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Dynamical Systems Approach of Internal Length in Fractional Calculus

Péter Balázs BÉDA

Budapest University of Technology and Economics
H-1111 Műegyetem rkp. 1-3. Budapest, Hungary
e-mail: bedap@kme.bme.hu

Conventionally, non-local properties are included in the constitutive equations in the form of strain gradient-dependent terms. In case of the second gradient dependence an internal material length can be obtained from the critical eigenmodes in instability problems. When non-locality is included by using fractional calculus, a generalized strain can be defined. Stability investigation can be also performed and internal length effects can be studied by analysing the critical eigenspace. Such an approach leads to classical results for second gradient, but new phenomena appear in the first gradient case.

Key words: rate and gradient dependence, fractional calculus, static and dynamic internal length.

1. Introduction

Non-local materials were already studied in the 1960s by several authors (for example [1]) as a part of continuum mechanics. Later, when material instability attracted more interest, non-local behavior appeared again in [2], because instability zones exhibited singular properties in local constitutive equations. Therefore, first and second gradients of the strain tensor $\nabla\varepsilon$, $\nabla^2\varepsilon$ are added to stress σ and strain ε in order to include non-locality in the constitutive equation

$$F(\sigma, \varepsilon, \nabla\varepsilon, \nabla^2\varepsilon, ...) = 0. \tag{1.1}$$

Internal length conventionally appears in (1.1) as the coefficient of second gradient term.

Over the last few years, fractional calculus has become the focus of non-local continuum theories. It was used in non-local elasticity to study the micromechanical effects in rod models [3] or to extend the basic functions of the classical continuum field theory to non-differentiable functions [4]. A general framework of fractional calculus as a generalization of gradient elasticity is presented in [5]; moreover, thermal effects may also be taken into account in the fractional continuum mechanics [6].

In the present study, a short mathematical explanation of internal length will be followed by the introduction of fractional strain. Then – as a new approach – material instability investigations are performed, in which the existence of finite and non-zero internal length is connected to the regularity of non-trivial critical eigenspace of an operator determined by the basic equation of continua. Here, necessary conditions can be formed for various constitutive equations to obtain regular non-trivial eigenfunctions.

2. Gradient materials and internal length

Most gradient theories concentrate on the second gradient. Let the constitutive equation be in rate form

$$\dot{\sigma} = \widetilde{c}_1 \dot{\varepsilon} + \widetilde{c}_2 \ddot{\varepsilon} - \widetilde{c}_3 \frac{\partial^2 \dot{\varepsilon}}{\partial x^2}, \tag{2.1}$$

where x is uniaxial space coordinate, overdot denotes time derivatives, and $\widetilde{c}_i$ $(i = 1, 2, 3)$ are material parameters. Then, the set of the basic equations of continua consists of (2.1), and the equation of motion together with the kinematic equation reads

$$\rho \dot{v} = \frac{\partial \sigma}{\partial x}, \qquad \dot{\varepsilon} = \frac{\partial v}{\partial x}.$$

As usual, velocity field is denoted by v and ρ is mass density. By transforming the equations into the velocity field and using new variables

$$y_1 = v, \qquad y_2 = \dot{v},$$

a dynamical system

$$\dot{y}_1 = y_2, \tag{2.2}$$

$$\dot{y}_2 = \left(c_1 \frac{\partial^2}{\partial x^2} - c_3 \frac{\partial^4}{\partial x^4} \right) \dot{y}_1 + c_2 \frac{\partial^2}{\partial x^2} y_2, \tag{2.3}$$

is obtained, where

$$c_1 = \frac{\widetilde{c}_i}{\rho}, \quad (i = 1, 2, 3).$$

Its characteristic equation for scalar variable λ reads

$$\lambda^2 y_1 - \lambda c_2 \frac{\partial^2}{\partial x^2} y_2 - \left(c_1 \frac{\partial^2}{\partial x^2} - c_3 \frac{\partial^4}{\partial x^4} \right) y_1 = 0. \tag{2.4}$$

At the loss of stability

$$\lambda = 0 \qquad \text{and} \qquad c_1 = c_{1\text{crit}} < 0,$$

then the critical eigenfunction of (2.4) is

$$y_1 = \exp\left(ix\sqrt{\frac{-c_{1\,\text{crit}}}{c_3}} \right). \tag{2.5}$$

Now (static) internal length can be identified as

$$l^s := \pi\sqrt{\frac{-c_{1\,\text{crit}}}{c_3}}. \tag{2.6}$$

3. Applied fractional calculus

Following the idea of [7], strain can be generalized to fractional derivatives of displacement u

$$\sigma = \tilde{c}_1 \frac{1}{2} \left({}^C D^\alpha_{a+} u(x) - {}^C D^\alpha_{a-} u(x) \right), \tag{3.1}$$

where ${}^C D^\alpha_{a+} u(x)$ and ${}^C D^\alpha_{a-} u(x)$ are left and right α-th Caputo fractional derivatives with respect to x for a rod of length $L - a$,

$${}^C D^\alpha_{a+} u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int\limits_a^x \frac{u(\xi) - u(a)}{(x-\xi)^\alpha} d\xi.$$

