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A New Tolerance Model of Dynamic Thermoelastic Problems for Thin Uniperiodic Cylindrical Shells

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The objects of consideration are thin linearly thermo-elastic Kirchhoff-Love-type circular cylindrical shells having a periodically micro-inhomogeneous structure in circumferential direction (*uniperiodic shells*). The aim of this note is to formulate and discuss a new non-asymptotic averaged model for the analysis of selected dynamic thermoelastic problems for these shells. Contrary to the starting exact shell equations with highly oscillating, non-continuous and periodic coefficients, the proposed *tolerance model equations* have constant coefficients depending also on a cell size. Hence, an important advantage of this model is that it makes it possible to investigate the effect of a period of inhomogeneity on the global shell thermodynamics (*the length-scale effect*). This effect is neglected in the known homogenized models derived by asymptotic methods.

Key words: periodic shells, thermoelasic problems, tolerance modelling, length-scale effect.

1. Formulation of the problem, starting equations

Thin linearly thermo-elastic Kirchhoff-Love-type circular cylindrical shells with a periodically micro-heterogeneous structure in circumferential direction are analysed. Shells of this kind are termed *uniperiodic*. At the same time, the shells under consideration have constant properties in axial direction. Periodic inhomogeneity means here periodically variable shell thickness and/or periodically variable inertial, elastic and thermal properties of the shell material. The period of inhomogeneity λ is assumed to be very large compared with the maximum shell thickness and very small as compared to the midsurface curvature radius as well as the length dimension of the shell midsurface in periodicity direction. It means that the shells under consideration are composed of a large number of identical elements and every such element, called *a periodicity cell*, can be treated as a thin shell, cf. Fig. 1.

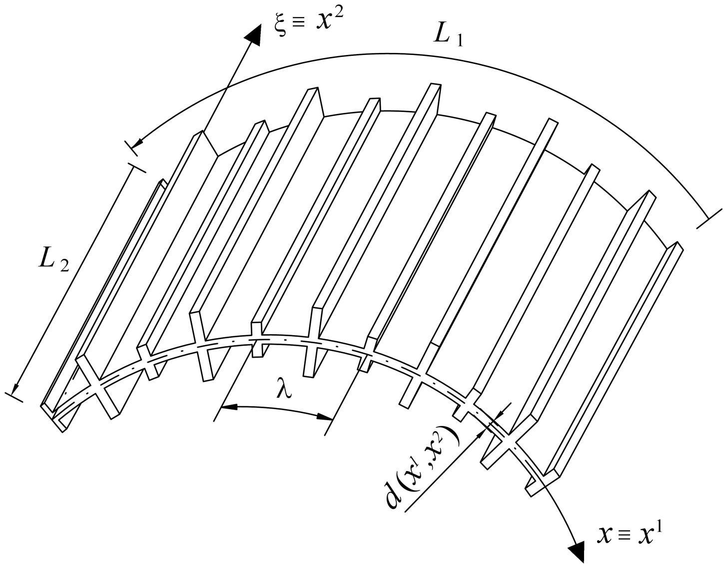

Fig. 1. An example of a shell with an uniperiodic microstructure.

The dynamic thermoelastic problems of such shells are described by partial differential equations with highly oscillating, non-continuous and periodic coefficients, so these equations are too complicated to apply to investigations of engineering problems. To obtain averaged equations with constant coefficients, a lot of different approximate modelling methods have been proposed. Periodic cylindrical shells (plates) are usually described using *homogenized models* derived by means of *asymptotic methods*, cf. [1]. Unfortunately, in the models of this kind *the effect of a microstructure size* (called *the length-scale effect*) on the overall shell behaviour is neglected. This effect can be taken into account using *the tolerance averaging technique*, cf. [2–5]. Some applications of this method to the modelling of mechanical and thermomechanical problems for various periodic and tolerance-periodic structures are shown in many works. The extended list of papers and books on this topic can be found in [3–5].

The aim of this contribution is to formulate and discuss *a new averaged tolerance 2-D model for the analysis of selected dynamic thermoelastic problems for the periodic cylindrical shells under consideration.* Contrary to the starting exact equations of the shell thermoelasticity with highly oscillating, non-continuous and periodic coefficients, *governing equations of the proposed averaged model have constant coefficients depending also on a microstructure size* λ. Hence, this model makes it possible to describe the effect of a length scale on the thermoelastic shell behaviour. The model will be derived applying *the tolerance modelling technique*, cf. [2–5].

