsilon,\xi,t) = w^0(z,\xi,t) + \varepsilon^2 \widetilde{g}_\varepsilon(x,z) W(z,\xi,t),
\end{aligned} \tag{2.1}
$$

where $\varepsilon = 1/m$, $m = 1,2,...$, $z \in \Delta_\varepsilon(x)$, $\Delta_\varepsilon(x) \equiv x + \Delta_\varepsilon$, $\Delta_\varepsilon \equiv (-\varepsilon\lambda/2,\, \varepsilon\lambda/2)$ (scaled cell), $x \in \Omega$, $(\xi,t) \in \Xi \times \mathrm{I}$.

Unknown functions u^0_α, w^0 and U_α, W in (2.1) are assumed to be continuous and bounded in Ω. Unknowns u^0_α, w^0 and U_α, W are called *macrodisplacements* and *fluctuation amplitudes*, respectively. They are independent of ε.

By $\widetilde{h}_\varepsilon(x,z) \equiv \widetilde{h}(x,z/\varepsilon)$ and $\widetilde{g}_\varepsilon(x,z) \equiv \widetilde{g}(x,z/\varepsilon)$ in (2.1) are denoted periodic approximations of highly oscillating *fluctuation shape functions* $h(\cdot)$ and $g(\cdot)$ in $\Delta(x)$. The *fluctuation shape functions* are assumed to be known in every problem under consideration. In this work, they have to satisfy conditions: $h \in O(\lambda)$, $\lambda\partial_1 h \in O(\lambda)$, $g \in O(\lambda^2)$, $\lambda\partial_1 g \in O(\lambda^2)$, $\lambda^2\partial_{11} g \in O(\lambda^2)$, $\langle\mu h\rangle = \langle\mu g\rangle = 0$.

Introducing decomposition (2.1) into (1.1), under limit passage $\varepsilon \to 0$ we obtained the averaged form of lagrangian (1.1). Then, applying the principle of stationary action we obtain *the governing equations of consistent asymptotic model for the tolerance-periodic shells under consideration.* These equations consist of partial differential equations for macrodisplacements u_α, w coupled with linear algebraic equations for fluctuation amplitudes U_α, W. After eliminating fluctuation amplitudes from the governing equations by means of

$$
\begin{aligned}
U_\gamma &= -(G^{-1})_{\gamma\eta}\Big[\langle\partial_1 h D^{1\eta\mu\vartheta}\rangle\,\partial_\vartheta u^0_\mu + r^{-1}\langle\partial_1 h D^{1\eta 11}\rangle\, w^0\Big], \\
W &= -E^{-1}\langle\partial_{11} g B^{11\gamma\delta}\rangle\,\partial_{\gamma\delta} w^0,
\end{aligned} \tag{2.2}
$$

where $G_{\alpha\gamma}(x) = \langle D^{\alpha 1\gamma 1}(\partial_1 h)^2\rangle(x)$, $E(x) = \langle B^{1111}(\partial_{11} g)^2\rangle(x)$, we arrive finally at *the asymptotic model equations expressed only in macrodisplacements* u_α, w

$$
\begin{aligned}
\partial_\beta(D^{\alpha\beta\gamma\delta}_h \partial_\delta u^0_\gamma + r^{-1} D^{\alpha\beta 11}_h w^0) - \langle\mu\rangle\, a^{\alpha\beta}\ddot{u}^0_\beta = 0, \\
\partial_{\alpha\beta}(B^{\alpha\beta\gamma\delta}_h \partial_{\gamma\delta} w^0) + r^{-1} D^{11\gamma\delta}_h \partial_\delta u^0_\gamma + r^{-2} D^{1111}_h w^0 + \langle\mu\rangle\, \ddot{w}^0 = 0,
\end{aligned} \tag{2.3}
$$

where

$$D_h^{\alpha\beta\gamma\delta}(x) \equiv \langle D^{\alpha\beta\gamma\delta}\rangle - \langle D^{\alpha\beta\eta 1}\partial_1 h\rangle (G^{-1})_{\eta\zeta}\langle \partial_1 h D^{1\zeta\gamma\delta}\rangle,$$

$$B_g^{\alpha\beta\gamma\delta}(x) \equiv \langle B^{\alpha\beta\gamma\delta}\rangle - \langle B^{\alpha\beta 11}\partial_{11} g\rangle E^{-1}\langle \partial_{11} g B^{11\gamma\delta}\rangle.$$

Coefficients of equations (2.3) are *slowly varying* in x but they are *independent of the microstructure cell size*. Hence, this model is not able to describe the length-scale effect on the overall shell dynamics and it will be referred to as *the macroscopic model*. The number and form of boundary/initial conditions for unknowns u_α, w are the same as in the classical shell theory governed by Eqs. (1.2).

