ENGINEERING TRANSACTIONS • *Engng. Trans.* • **64**, 4, 441–448, 2016
Polish Academy of Sciences • Institute of Fundamental Technological Research (IPPT PAN)
National Engineering School of Metz (ENIM) • Poznan University of Technology

Self-Equilibrium Geometry of the Class-Theta Tetrahedral Tensegrity Module

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Tensegrity structures are mechanically stable because of the way in which they balance and distribute the mechanical stress, which is not principally a result of the strength of the individual components. This category of structures has the property that, even before the application of any external load, the members of the structure are already in tension or compression, that is, they are prestressed. There are different methods to find the equilibrium state attained by a tensegrity system for a given connectivity of the constituent cables and bars. In this paper, the tensegrity tetrahedron, characteristically shaped and free-standing, i.e., with no external loads, is investigated.

Key words: analytical modeling, structural system, tensegrity.

1. Introduction

The relationship between forms and forces is one of the main topics of structural morphology. This harmonious, coexisting relationship is very strong for systems in tensegrity state, commonly called tensegrity systems. Our present interest concerns mainly engineering structures.

The geometrical form with which a tensegrity system is built is extremely important to the equilibrium state of the structure. The preferred geometry of a tensegrity system can be established numerically. Compressed bars and tensioned cables form highly complex geometrical shapes made up of triangles, as shown in Fig. 1. In the classic examples below, in Fig. 1a an elementary three-bar unit or module or cell used to form a tensegrity structure is presented, and in Fig. 1b it is an elementary four-bar tensegrity unit. This latter unit is composed of eight nodes, four bars, and twelve cables. The compressed members are of equal length (b) and the tension members are of equal length (l). The geometry of the system relies on a specific ratio (b/l) based on the lengths of the bars and

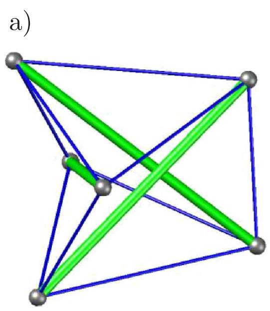

FIG. 1. The classic examples of elementary tensegrity units.

cables, and the relative rotation (α) of the upper and lower square. Self-stress equilibrium geometry for the four-bar tensegrity cell is represented in Fig. 2a. The upper square composed of cables and the lower square are relatively rotated with a 45° angle.

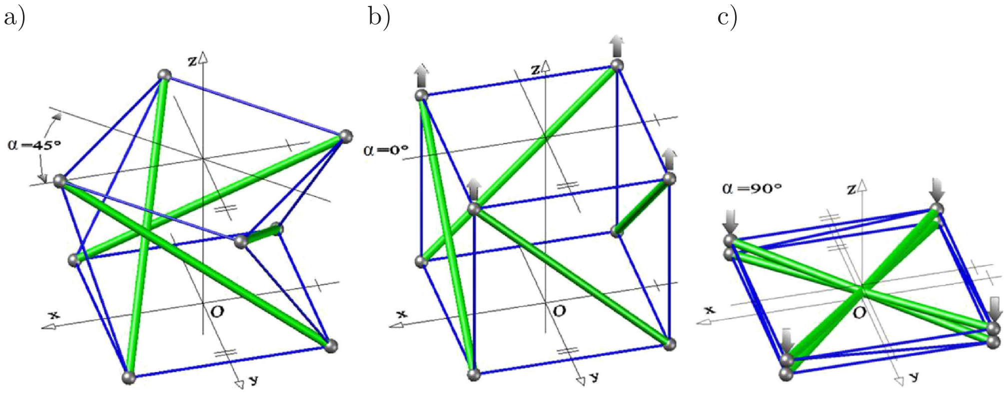

FIG. 2. Funicular shapes of the four-bar cell: a) self-stress equilibrium geometry, b) funicular geometry under upward forces, c) funicular geometry under downward forces.

The research of funicular shapes is fundamental when a description of mobile structures is required. We take as an example the tensegrity cell in Fig. 2a. If the length (b) is too small, then the system becomes unstable, yet it can reach a funicular shape under external vertical actions exerted on the four upper nodes.