We assume that x^1 and x^2 are coordinates parametrizing the shell midsurface Min circumferential and axial directions, respectively. We denote $x \equiv x^1 \in \Omega \equiv (0, L_1)$ and $\xi \equiv x^2 \in \Xi \equiv (0, L_2)$, where L_1, L_2 are length dimensions of M, cf. Fig. 1. Let $O\,\overline{x}^1\overline{x}^2\overline{x}^3$ stand for a Cartesian orthogonal coordinate system in the physical space R^3 and denote $\overline{\mathbf{x}} \equiv (\overline{x}^1, \overline{x}^2, \overline{x}^3)$. A cylindrical shell midsurface M is given by $M \equiv \left\{\overline{\mathbf{x}} \in R^3 : \overline{\mathbf{x}} = \overline{\mathbf{r}}\left(x^1, x^2\right),\ \left(x^1, x^2\right) \in \Omega \times \Xi\right\}$, where $\overline{\mathbf{r}}(\cdot)$ is the smooth function such that $\partial\overline{\mathbf{r}}/\partial x^1 \cdot \partial\overline{\mathbf{r}}/\partial x^2 = 0$, $\partial\overline{\mathbf{r}}/\partial x^1 \cdot \partial\overline{\mathbf{r}}/\partial x^1 = 1$, $\partial\overline{\mathbf{r}}/\partial x^2 \cdot \partial\overline{\mathbf{r}}/\partial x^2 = 1$. It means that on M we have introduced *the orthonormal parametrization*. Sub- and superscripts $\alpha, \beta, \ldots$ run over 1, 2 and are related to x^1, x^2, summation convention holds. Partial differentiation related to x^α is represented by ∂_α. Moreover, it is denoted $\partial_{\alpha\ldots\delta} \equiv \partial_\alpha \ldots \partial_\delta$. Let $a_{\alpha\beta}$ and $b_{\alpha\beta}$ stand for the midsurface first and second covariant metric tensors, respectively. Under orthonormal parametrization introduced on M, $a_{11} = a_{22} = 1$, $a_{12} = a_{21} = 0$ and $b_{22} = b_{12} = b_{21} = 0$, $b_{11} = -r^{-1}$.

The time coordinate is denoted by $t \in \mathrm{I} \equiv [t_0, t_1]$. Let $d(x)$ and r stand for the shell thickness and the midsurface curvature radius, respectively.

The basic cell Δ and *a cell* $\Delta(x)$ *with the centre at point* $x \in \Omega_\Delta$ are defined by: $\Delta \equiv [-\lambda/2,\ \lambda/2]$, $\Delta(x) \equiv x + \Delta$, $x \in \Omega_\Delta$, $\Omega_\Delta \equiv \{x \in \Omega : \Delta(x) \subset \Omega_\Delta\}$, where λ is a cell length dimension in x-direction. *The microstructure length parameter* λ satisfies conditions: $\lambda/d_{\max} \gg 1$, $\lambda/r \ll 1$ and $\lambda/L_1 \ll 1$. Setting $z \equiv z^1 \in [-\lambda/2,\ \lambda/2]$, we assume that cell Δ has a symmetry axis: for $z = 0$. It is also assumed that inside the cell not only the geometrical but also elastic, inertial and thermal properties of the shell are described by symmetric (i.e. even) functions of argument z.

Denote by $u_\alpha = u_\alpha(x, \xi, t)$, $w = w(x, \xi, t)$, $(x, \xi, t) \in \Omega \times \Xi \times \mathrm{I}$, the shell displacements in directions tangent and normal to M, respectively. Elastic properties of the shells are described by shell stiffness tensors $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$. Let $\mu(x)$ stand for a shell mass density per midsurface unit area. Let $f^\alpha(x, \xi, t)$, $f(x, \xi, t)$ be external forces per midsurface unit area, respectively tangent and normal to M. Denote by $\theta(x, \xi, t)$ the temperature field treated as the temperature increment from a certain constant reference temperature T_0 (by reference temperature we shall mean the zero stress temperature). It is assumed that $\theta/T_0 \ll 1$. Let $\overline{d}^{\alpha\beta}(x)$ stand for the membrane thermal stiffness tensor (tensor of thermo-elastic moduli: $\overline{d}^{\alpha\beta} = D^{\alpha\beta\gamma\delta}\alpha_{\gamma\delta}$, where $\alpha_{\gamma\delta}$ are coefficients of thermal expansion). Denote by $K^{\alpha\beta}(x)$ and by $c(x)$ the tensor of heat conductivity and the specific heat, respectively. The heat sources will be neglected. For uniperiodic shells, $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$, $\mu(x)$, $\overline{d}^{\alpha\beta}(x)$, $K^{\alpha\beta}(x)$, $c(x)$ are highly oscillating, non-continuous and periodic functions in x.

It is assumed that the temperature along the shell thickness is constant. From this restriction it follows that only the coupling between temperature field

θ and membrane stresses occurs (describing by tensor $\overline{d}^{\alpha\beta}(x)$) while the coupling of temperature and bending stresses is absent.

The starting equations are the well known governing equations of linear Kirchhoff-Love theory of thin elastic cylindrical shells combined with Duhamel-Neumann thermo-elastic constitutive relations and coupled with the known linearized Fourier heat conduction equation in which the heat sources are neglected. Thus, the starting equations consist of:

a) the Duhamel-Neumann stress-strain-temperature relations

$$n^{\alpha\beta}(x,\xi,t) = D^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} - \overline{d}^{\alpha\beta}\theta, \qquad m^{\alpha\beta}(x,\xi,t) = B^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta}, \tag{1.1}$$

where

$$\varepsilon_{\alpha\beta}(x,\xi,t) = \frac{1}{2}(\partial_\beta u_\alpha + \partial_\alpha u_\beta) - b_{\alpha\beta}w, \quad \kappa_{\alpha\beta}(x,\xi,t) = -\partial_{\alpha\beta}w,$$