In the first step of combined modelling it is assumed that within the asymptotic model, solutions u_α^0, w^0 to the problem under consideration are known. Hence, there are also known functions $u_{0\alpha} = u_\alpha^0 + hU_\alpha$ and $w_0 = w^0 + gW$, where U_α, W are given by means of (2.2).

The second step is based on *the tolerance averaging* of lagrangian (1.1) under so-called *superimposed decomposition.*

The fundamental concepts of the tolerance approach under consideration are those of *two tolerance relations between points and real numbers determined by tolerance parameters, slowly-varying functions, tolerance-periodic functions, fluctuation shape functions and the averaging operation*, cf. [3–5].

A continuous, bounded and differentiable function $F(\cdot)$ defined in $\overline{\Omega} \equiv [0,\ L_1]$ is called *slowly-varying of the R-th kind* with respect to cell Δ and tolerance parameters δ, $F \in SV_\delta^R(\Omega,\ \Delta)$, if it can be treated (together with its derivatives up to the R-th order) as constant on an arbitrary cell. Nonnegative integer R is assumed to be specified in every problem under consideration. An integrable and bounded function $f(\cdot)$ defined in $\overline{\Omega} \equiv [0,\ L_1]$ is called *tolerance-periodic of the R-th kind* with respect to cell Δ and tolerance parameters δ, $f \in SV_\delta^R(\Omega,\ \Delta)$, if it can be treated (together with its derivatives up to the R-th order) as periodic on an arbitrary cell.

The *averaging operator* for an arbitrary function $f(\cdot)$ being integrable and bounded in every cell is defined by:

$$\langle f\rangle(x) \equiv \frac{1}{\lambda}\int_{x-\lambda/2}^{x+\lambda/2} \tilde{f}(x,z)dz, \qquad z \in \Delta(x), \qquad x \in \Omega, \tag{2.4}$$

where $\tilde{f}(x,z)$ is a periodic approximation of $f(x)$ in $\Delta(x)$.

The tolerance modelling is based on two assumptions. The first of them is called *the tolerance averaging approximation* (tolerance relations which make it possible to neglect terms of an order of tolerance parameters δ), cf. [3–5]. The second one is termed *the micro-macro decomposition.* In the problem under consideration, we introduce *the extra micro-macro decomposition* superimposed on the known solutions $u_{0\alpha}$, w_0 obtained within the macroscopic model

$$
\begin{aligned}
u_{c\alpha}(x,\xi,t) &= u_{0\alpha}(x,\xi,t) + c(x)Q_\alpha(x,\xi,t),\\
w_b(x,\xi,t) &= w_0(x,\xi,t) + b(x)V(x,\xi,t),
\end{aligned}
\tag{2.5}
$$

where *fluctuation amplitudes* Q_α, V are *the new slowly-varying unknowns*, i.e. $Q_\alpha \in SV^1_\delta(\Omega,\Delta)$, $V \in SV^2_\delta(\Omega,\Delta)$. Functions $c(x)$ and $b(x)$ are *the new* tolerance-periodic, continuous and highly-oscillating *fluctuation shape functions* which are assumed to be known in every problem under consideration. These functions have to satisfy conditions: $c \in O(\lambda)$, $\lambda\partial_1 c \in O(\lambda)$, $b \in O(\lambda^2)$, $\lambda\partial_1 b \in O(\lambda^2)$, $\lambda^2\partial_{11} b \in O(\lambda^2)$, $\langle \mu c\rangle = \langle \mu b\rangle = 0$.

We substitute the right-hand sides of (2.5) into (1.1). The resulting lagrangian is denoted by L_{cb}. Then, we average L_{cb} over cell Δ using averaging formula (2.4) and applying *the tolerance averaging approximation.* As a result we obtain function $\langle L_{cb}\rangle$ called *the tolerance averaging of starting lagrangian* (1.1) *in* Δ *under superimposed decomposition* (2.5). Then, applying the principle of stationary action, we obtain the system of Euler-Lagrange equations for Q_α, V, which can be written in explicit form as

$$
\begin{aligned}
&\underline{\langle D^{\alpha 22\delta}(c)^2\rangle}(x)\partial_{22}Q_\delta - \langle D^{\alpha 11\delta}(\partial_1 c)^2\rangle(x)Q_\delta - \underline{\langle \mu(c)^2\rangle}(x)a^{\alpha\beta}\ddot{Q}_\beta\\
&\qquad = r^{-1}\langle D^{\alpha 111}\partial_1 c\, w_0\rangle(x) + \langle D^{\alpha\beta\gamma 1}\partial_1 c\,\partial_\beta u_{0\gamma}\rangle(x),
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
&\underline{\langle B^{2222}(b)^2\rangle}(x)\partial_{2222}V + [\underline{2\langle B^{1122}b\partial_{11}b\rangle}(x) - 4\underline{\langle B^{1212}(\partial_1 b)^2\rangle}(x)]\partial_{22}V\\
&\qquad + \langle B^{1111}(\partial_{11}b)^2\rangle(x)V + \underline{\langle \mu(b)^2\rangle}(x)\ddot{V} = -\langle B^{\alpha\beta 11}\partial_{11}b\,\partial_{\alpha\beta}w_0\rangle(x),
\end{aligned}
\tag{2.7}
$$