It can be pointed out that this self-stress equilibrium geometry corresponds to the maximum value 1.553774 of the curve, which is relating the ratio (b/l) to the relative rotation (α) of the upper and lower square, as shown in Fig. 3. Any other lower value of the ratio (b/l) gives two geometries, which are subjected to mechanisms. So the ratio 1.414213 characterizes the two geometries for which funicular shapes are obvious, Fig. 2b and Fig. 2c, and corresponding to relative rotations $\alpha = 0°$ and $\alpha = 90°$.

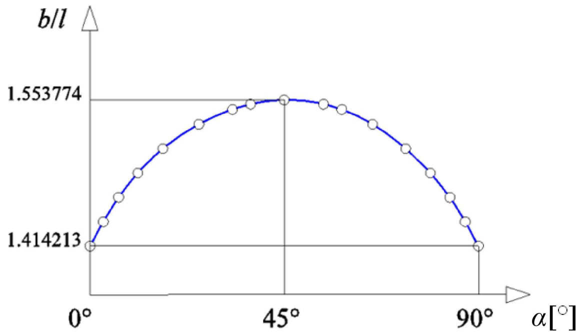

FIG. 3. Relationship between α and b/l with reference to the four-bar tensegrity cell.

1.1. Space filling tetrahedron

Space-filling means that the combination of like or complementary figures in a three-dimensional packing is continuously repeated in such a way that there is no unoccupied space. Is it possible to subdivide a space into congruent and disjoint tetrahedra? This problem is not trivial as it is in 2D since, unlike equilateral triangles, the regular tetrahedra cannot be fitted together to fill space. The paper [1] demonstrates in full the nature of the cube and the rhombic dodecahedron, respectively, as being in general made up of the homogeneous tetrahedra. Starting from here, the space-filling tetrahedron $\mathbf{T}_2$ is crucial to the researchconducted in this paper.

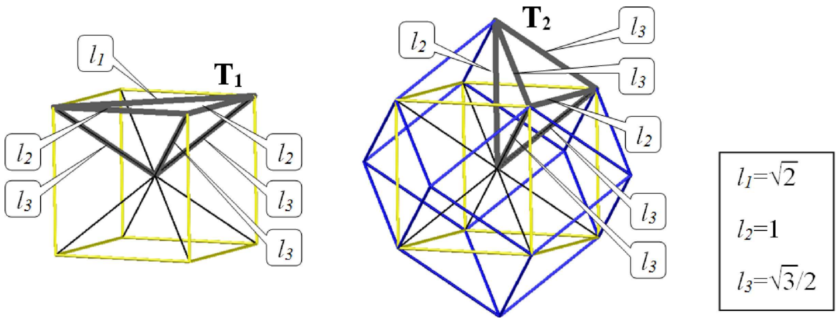

FIG. 4. Dissection of the cube as well as the rhombic dodecahedron into congruent and disjoint tetrahedra [5].

1.2. The Class ***k*** *and Class* ***Θ*** *tensegrity systems*

The *Class* ***k*** tensegrity system, according to the strictest definitions, is a structural unit based on the use of isolated components in compression inside a net of continuous tension. The tensegrity systems depicted in Figs. 1, 2a, 5a and 5b represent strictly speaking the *Class* $\boldsymbol{k} = 1$, since each node is connected to one compressive member only [2, 3]. A tensegrity system with as many as $\boldsymbol{k}$ rigid components in contact is a *Class* $\boldsymbol{k}$ tensegrity system. Figure 5a illustrates the simplest example of linear tensegrity system: one bar and one cable in tension. Other examples in relation to $\boldsymbol{k} > 1$ are shown in Figs. 5c and 5d.

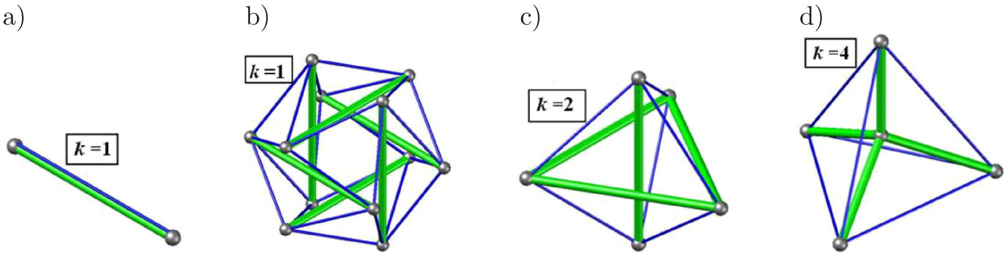

Fig. 5. Some examples of the *Class* $\boldsymbol{k}$ tensegrity systems: a) and b) *Class* $\boldsymbol{k} = 1$, c) *Class* $\boldsymbol{k} = 2$, d) *Class* $\boldsymbol{k} = 4$.