Thus, by evaluating the right-hand side of (3.1)

$$\begin{aligned} {}^C D^\alpha_{a+} u(x) - {}^C D^\alpha_{a-} u(x) &= \frac{1}{\Gamma(1-\alpha)} \\ &\cdot \left(\int\limits_a^x (x-\xi)^{-\alpha} \frac{\partial u(\xi)}{\partial x} d\xi + \int\limits_x^L (-x+\xi)^{-\alpha} \frac{\partial u(\xi)}{\partial x} d\xi \right). \end{aligned} \tag{3.2}$$

In (3.2), the non-locality is obvious: value of the derivatives of function u at position x is determined by all values before and after that position. In rate form, (3.1) reads as

$$\dot{\sigma} = \tilde{c}_1 \frac{1}{2} \left({}^C D^\alpha_{a+} v(x) - {}^C D^\alpha_{a-} v(x) \right), \tag{3.3}$$

thus the equations in velocity field result in

$$\rho\ddot{v} = \frac{1}{2}\widetilde{c}_1\frac{\partial^2}{\partial x^2}\left({}^C D^{\alpha}_{a+}v(x) - {}^C D^{\alpha}_{a-}v(x)\right). \tag{3.4}$$

By adding a strain rate-dependent term to the constitutive equation (3.3), the equation for the velocity field (3.4) has the form

$$\rho\ddot{v} = \widetilde{c}_2\frac{\partial^2}{\partial x^2}\dot{v}(x) + \frac{1}{2}\widetilde{c}_1\frac{\partial^2}{\partial x^2}\left({}^C D^{\alpha}_{a+}v(x) - {}^C D^{\alpha}_{a-}v(x)\right). \tag{3.5}$$

The characteristic equation of (3.5) is

$$\rho\lambda^2 - \widetilde{c}_2\frac{\partial^2}{\partial x^2}\lambda v(x) - \frac{1}{2}\widetilde{c}_1\frac{\partial^2}{\partial x^2}\left({}^C D^{\alpha}_{a+}v(x) - {}^C D^{\alpha}_{a-}v(x)\right) = 0. \tag{3.6}$$

Thus, the stability boundary is

$$\widetilde{c}_1\frac{\partial^2}{\partial x^2}\left({}^C D^{\alpha}_{a+}v(x) - {}^C D^{\alpha}_{a-}v(x)\right) = 0. \tag{3.7}$$

The condition that allows to obtain the internal length is the existence of non-trivial solutions of (3.7).

For the sake of simplicity, the conventional uniaxial kinematic equation is generalized to a fractional form. While the fractional derivative rule of exponential function is used, there is no need to specify the type of the derivative. Let us simply write

$$\dot{\varepsilon} = \frac{\partial^{\alpha}v}{\partial x^{\alpha}}.$$

A classical second gradient-dependent constitutive equation is used. In rate form with material parameters B and C it reads

$$\dot{\sigma} = B\dot{\varepsilon} + C\frac{\partial^2}{dx^2}\dot{\varepsilon}.$$

Now the equation of the velocity field is

$$\rho\ddot{v} = B\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial}{\partial x}v + C\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^3}{\partial x^3}v. \tag{3.8}$$

For periodic perturbations of frequency ω the characteristic equation of (3.8) has zero solution, if

$$(B - \omega^2 C)\frac{\partial^{\alpha}}{\partial x^{\alpha}}\exp(i\omega x) = 0.$$

Thus

$$\frac{B}{C} = \omega^2 \Rightarrow \omega = \sqrt{\frac{B}{C}}$$

and the non-trivial eigenfunctions are of form (2.5)

$$\exp\left(i\sqrt{\frac{B}{C}}x\right).$$

Thus, internal lengths are the same for both fractional and conventional strain tensors.

Now we study a Malvern-Cristescu type constitutive equation in the following form

$$\sigma + D\dot{\sigma} = E\varepsilon + H\dot{\varepsilon},$$

where material parameters are denoted by D, E and H. Now, the characteristic equation for periodic perturbations reads

$$-\lambda^2\left(\lambda + \frac{1}{D}\right) + \lambda\frac{H}{D}\omega^\alpha \exp\Big(i\frac{\pi}{2}\alpha\Big) + \frac{1}{\rho}\frac{E}{D}\omega^{1+\alpha}\exp\Big(i\frac{\pi}{2}(1+\alpha)\Big) = 0. \tag{3.9}$$