Equations (2.6) and (2.7) together with *the micro-macro decomposition* (2.5) constitute *the superimposed microscopic model* (i.e. microscopic model imposed on the macroscopic model obtained in the first step of combined modelling). Coefficients of the derived model equations are *smooth and slowly-varying* in x and some of them *depend on a cell size* λ (underlined terms). The right-hand sides of (2.6) and (2.7) are known under assumption that $u_{0\alpha}$, w_0 were determined in the first step of modelling. The basic unknowns Q_α, V of the model equations must be *the slowly-varying functions in tolerance periodicity direction.* This requirement can be verified only *a posteriori and it determines the range of the physical applicability of the model.* The boundary conditions for Q_α, V should be defined only on boundaries $\xi = 0$, $\xi = L_2$. The form of initial/boundary conditions are the same as in the classical shell theory governed by Eqs. (1.2).

It can be shown, that under assumption that fluctuation shape functions h, g of macroscopic model coincide with those of microscopic model we can obtain microscopic model equations, which are independent of the solutions obtained in the framework of the macroscopic model

$$\underline{\langle D^{\alpha 22\delta}(h)^2\rangle}(x)\partial_{22}Q_\delta - \langle D^{\alpha 11\delta}(\partial_1 h)^2\rangle(x)Q_\delta - \underline{\langle \mu(h)^2\rangle}(x)a^{\alpha\beta}\ddot{Q}_\beta = 0, \tag{2.8}$$

$$\begin{aligned}\underline{\langle B^{2222}(g)^2\rangle}(x)\partial_{2222}V + [2\underline{\langle B^{1122}g\partial_{11}b\rangle}(x) - 4\underline{\langle B^{1212}(\partial_1 g)^2\rangle}(x)]\partial_{22}V \\ + \langle B^{1111}(\partial_{11}g)^2\rangle(x)V + \underline{\langle \mu(g)^2\rangle}(x)\ddot{V} = 0.\end{aligned} \tag{2.9}$$

It means, that an important advantage of the combined model is that it makes *it possible to describe selected problems of the shell micro-dynamics* (e.g. the free micro-vibrations, propagation of waves related to the micro-fluctuation amplitudes) *independently of the shell macro-dynamics*. Moreover, micro-dynamic behaviour of the shell in the axial and circumferential directions can be analysed independently of its micro-dynamic behaviour in the direction normal to the shell midsurface.

3. Remarks and conclusions

The combined asymptotic-tolerance modelling procedure, proposed by Woźniak in [5], is applied to the known partial differential equations of Kirchhoff-Love-type, thin, linearly elastic cylindrical shells with transversally graded macrostructure and tolerance-periodic microstructure in circumferential direction.

Governing equations of the combined asymptotic-tolerance model, proposed in this contribution, consist of *macroscopic model equations* (2.3) formulated by means of *the consistent asymptotic procedure* which are combined with *superimposed microscopic model equations* (2.6), (2.7) derived by applying *the tolerance* (*non-asymptotic*) *modelling technique* (cf. [3–5]) and under assumption that in the framework of the macroscopic model the solution to the problem under consideration is known.

In contrast to exact shell equations (1.2) with discontinuous, highly oscillating and tolerance periodic coefficients, the combined model equations proposed here *have smooth and slowly-varying coefficients*. Moreover, *some coefficients of the superimposed microscopic model equations depend on a cell size* λ. Thus, the combined model makes it possible to analyse *the length-scale effect*.

It can be shown, that under special conditions imposed on the fluctuation shape functions we can obtain microscopic model equations (2.8), (2.9), which are independent of the solutions obtained in the framework of the macroscopic model. It means, that *an important advantage of the combined model is that* *it makes it possible to separate the macroscopic description of some special dynamic problems from the microscopic description of these problems.*

Microscopic model equations (2.8), (2.9) also describe certain *time-boundary and space-boundary phenomena* strictly related to the specific form of initial and boundary conditions imposed on unknown fluctuation amplitudes Q_α, V. That is why, these equations are referred to as *the boundary layer equations*, where the term "boundary" is related both to time and space.

Some applications of the combined model proposed here will be shown in forthcoming papers.

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