It is possible to design a separate set of cables inside the cable-bar elementary cell and to establish a self-stress state of equilibrium [4, 5]. Each of the basic tensegrity systems termed *Class* $\boldsymbol{\Theta}$ possesses an external and internal set of tension components. The shape of Greek capital letter $\boldsymbol{\Theta}$ (Theta) reflects two sets of such components. This notation corresponds to Skelton's *Class* $\boldsymbol{k}$ tensegrity structure. Refer to the space-filling tetrahedron $\mathbf{T}_2$ from Fig. 4, the representative specimens of possible topologies of the *Class* $\boldsymbol{k}$ and *Class* $\boldsymbol{\Theta}$ tensegrity systems are shown in Fig. 6.

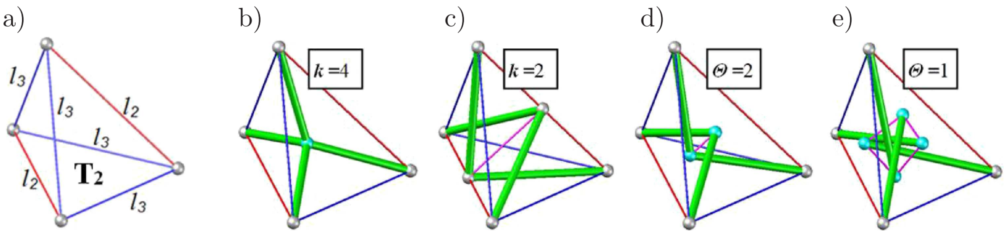

Fig. 6. The representative specimens of possible topologies of the *Class* $\boldsymbol{k}$ and *Class* $\boldsymbol{\Theta}$ tensegrity systems based on the space-filling tetrahedron $\mathbf{T}_2$.

1.3. The mother, coplanar and expanded configuration

Figure 7a exemplifies a mother/initial configuration of internal nodes. The tensegrity module, shown in Fig. 7b, is an example of coplanar configuration, in which internal nodes occupy the appropriate faces of tetrahedral cell. As shown in Figs. 7c and 7d, it is hypothetically possible to build up the coplanar configuration toward a few expanded configurations.

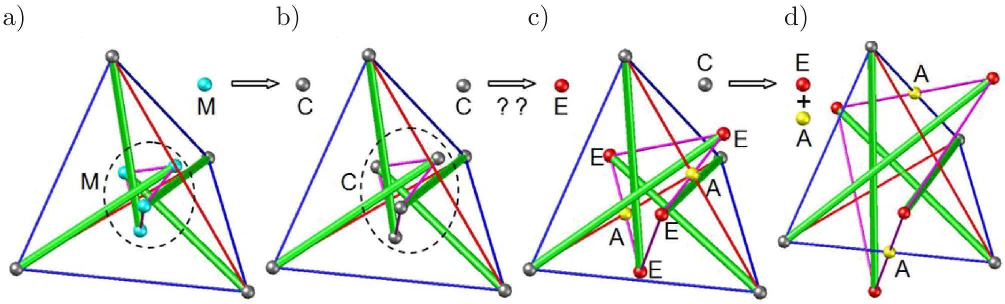

FIG. 7. M-mother/initial configuration of internal nodes, C-coplanar configuration of internal nodes, E-expanded configuration of internal nodes. Additional nodes labeled A are necessary for the existence of currently obtained expanded configurations in the self-equilibrium state.

2. Mathematical model of the class $\boldsymbol{\Theta} = 1$ tensegrity module

Numerical form finding of the *Class* $\boldsymbol{\Theta} = 1$ tensegrity tetrahedron starts from the mother configuration, similar to the example in Figs. 6e and 8b. Consider a space-filling tetrahedron of one edge length $2l$ centered at the origin of x axis, and its second edge length $2l$ that is parallel to the y axis of Cartesian system, as shown in Fig. 8a. If the Cartesian coordinates of one edge or cable are $(-l, 0, 0)$ and $(l, 0, 0)$, then its second edge or cable coordinates will be respectively $(0, -l, l)$ and $(0, l, l)$. The coordinates of the other bar ends are also recorded in Eq. (2.1) and shown in Fig. 8.