b) the dynamic equilibrium equations

$$\partial_\beta n^{\alpha\beta} - \mu a^{\alpha\beta}\ddot{u}_\beta + f^\alpha = 0, \qquad \partial_{\alpha\beta}m^{\alpha\beta} + b_{\alpha\beta}n^{\alpha\beta} - \mu\ddot{w} + f = 0,$$

which after combining with (1.1) are expressed in displacement fields u_α, w and temperature field θ

$$\begin{gathered}\partial_\beta(D^{\alpha\beta\gamma\delta}\partial_\delta u_\gamma) + r^{-1}\partial_\beta(D^{\alpha\beta11}w) - \partial_\beta(\overline{d}^{\alpha\beta}\theta) - \mu a^{\alpha\beta}\ddot{u}_\beta + f^\alpha = 0,\\ r^{-1}D^{\alpha\beta11}\partial_\beta u_\alpha + \partial_{\alpha\beta}(B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w) - r^{-1}\overline{d}^{11}\theta + r^{-2}D^{1111}w + \mu\ddot{w} - f = 0,\end{gathered} \tag{1.2}$$

c) the linearized heat conduction equation based on the Fourier law

$$\partial_\alpha(K^{\alpha\beta}\partial_\beta\theta) - c\dot{\theta} = T_0(\overline{d}^{\alpha\beta}\partial_\alpha\dot{u}_\beta + r^{-1}\overline{d}^{11}\dot{w}). \tag{1.3}$$

Equations (1.2) and (1.3) describe the dynamic thermoelastic problems for the shells under consideration. Coefficients of these equations are highly oscillating, non-continuous and periodic functions in x. Applying the tolerance modelling technique (cf. [4, 5]) to (1.2) and (1.3), we will derive the averaged tolerance model equations with constant coefficients depending also on a microstructure size.

2. Modelling procedure, equations of tolerance model

The fundamental concepts of the tolerance approach under consideration are those of *two tolerance relations between points and real numbers determined by tolerance parameters, slowly-varying functions, tolerance-periodic functions, fluctuation shape functions and the averaging operation*, cf. [3–5].

Below, the mentioned above concepts and assumptions will be specified with respect to one-dimensional region $\Omega = (0, L_1)$.

Tolerance between points. Let λ be a positive real number. Points x, y belonging to $\Omega = (0, L_1)$ are said to be in tolerance determined by λ, if and only if the distance between points x, y does not exceed λ, i.e. $\|x - y\| \leqslant \lambda$.

Tolerance between real numbers. Let $\widetilde{\delta}$ be a positive real number. Real numbers μ, ν are said to be in tolerance determined by $\widetilde{\delta}$, if and only if $|\mu - \nu| \leqslant \widetilde{\delta}$.

The above relations are denoted by: $x \overset{\lambda}{\approx} y$, $\mu \overset{\widetilde{\delta}}{\approx} \nu$. Positive parameters λ, $\widetilde{\delta}$ are called *tolerance parameters*.

Let $F(\cdot)$ be a function defined in $\overline{\Omega} = [0, L_1]$, which is continuous, bounded and differentiable in $\overline{\Omega}$ together with their derivatives up to the R-th order. Nonnegative integer R is assumed to be specified in every problem under consideration. Note, that function F can also depend on $\xi \in \overline{\Xi} = [0, L_2]$ and time coordinate t as parameters. Let $\delta \equiv (\lambda, \delta_0, \delta_1, .., \delta_R)$ be the set of tolerance parameters. The first of them is related to the distances between points in Ω, the second one is related to the differences between values of function $F(\cdot)$ at points x, y belonging to Ω, such that $\|x - y\| \leqslant \lambda$, and the k-th one to the differences between values of the k-th derivative of $F(\cdot)$, $k = 1, ..., R$, at points x, y belonging to Ω, such that $\|x - y\| \leqslant \lambda$. A function $F(\cdot)$ is called *slowly-varying of the R-th kind* with respect to cell Δ and tolerance parameters δ, $F \in SV_\delta^R(\Omega, \Delta)$, if and only if

$$\big(\forall (x, y) \in \Omega^2\big)\Big[(x \overset{\lambda}{\approx} y) \Rightarrow F(x) \overset{\delta_0}{\approx} F(y) \ \text{ and } \ \partial_1^k F(x) \overset{\delta_k}{\approx} \partial_1^k F(y), \ \ k = 1, 2, ..., R\Big],$$

where $\partial_1^k F(\cdot)$ stands for the k-th derivative of $F(\cdot)$ in Ω. Roughly speaking, *slowly-varying* function $F(\cdot)$ can be treated (together with its derivatives up to the R-th order) as constant on a cell in the framework of tolerance determined by the pertinent tolerance parameters.

In the applications of the tolerance modelling, tolerance parameter λ is known *a priori* as a certain microstructure length, whereas values of tolerance parameters $\delta_0, \delta_1, ..., \delta_R$ can be determined only *a posteriori*, i.e. after obtaining solution to the initial-boundary value problem under consideration.

An integrable and bounded function $f(\cdot)$ defined in $\overline{\Omega} \equiv [0, L_1]$ is called *tolerance-periodic of the R-th kind* with respect to cell Δ and tolerance parameters δ, $f \in SV_\delta^R(\Omega, \Delta)$, if it can be treated (together with its derivatives up to the R-th order) as periodic on a cell.