Let us assume that the system undergoes a dynamic bifurcation ($\mathrm{Re}\lambda = 0$), then $\lambda = i\beta$ and (3.9) has the following form

$$i\beta^3 + \frac{1}{D}\beta^2 + \left(\frac{1}{\rho}\frac{E}{D}\omega^{1+\alpha} + \beta\frac{H}{D}\omega^\alpha\right)\exp\Big(i\frac{\pi}{2}(1+\alpha)\Big) = 0. \tag{3.10}$$

Calculating the real and imaginary parts of (3.10) we have

$$-\frac{1}{D}\beta^2 + \left(\frac{1}{\rho}\frac{E}{D}\omega^{1+\alpha} + \beta\frac{H}{D}\omega^\alpha\right)\sin\left(\frac{\pi}{2}\alpha\right) = 0 \tag{3.11}$$

and

$$\beta^3 + \left(\frac{1}{\rho}\frac{E}{D}\omega^{1+\alpha} + \beta\frac{H}{D}\omega^\alpha\right)\cos\left(\frac{\pi}{2}\alpha\right) = 0. \tag{3.12}$$

From (3.11) β can be expressed as

$$\beta_{1,2} = \frac{-H\sin\left(\frac{\pi}{2}\alpha\right)\omega^\alpha \pm \sqrt{H^2\sin^2\left(\frac{\pi}{2}\alpha\right)\omega^{2\alpha} + 4\frac{E}{\rho}\sin\left(\frac{\pi}{2}\alpha\right)\omega^{1+\alpha}}}{-2}.$$

Then, β should be substituted into (3.12) and an equation consisting the value α, the material parameters E, D, H, ρ and perturbation frequency ω is obtained.

For the sake of simplicity, let us assume that $H = 0$, then this equation is

$$\left(\frac{E}{\rho}\sin\left(\frac{\pi}{2}\alpha\right)\omega^{i+\alpha}\right)^{3/2} + \frac{1}{\rho}\frac{E}{D}\omega^{i+\alpha}\cos\left(\frac{\pi}{2}\alpha\right) = 0. \tag{3.13}$$

When nontrivial solutions of (3.13) are searched for,

$$\omega = \left(\frac{\cos\left(\frac{\pi}{2}\alpha\right)}{D} \frac{1}{\sqrt{\frac{E}{\rho}\sin\left(\frac{\pi}{2}\alpha\right)}} \right)^{\frac{2}{1+\alpha}} \tag{3.14}$$

is obtained. From (3.14) a ω can be calculated, which is the "critical" perturbing frequency and a dynamic internal length can be defined as before.

In Fig. 1 we can see how the order of the derivative (α) and stress rate factor D act on critical perturbing frequency (and consequently on dynamic internal length).

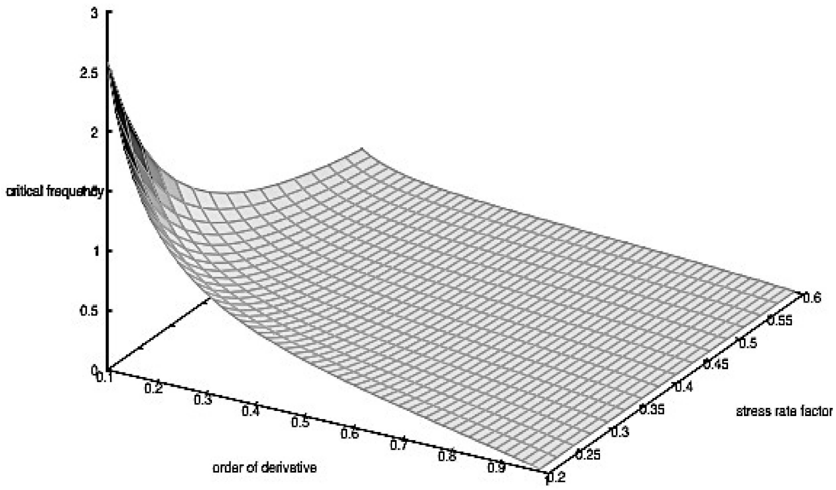

FIG. 1. The effect of both α and D.

4. Summary

By using fractional derivatives in generalization of the material strain stability analysis was performed for a few types of constitutive equations. In this study, as the main difference to classical works the internal length was obtained by calculating the critical eigenfunctions at the loss of stability. From this new approach a general condition can be given for the existence of internal length effect. In case of periodic perturbations and second gradient-dependent constitutive equation, static internal lengths are the same for both fractional and conventional strains. For the Malvern-Cristescu equation in the form $\sigma + D\dot{\sigma} = E\varepsilon + H\dot{\varepsilon}$ with fractional strain, a dynamic internal length can be defined. The values of the critical perturbing frequency and dynamic internal length decrease as the order of the derivative in strain or the stress rate intensity factor (D) increase. Such a dynamic internal length disappears when the order of the derivative in strain is one (the same as the classical result).

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