$$
\begin{aligned}
&XB1 \rightarrow (-l, 0, 0)\,,\\
&XB2 \rightarrow (l, 0, 0)\,,\\
&YT1 \rightarrow (0, -l, l)\,,\\
&YT2 \rightarrow (0, l, l)\,,\\
&TB1 \rightarrow \left(\frac{1}{2}c\cos\alpha, \frac{1}{2}c\sin\alpha, \frac{1}{2}\left(l + c\cos\left(\frac{\pi}{4} - \alpha\right)\right)\right),\\
&TB2 \rightarrow \left(-\frac{1}{2}c\cos\alpha, -\frac{1}{2}c\sin\alpha, \frac{1}{2}\left(l + c\cos\left(\frac{\pi}{4} - \alpha\right)\right)\right),\\
&BT1 \rightarrow \left(\frac{1}{2}c\sin\alpha, \frac{1}{2}c\cos\alpha, \frac{1}{2}\left(l - c\cos\left(\frac{\pi}{4} - \alpha\right)\right)\right),\\
&BT2 \rightarrow \left(-\frac{1}{2}c\sin\alpha, -\frac{1}{2}c\cos\alpha, \frac{1}{2}\left(l - c\cos\left(\frac{\pi}{4} - \alpha\right)\right)\right).
\end{aligned} \tag{2.1}
$$

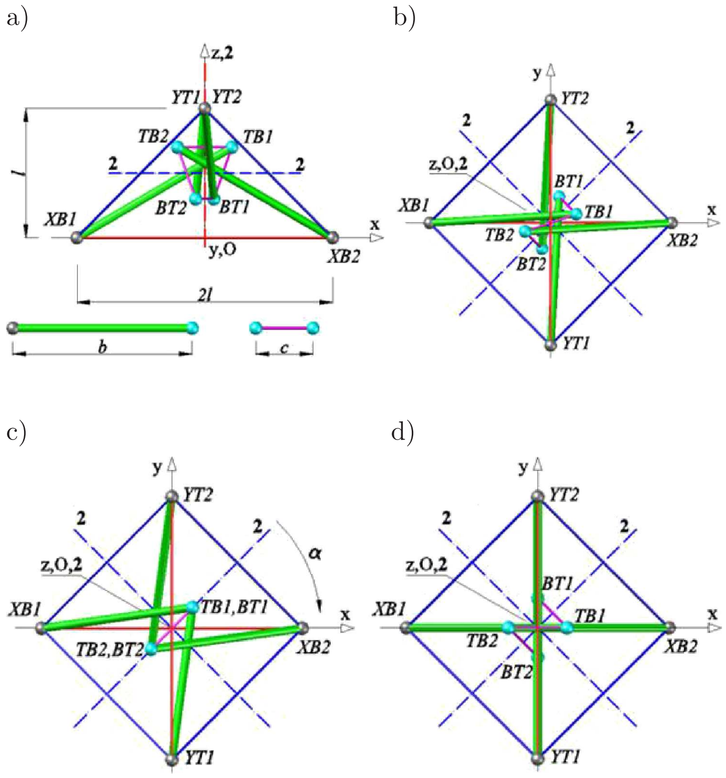

FIG. 8. Model of the *Class* $\boldsymbol{\Theta} = 1$ tensegrity tetrahedron: a) front view of the mother configuration, b) top view of the mother configuration, c) top view of model in the first extreme position for $\alpha = 0°$, d) top view of model in the second extreme position for $\alpha = 45°$.

The value b, that is, the length each of four bars can be determined by subtracting the coordinates of one bar end (e.g., *XB1*) from the coordinates of the other end (*TB1*). Therefore, the length of the bar between *TB1* and *XB1* is

$$b = \left[\frac{1}{4}c^2 + l^2 + \frac{1}{4}\left(l + c\cos\alpha\right)^2 + lc\cos\left(\frac{\pi}{4} - \alpha\right)\right]^{1/2}. \tag{2.2}$$

Thus, the relationship (2.2) is the function of several variables: l, c and α. Only when the distances between *TB1* and *XB1*, *TB2*, *XB2*, etc. achieve a maximum, then all the elements are stressed and the tensegrity system can be stable, see Fig. 9. The quality of theoretical results depicted in Fig. 10 was confirmed by both physical and 3D graphical models [6].