Let $h(\cdot)$ be a periodic highly oscillating function defined in $\overline{\Omega} = [0, L_1]$, which is continuous together with derivatives $\partial_1^k h$, $k = 1, ..., R-1$, and has a continuous or a piecewise continuous bounded derivative $\partial_1^R h$. Periodic function $h(\cdot)$ will be called *the fluctuation shape function*, $h(\cdot) \in FS^R(\Omega,\Delta)$, if it depends on λ as a parameter and satisfies conditions:

$$h \in O\left(\lambda^R\right), \qquad \partial_1^k h \in O\left(\lambda^{R-k}\right),$$

$$k = 1,2,...,R, \qquad \int\limits_{\Delta(x)} \mu(z)\, h(z) dz = 0, \qquad z \in \Delta(x), \qquad x \in \Omega_\Delta,$$

where $\mu(\cdot)$ is a certain positive valued periodic function defined in $\overline{\Omega}$.

The *averaging operator* for an arbitrary function $f(\cdot)$ being integrable and bounded in every cell is defined by:

$$\langle f > (x)\rangle \equiv \frac{1}{\lambda}\int\limits_{x-\lambda/2}^{x+\lambda/2} f(z)dz, \qquad z \in \Delta(x), \qquad x \in \Omega_\Delta. \tag{2.1}$$

The tolerance modelling is based on two assumptions. The first of them is called *the tolerance averaging approximation*.

Let $f(\cdot)$ be an arbitrary integrable tolerance-periodic functions defined in $\overline{\Omega} = [0, L_1]$ and let $F(\cdot) \in SV_\delta^1(\Omega,\Delta)$, $G(\cdot) \in SV_\delta^2(\Omega,\Delta)$ and $h(\cdot) \in FS^1(\Omega,\Delta)$, $g(\cdot) \in FS^2(\Omega,\Delta)$. *The tolerance averaging approximation has the form*

$$\left\langle f\, \partial_1^R F\right\rangle (x) = \langle f\rangle \partial_1^R F(x) + O(\delta), \qquad R = 0,1, \qquad \partial_1^0 F \equiv F,$$

$$\left\langle f\, \partial_1^R G\right\rangle (x) = \langle f\rangle \partial_1^R G(x) + O(\delta), \qquad R = 0,1,2, \qquad \partial_1^0 G \equiv G,$$

$$\langle f\, \partial_1 (hF)\rangle (x) = \langle f\, \partial_1 h\rangle (x) F(x) + O(\delta),$$

$$\langle f\, \partial_1 (gG)\rangle (x) = \langle f\, \partial_1 g\rangle (x) G(x) + O(\delta),$$

$$\left\langle f\, \partial_1^2 (gG)\right\rangle (x) = \langle f\, \partial_1^2 g\rangle (x) G(x) + O(\delta).$$

In the course of modelling, terms $O(\delta)$ will be neglected. Let us observe that the slowly-varying functions can be regarded as invariant under averaging.

The second assumption is termed *the micro-macro decomposition*. In the problem under consideration, *the micro-macro decomposition* is assumed in the form

$$\begin{aligned} u_\alpha(x,\xi,t) &= u_\alpha^0(x,\xi,t) + h(x) U_\alpha(x,\xi,t), \\ w(x,\xi,t) &= w^0(x,\xi,t) + g(x) W(x,\xi,t), \\ \theta(x,\xi,t) &= \theta^0(x,\xi,t) + b(x)\Theta(x,\xi,t), \end{aligned} \tag{2.2}$$

where *macrodisplacements* u^0_α, w^0 and *macrotemperature* θ^0 as well as *displacement fluctuation amplitudes* U_α, W and *temperature fluctuation amplitude* Θ are *the new slowly-varying unknowns*, i.e. u^0_α, U_α, θ^0, $\Theta \in SV^1_\delta(\Omega, \Delta)$, w^0, $W \in SV^2_\delta(\Omega, \Delta)$. *Fluctuation shape functions for displacements* $h(x)$, $g(x)$ and *fluctuation shape function for temperature* $b(x)$ are the known in every problem under consideration, λ-periodic, continuous and highly-oscillating functions. They depend on λ as a parameter and in this work they have to satisfy conditions: $h \in O(\lambda)$, $\lambda\partial_1 h \in O(\lambda)$, $g \in O(\lambda^2)$, $\lambda\partial_1 g \in O(\lambda^2)$, $\lambda^2\partial_{11} g \in O(\lambda^2)$, $b \in O(\lambda)$, $\lambda\partial_1 b \in O(\lambda)$, $\langle \mu\, h\rangle = \langle \mu\, g\rangle = \langle cb\rangle = 0$.

We substitute the right-hand sides of (2.2) into (1.2), (1.3). For decomposition (2.2), the governing Eqs. (1.2), (1.3) do not hold, i.e. there exist residual fields defined by

$$
\begin{aligned}
p^\alpha &\equiv \partial_\beta(D^{\alpha\beta\gamma\delta}\partial_\delta(u^0_\gamma + hU_\gamma)) + r^{-1}\partial_\beta(D^{\alpha\beta 11}(w^0 + gW)) \\
&\qquad - \partial_\beta(\overline{d}^{\alpha\beta}(\theta^0 + b\Theta)) - \mu a^{\alpha\beta}(\ddot{u}^0_\beta + h\ddot{U}_\beta) + f^\alpha, \\
p &\equiv r^{-1}D^{\alpha\beta 11}\partial_\beta(u^0_\alpha + hU_\alpha) + \partial_{\alpha\beta}(B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}(w^0 + gW)) \\
&\qquad - r^{-1}\overline{d}^{11}(\theta^0 + b\Theta) + r^{-2}D^{1111}(w^0 + gW) + \mu(\ddot{w}^0 + g\ddot{W}) - f, \\
s &\equiv \partial_\alpha(K^{\alpha\beta}\partial_\beta(\theta^0 + b\Theta)) - c(\dot{\theta}^0 + b\dot{\Theta}) \\
&\qquad - T_0(\overline{d}^{\alpha\beta}\partial_\alpha(\dot{u}^0_\beta + h\dot{U}_\beta) + r^{-1}\overline{d}^{11}(\dot{w}^0 + g\dot{W})).
\end{aligned}
\tag{2.3}
$$

Following [2], we introduce *the residual orthogonality assumption* which states that residual fields (2.3) have to satisfy the following orthogonality conditions

$$
\langle p^\alpha\rangle = 0, \quad \langle p^\alpha h\rangle = 0, \quad \langle p\rangle = 0, \quad \langle pg\rangle = 0, \quad \langle s\rangle = 0, \quad \langle sb\rangle = 0,
\tag{2.4}
$$

for almost every $(x, \xi) \in \Omega \times \Xi$ and every $t \in \mathrm{I} \equiv [t_0, t_1]$. Tolerance operation $\langle\cdot\rangle$ on cell Δ is defined by (2.1).

Conditions (2.4), on the basis of *the tolerance averaging approximation*, lead to the system of averaged equations for unknowns u^0_α, w, U^0_α, W, θ^0, Θ. Under the extra approximation $1 + \lambda/r \approx 1$, this system can be written in the form of:
a) the stress-strain-temperature relations

$$
\begin{aligned}
N^{\alpha\beta} &= \left\langle D^{\alpha\beta\gamma\delta}\right\rangle \partial_\delta u^0_\gamma + r^{-1}\left\langle D^{\alpha\beta 11}\right\rangle w^0 + \left\langle D^{\alpha\beta\gamma 1}\partial_1 h\right\rangle U_\gamma \\
&\qquad + \underline{\left\langle D^{\alpha\beta\gamma 2}h\right\rangle}\partial_2 U_\gamma - \left\langle \overline{d}^{\alpha\beta}\right\rangle\theta^0 - \underline{\left\langle\overline{d}^{\alpha\beta}b\right\rangle}\Theta, \\
M^{\alpha\beta} &= \left\langle B^{\alpha\beta\gamma\delta}\right\rangle\partial_{\gamma\delta}w^0 + \left\langle B^{\alpha\beta 11}\partial_{11}g\right\rangle \\
&\qquad + 2\underline{\left\langle B^{\alpha\beta 12}\partial_1 g\right\rangle}\partial_2 W + \underline{\left\langle B^{\alpha\beta 22}g\right\rangle}\partial_{22}W,
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
H^{\beta} &= \left\langle \partial_1 h D^{\beta 1\gamma\delta} \right\rangle \partial_{\delta} u^0_{\gamma} - \underline{\left\langle h D^{\beta 2\gamma\delta} \right\rangle} \partial_{2\delta} u^0_{\gamma} + \left\langle \partial_1 h D^{\beta 11\gamma} \partial_1 h \right\rangle \\
&\quad U_{\gamma} - \left\langle h D^{\beta 22\gamma} h \right\rangle \partial_{22} U_{\gamma} + r^{-1} \left\langle \partial_1 h D^{\beta 111} \right\rangle w^0 - \left\langle \partial_1 h\, \overline{d}^{\beta 1} \right\rangle \theta^0 \\
&\quad + \underline{\left\langle \overline{d}^{\beta 2} h \right\rangle} \partial_2 \theta^0 - \underline{\left\langle \overline{d}^{\beta 1} \partial_1 h\, b \right\rangle} \Theta + \underline{\left\langle \overline{d}^{\beta 2} b h \right\rangle} \partial_2 \Theta, \\
G &= \left\langle \partial_{11} g B^{11\alpha\beta} \right\rangle \partial_{\alpha\beta} w^0 - 2 \underline{\left\langle \partial_1 g B^{\alpha\beta 12} \right\rangle} \partial_{\alpha\beta 2} w^0 \\
&\quad + \underline{\left\langle g B^{\alpha\beta 22} \right\rangle} \partial_{\alpha\beta 22} w^0 + \left\langle (\partial_{11} g)^2 B^{1111} \right\rangle W + \Big(\underline{2 \left\langle \partial_{11} g B^{1122} g \right\rangle} \\
&\quad - 4 \underline{\left\langle (\partial_1 g)^2 B^{1212} \right\rangle} \Big) \partial_{22} W + \underline{\left\langle (g)^2 B^{2222} \right\rangle} \partial_{2222} W,
\end{aligned}
\tag{2.6}
$$