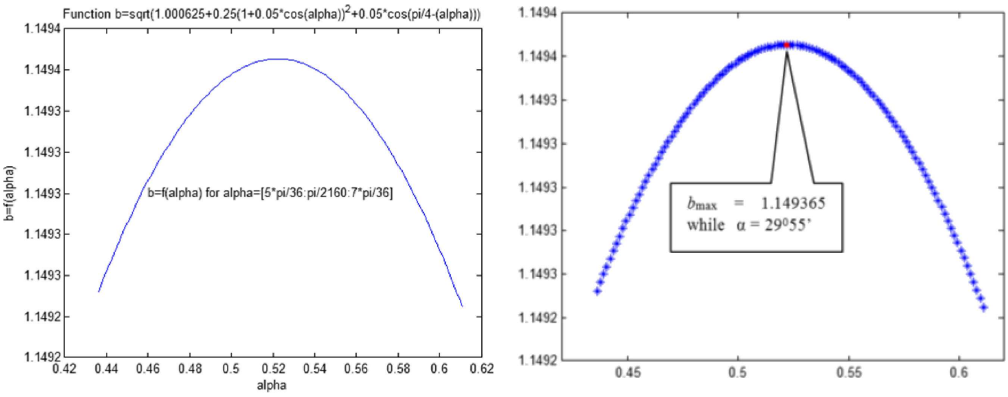

FIG. 9. The Matlab graph $b_{\max} = f(\alpha)$ for $c = 0.05$.

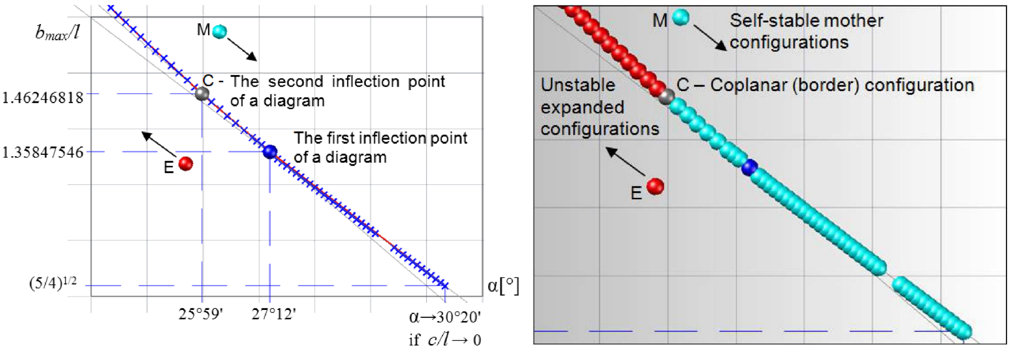

FIG. 10. Diagram of the maximal length of bars $b_{\max}$ (if $l = 1$) as a function of the geometric parameters: either the given angle α or the given length of internal cables c interchangeably.

REFERENCES

1. BIENIEK Z., *Space-filling tetrahedra*, Symmetry: Art and Science, The Journal of the International Society for the Interdisciplinary Study of Symmetry, Special Issue for the Conference of ISIS Symmetry Wrocław – Kraków, Poland, Lugosi G., Nagy D. [Eds.], September 14–19, 2009/1–4, pp. 44–47, 2009.

2. SKELTON R.E., HELTON J.W., ADHIKARI R., PINAUD J.P., CHAN W., *An introduction to the mechanics of tensegrity structures, dynamics and control of aerospace systems*, University of California, San Diego, CRC Press LLC, 2002.

3. SKELTON R.E., DE OLIVEIRA M.C., *Tensegrity systems*, Springer, 2009.

4. BIENIEK Z., *Chosen ideas of geometrical shaping of modular tensegrity structures*, Structural Engineers World Congress, Como, Italy, Congress paper on CD, 2011.

5. Bieniek Z., *Examples of cable-bar modular structures based on the Class-Theta tensegrity systems*, Journal of Civil Engineering and Architecture, **9**: 1452–1462, 2015.

6. Bieniek Z., *A mathematical model of the Class Theta tetrahedral tensegrity module*, Lightweight Structures in Civil Engineering – Contemporary Problems, XXI LSCE, Local Seminar of IASS Polish Chapter, Tarczewski R., Bieniek Z. [Eds.], Rzeszów University of Technology, pp. 9–16, 2015.

Received October 16, 2016; accepted version October 29, 2016.