b) the dynamic equilibrium equations

$$
\begin{aligned}
\partial_{\alpha} N^{\alpha\beta} - \langle \mu \rangle a^{\alpha\beta} \ddot{u}^0_{\alpha} + \langle f^{\beta} \rangle &= 0, \\
\partial_{\alpha\beta} M^{\alpha\beta} + r^{-1} N^{11} + \langle \mu \rangle \ddot{w}^0 - \langle f \rangle &= 0, \\
\underline{\left\langle \mu (h)^2 \right\rangle} a^{\alpha\beta} \ddot{U}_{\alpha} + H^{\beta} - \underline{\left\langle f^{\beta} h \right\rangle} &= 0, \\
\underline{\left\langle \mu (g)^2 \right\rangle} \ddot{W} + G - \underline{\langle f g \rangle} &= 0,
\end{aligned}
\tag{2.7}
$$

c) the heat conduction equations

$$
\begin{aligned}
&\left\langle K^{\alpha\beta} \right\rangle \partial_{\alpha\beta} \theta^0 + \left\langle K^{1\beta} \partial_1 b \right\rangle \partial_{\beta} \Theta + \underline{\left\langle K^{2\beta} b \right\rangle} \partial_{2\beta} \Theta - \langle c \rangle \dot{\theta}^0 = \left\langle T_0\, \overline{d}^{\alpha\beta} \right\rangle \partial_{\alpha} \dot{u}^0_{\beta} \\
&\qquad + \left\langle T_0\, \overline{d}^{1\beta} \partial_1 h \right\rangle \dot{U}_{\beta} + \underline{\left\langle T_0\, \overline{d}^{2\beta} h \right\rangle} \partial_2 \dot{U}_{\beta} + r^{-1} \left\langle T_0\, \overline{d}^{11} \right\rangle \dot{w}^0, \\
&\underline{\left\langle K^{2\beta} b \right\rangle} \partial_{2\beta} \theta^0 - \left\langle K^{1\beta} \partial_1 b \right\rangle \partial_{\beta} \theta^0 + \underline{\left\langle K^{22} (b)^2 \right\rangle} \partial_{22} \Theta - \left\langle K^{11} (\partial_1 b)^2 \right\rangle \Theta \\
&\qquad - \underline{\left\langle c (b)^2 \right\rangle} \dot{\Theta} = \underline{\left\langle T_0\, b\, \overline{d}^{\alpha\beta} \right\rangle} \partial_{\alpha} \dot{u}^0_{\beta} + \underline{\left\langle T_0\, \overline{d}^{1\beta} b\, \partial_1 h \right\rangle} \dot{U}_{\beta} + \underline{\left\langle T_0\, \overline{d}^{2\beta} b\, h \right\rangle} \partial_2 \dot{U}_{\beta}.
\end{aligned}
\tag{2.8}
$$

Equations (2.5)–(2.8) together with *the micro-macro decomposition* (2.2) constitute *the tolerance model for the analysis of selected dynamic thermoelasic problems for uniperiodic shells under consideration*. Coefficients of the derived model equations are *constant* and some of them *depend on microstructure length parameter* λ (underlined terms).

3. An example of applications

In this section as an example of applications of Eqs. (2.5)–(2.8) we shall investigate the effect of a cell size λ on the initial distributions of temperature micro-fluctuations in the uniperiodic shells under consideration. An example of a shell with an uniperiodic structure is shown in Fig. 1.

In order to analyse this problem, we assume that the external forces f^{β}, f are equal to zero. We neglect the forces of inertia $\langle\mu\rangle\, a^{\alpha\beta}\ddot{u}^0_\alpha$, $\left\langle\mu(h)^2\right\rangle a^{\alpha\beta}\ddot{U}_\alpha$ in directions tangential to the shell midsurface as sufficiently small when compared to the forces of inertia $\langle\mu\rangle\, \ddot{w}^0$, $\left\langle\mu(g)^2\right\rangle \ddot{W}$ in direction normal to the shell midsurface. At the same time we also neglect terms containing the first time derivatives of macrodisplacements $u^0_\alpha(\cdot,t)$ and of displacement fluctuation amplitudes $U_\alpha(\cdot,t)$ as sufficiently small when compared to terms containing the first time derivatives of kinematical unknowns $w^0(\cdot,t)$, $W(\cdot,t)$.

The investigated problem is rotationally symmetric with a period λ/r; hence u^0_1, $U_1 = 0$ and the remaining unknowns of the tolerance model u^0_2, U_2, w^0, W, θ^0, Θ (but not total displacements u_2, w and total temperature field θ in decomposition (2.2)!) are independent of x-midsurface parameter.

Taking into account the symmetric form of a cell $\Delta \equiv [-\lambda/2,\, \lambda/2]$, we assume that fluctuation shape functions for displacements $h(\cdot) \in FS^1(\Omega,\Delta)$ and for temperature $b(\cdot) \in FS^1(\Omega,\Delta)$ are odd with respect to $z \in [-\lambda/2,\, \lambda/2]$ (the cell has a symmetry axis for $z = 0$) whereas fluctuation shape function for displacements $g(\cdot) \in FS^2(\Omega,\Delta)$ is even with respect to z.

We restrict considerations to uniperiodic shells composed of homogeneous, isotropic constituents. In this case $\overline{d}^{12} = \overline{d}^{21} = 0$, $\overline{d}^{11} = \overline{d}^{22}$ and $K^{12} = K^{21} = 0$, $K^{11} = K^{22}$.

Under assumptions given above, the system of tolerance model Eqs. (2.7)–(2.8) separates into the following system of five equations for $u^0_2(\xi,t)$, $w^0(\xi,t)$, $U_2(\xi,t)$, $W(\xi,t)$, $\theta^0(\xi,t)$

$$
\begin{aligned}
&\left\langle D^{2222}\right\rangle \partial_{22}u^0_2 + r^{-1}\left\langle D^{2211}\right\rangle \partial_2 w^0 - \left\langle \overline{d}^{22}\right\rangle \partial_2\theta^0 = 0,\\
&\left\langle B^{2222}\right\rangle \partial_{2222}w^0 + \left\langle B^{2211}\partial_{11}g\right\rangle \partial_{22}W + \underline{\left\langle B^{2222}g\right\rangle}\partial_{2222}W + \langle\mu\rangle\, \ddot{w}^0 = 0,\\
&\underline{\left\langle (h)^2 D^{2222}\right\rangle}\partial_{22}U_2 - \left\langle (\partial_1 h)^2 D^{2112}\right\rangle U_2 - \left\langle \overline{d}^{22} bh\right\rangle \partial_2\Theta = 0,\\
&\left\langle \partial_{11}gB^{1122}\right\rangle \partial_{22}w^0 + \underline{\left\langle gB^{2222}\right\rangle}\partial_{2222}w^0 + \left\langle (\partial_{11}g)^2 B^{1111}\right\rangle W\\
&\quad + (2\,\underline{\left\langle \partial_{11}gB^{1122}g\right\rangle} - 4\underline{\left\langle (\partial_1 g)^2 B^{1212}\right\rangle})\partial_{22}W + \underline{\left\langle (g)^2 B^{2222}\right\rangle}\partial_{2222}W\\
&\qquad + \underline{\left\langle \mu(g)^2\right\rangle}\ddot{W} = 0,\\
&\left\langle K^{22}\right\rangle \partial_{22}\theta^0 - \langle c\rangle\, \dot{\theta}^0 = r^{-1}\left\langle T_0\overline{d}^{11}\right\rangle \dot{w}^0,
\end{aligned}
\tag{3.1}
$$

and independent equation for temperature fluctuation amplitude $\Theta(\xi, t)$

$$\underline{\left\langle K^{22}(b)^2\right\rangle}\partial_{22}\Theta - \left\langle K^{11}(\partial_1 b)^2\right\rangle \Theta - \underline{\left\langle c(b)^2\right\rangle}\dot{\Theta} = 0. \tag{3.2}$$

The underlined averages in (3.1) and (3.2) depend on microstructure length parameter λ.

The subsequent analysis will be restricted to Eq. (3.2) describing micro-fluctuations of temperature field in axial direction caused by periodic structure of the shells under consideration.

We shall investigate the problem of time decaying of the temperature fluctuation amplitude $\Theta(\xi, t)$ setting

$$\Theta(\xi, t) = \Theta^*(\xi)\exp(-\gamma t), \qquad t \geqslant 0,$$

with $\gamma > 0$ as a time decaying coefficient. Function $\Theta^*(\xi)$ represents an initial distribution of temperature micro-fluctuations, i.e. $\Theta(\xi, t = 0) = \Theta^*(\xi)$.

Hence, under denotations

$$\widetilde{k}^2 \equiv \frac{\left\langle K^{11}(\partial_1 b)^2\right\rangle}{\lambda^2\left\langle K^{22}(\overline{b})^2\right\rangle}, \qquad \gamma_* \equiv \frac{\left\langle K^{11}(\partial_1 b)^2\right\rangle}{\lambda^2\left\langle c\,(\overline{b})^2\right\rangle},$$

where $\overline{b}(\cdot) = \lambda^{-1}b(\cdot)$, Eq. (3.2) yields

$$\partial_{22}\Theta^*(\xi) - \widetilde{k}^2[1 - (\gamma/\gamma_*)]\Theta^*(\xi) = 0, \tag{3.3}$$

where γ_* is *a certain new time decaying coefficient depending on microstructure length parameter* λ. It can be shown that averages $\left\langle K^{11}(\partial_1 b)^2\right\rangle$, $\left\langle K^{22}(\overline{b})^2\right\rangle$, $\left\langle c(\overline{b})^2\right\rangle$ are greater than zero; hence $\widetilde{k}^{\,2} > 0$ and $\gamma_* > 0$. The boundary conditions for $\Theta^*(\xi)$ are assumed in the form

$$\Theta^*(\xi = 0) = \Theta_0^*, \qquad \Theta^*(\xi = L_2) = 0,$$

where Θ_0^* is the known constant.

The solution to Eq. (3.3) depends on relations between time decaying coefficients γ and γ_*. The following special cases can be taken into account.

1) If $0 < \gamma \ll \gamma_*$ and setting $\widetilde{k}_\gamma^2 \equiv \widetilde{k}^2[1 - (\gamma/\gamma_*)]$ then

$$\Theta^*(\xi) = \Theta_0^*\exp(-\widetilde{k}_\gamma\xi);$$

in this case the temperature micro-fluctuations are strongly decaying near the boundary $\xi = 0$. It means that the micro-fluctuations can be treated as equal to zero outside a certain narrow layer near boundary $\xi = 0$. Thus, Eq. (3.2) being a starting point in the thermal problem under consideration makes it possible to investigate *the boundary layer phenomena*.

2) If $0 \ll \gamma < \gamma_*$ then

$$\Theta^*(\xi) = \Theta_0^*[\exp(-\widetilde{k}_\gamma \xi)(1-\exp(-2\widetilde{k}_\gamma L_2))^{-1} + \exp(\widetilde{k}_\gamma \xi)(1-\exp(2\widetilde{k}_\gamma L_2))^{-1}];$$

the initial micro-fluctuations decay exponentially but not strongly.

3) If $\gamma = \gamma_*$ then

$$\Theta^*(\xi) = \Theta_0^*(1-\xi/L_2);$$

we deal with a linear decaying of temperature micro-fluctuation amplitude.

4) If $\gamma > \gamma_*$ and setting $\kappa^2 \equiv \widetilde{k}^2[(\gamma/\gamma_*)-1] \neq (n\pi)^2(L_2)^{-2}$ then

$$\Theta^*(\xi) = \Theta_0^* \sin(\kappa(L_2-\xi))(\sin(\kappa L_2))^{-1};$$

the temperature micro-fluctuations oscillate.

5) If $\gamma > \gamma_*$ and $\kappa^2 \equiv \widetilde{k}^2[(\gamma/\gamma_*)-1] = (n\pi)^2(L_2)^{-2}$ then *the solution doesn't exist.*

The above effect cannot be analysed in the framework of the asymptotic models commonly used for investigations of thermoelastic problems for micro-periodically shells under consideration. It can be observed that within the asymptotic models neglecting the length-scale terms, Eq. (3.2) reduces to equation $\langle K^{11}(\partial_1 b)^2 \rangle\, \Theta = 0$, which has only trivial solution $\Theta = 0$.

Notice that in the problem under consideration, for an arbitrary but fixed time argument t the shape of temperature micro-fluctuation amplitude $\Theta(\xi,t)$ is the same as the form of initial temperature micro-fluctuation amplitude $\Theta^*(\xi)$.

4. Remarks and conclusions

The tolerance modelling procedure is applied to the known partial differential equations describing dynamic thermoelastic problems for Kirchhoff-Love-type thin linearly elastic cylindrical shells with periodic microstructure in circumferential direction.

In contrast to exact thermoelastic shell Eqs. (1.2), (1.3) with discontinuous, highly oscillating and periodic coefficients, the tolerance model Eqs. (2.5)–(2.8) proposed here *have constant coefficients depending also on a cell size* λ. Hence, this model makes it possible to analyse the effect of a microstructure size on the global thermodynamic shell behaviour (*the length-scale effect*). This effect is neglected in the known homogenized models derived by asymptotic methods.

The resulting model equations are uniquely determined by the highly oscillating, periodic *fluctuation shape functions* representing oscillations of temperature and displacement fields inside a cell caused by a periodic structure of the shells. These functions have to be known in every problem under consideration.

The number and form of boundary/initial conditions for unknown macrodisplacements u^0_α, w^0 and macrotemperature θ^0 are the same as in the classical shell theory governed by Eqs. (1.2), (1.3). The boundary conditions for kinematic fluctuation amplitudes Q_α, V and thermal fluctuation amplitude Θ should be defined only on boundaries $\xi = 0$, $\xi = L_2$. The form of initial/boundary conditions for Q_α, V and Θ are the same as in the classical shell theory.

Solutions to the initial-boundary value problems have the physical sense only if the basic unknowns u^0_α, w, Q_α, V, θ^0, Θ of the tolerance model are slowly-varying functions in periodicity direction. This requirement can be verified only *a posteriori* and it determines the range of the physical applicability of the model.

As an example of applications of tolerance model Eqs. (2.5)–(2.8), a certain special problem dealing with time decaying of initial micro-fluctuations of temperature field was analysed. It was shown that in the uniperiodic shells under consideration the form of initial temperature micro-fluctuations depends on relations between the given time decaying coefficient $\gamma > 0$ and *a certain time decaying coefficient* γ_* *depending on microstructure length parameter* λ. The initial temperature micro-fluctuations *decay exponentially* for $0 < \gamma < \gamma_*$. They *decay linearly* for $\gamma = \gamma_*$. If $\gamma > \gamma_*$ then the temperature micro-fluctuations*oscillate*. Moreover, if $0 < \gamma \ll \gamma_*$ then the micro-fluctuation amplitude is strongly decaying near the boundary $\xi = 0$. It means that the temperature micro-fluctuations can be treated as equal to zero outside a certain narrow layer near boundary $\xi = 0$. Thus, we have shown that the tolerance model proposed here makes it possible to analyse *the boundary layer phenomena. All the effects mentioned above cannot be investigated in the framework of the asymptotic models.*

Some other applications of the tolerance model proposed here to investigate special thermoelastic or only thermal problems for uniperiodic shells under consideration dealing with the effect of a cell size on the thermoelastic behaviour of the shells, e.g. influence of a period length on vibration caused by a certain thermal load, effect of the microstructure size on distribution of the averaged and fluctuating parts of temperature and displacement fields, will be shown in forthcoming papers.

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