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On Shape and Material Optimization of Isotropic Bodies

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This paper deals with the free material design and its two constrained versions constructed by imposing isotropy with (i) independent bulk and shear moduli, and (ii) fixed Poisson's ratio. In the latter case, the Young modulus is the only design variable. The moduli are viewed as non-negative, thus allowing for the appearance of void domains within the design domain. The paper shows that all these methods reduce to one stress-based problem in which the norm involved reflects the type of the constraints imposed.

Key words: optimum design, elastic moduli, free material design.

1. Introduction

The *free material design* (FMD) method put forward originally in Bendsøe *et al.* [1] leads to a simultaneous designing of the anisotropy and the material placement. In this method, the minimized merit function is the global compliance, while the design variable is the field of Hooke's tensor $\mathbf{C}$, subject to the cost condition expressed by the integral of the trace of tensor $\mathbf{C}$ over the design domain. The tensor $\mathbf{C}$ is subject to symmetry required in elasticity and to the conditions of positive semi-definiteness. A natural starting point for this theory is the spectral representation of the Hooke tensors, as proposed by Rychlewski [10]. The FMD method delivers a tool for cutting out a material domain from a design domain, thus linking the material and shape optimization. The material domain turns out to be the effective domain of the solution to the auxiliary problem:

$$\min \left\{ \int_{\Omega} d\rho\left(\boldsymbol{\tau}\right) \mid \boldsymbol{\tau} \quad \text{statically admissible stress fields} \right\} (\mathrm{P}).$$

Problem (P) has been derived in [4, 6], while its mathematical background is described by Bouchitté and Buttazzo in [2] concerning the optimal mass distribution. It becomes apparent that problem (P) should be, in general, expressed in terms of the theory of Radon measures. In a regular case, the integrand in (P) has the form $d\rho(\boldsymbol{\tau}) = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} dx$, where dx is the Lebesgue measure.

It turns out that problem (P) also appears in other versions of the FMD method, in which additional symmetry conditions are imposed on tensor $\mathbf{C}$. In the present paper, two versions of the FMD method are discussed: (i) the isotropic material design (IMD) method of designing an isotropic material of independent varying bulk and shear moduli, and (ii) the Young modulus design (YMD) method, in which Poisson's ratio is fixed while Young's modulus is the design variable. The unknown moduli are viewed as non-negative scalar fields. In both the methods, the minimum compliance problem reduces to the problems of the form similar to (P), with different integrands. Yet, in each case the integrand is expressed by a norm of the stress field. Only this property decides that the problem (P), in all its forms, determines the shape of a body as the effective domain (or a support, if it is a measure) of the solution. In this manner, we prove that the FMD, IMD, YMD methods (as well as the cubic material design method (CMD) proposed in [8]) solve two following problems simultaneously: optimal shape design and material optimal layout.

A conventional notation is applied- the design domain in R^n is denoted by Ω; in the case of $n = 3$, the domain is parameterized by the Cartesian system (x_1, x_2, x_3) with the orthogonal basis $\mathbf{e}_i$, $i = 1, 2, 3$. The set of second rank symmetric tensors is denoted by E_s^2. The set of fourth rank tensors satisfying the symmetries $C_{ijkl} = C_{klij}$, $C_{ijkl} = C_{jikl}$ is denoted by E_s^4. The trace of $\mathbf{C}$ is defined by: $\mathrm{tr}\mathbf{C} = C_{ijij}$. The identity tensor in E_s^4 is represented by $\mathbf{I} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$. The scalar product of $\boldsymbol{\sigma},\ \boldsymbol{\varepsilon} \in E_s^2$ is defined by $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \sigma_{ij}\, \varepsilon_{ij}$, where repetition of indices imply summation. The Euclidean norm of $\boldsymbol{\sigma} \in E_s^2$ is defined by $\|\boldsymbol{\sigma}\| = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{1/2}$. Comma implies partial differentiation, e.g, $\partial(\cdot)/\partial x_i = (\cdot)_{,i}$. The symmetric part of the gradient of the vector field $\mathbf{v}$ is denoted by $\boldsymbol{\varepsilon}_{ij}(\mathbf{v}) = (\nu_{i,j} + \nu_{j,i})/2$.

2. Free material design (FMD) revisited

Consider a non-homogeneous anisotropic linearly elastic body, occupying a given domain Ω, supported on the part Γ_2 of the boundary and subject to tractions $\mathbf{T}$ on the remaining part of the boundary Γ_1. The deformed configuration is given by the displacement field $\mathbf{u}$. The compliance is expressed as

$$\wp = f(\mathbf{u}), \qquad f(\mathbf{v}) = \int_{\Gamma_1} \mathbf{T} \cdot \mathbf{v}, \tag{2.1}$$

where $\mathbf{v} = (v_1, v_2, v_3)$ represents a virtual displacement field. The compliance will be treated as a functional of argument $\mathbf{C} = (C_{ijkl})$ – a tensor field of elastic moduli on the design domain. We assume that the mentioned symmetry properties are satisfied together with the assumption of positive semi-definiteness, denoted by $\mathbf{C} \geqslant \mathbf{0}$. Let us define the function:

$$j(\boldsymbol{\eta}) = \frac{1}{2}\boldsymbol{\eta} \cdot (\mathbf{C}\boldsymbol{\eta}), \qquad \boldsymbol{\eta} \in E_s^2 \tag{2.2}$$

and its Fenchel transform

$$j^*(\boldsymbol{\tau}) = \max\{\boldsymbol{\tau} \cdot \boldsymbol{\eta} - j(\boldsymbol{\eta}) \,\big|\, \boldsymbol{\eta} \in E_s^2\}. \tag{2.3}$$

The tensor field $\mathbf{C}$ satisfying the point-wise conditions: $\mathbf{C} \in E_s^4$, $\mathbf{C} \geqslant \mathbf{0}$, and the global condition

$$\int_\Omega \operatorname{tr} \mathbf{C}\, dx = \Lambda \tag{2.4}$$

is the main design variable. The set of statically admissible stresses $\boldsymbol{\tau} = (\tau_{ij})$ is denoted by $\Sigma_T(\Omega)$. The compliance of the body of a given distribution of elastic moduli is expressed by

$$\wp(\mathbf{C}) = \inf\left\{ \int_\Omega j^*(\boldsymbol{\tau}) \,|\boldsymbol{\tau} \in \Sigma_T(\Omega) \right\}. \tag{2.5}$$

We consider the optimum design problem

$$Y = \inf\{\wp(\mathbf{C}) \,\big|\, \mathbf{C}(x) \in E_s^4, \mathbf{C}(x) \geqslant \mathbf{0} \quad \text{a.e. in } \Omega, \quad \mathbf{C} \text{ satisfies (2.4)}\}. \tag{2.6}$$

One can prove that

$$Y = \frac{Z^2}{\Lambda}, \qquad Z = \inf\left\{ \int_\Omega \|\boldsymbol{\tau}\| \,|\boldsymbol{\tau} \in \Sigma_T(\Omega) \right\} \ (\mathrm{P_{FMD}}). \tag{2.7}$$

The problem dual to $(\mathrm{P_{FMD}})$ reads

$$\begin{aligned} Z = \sup\{f(\mathbf{v}) \,|\, &\text{kinematically admissible } \mathbf{v}, \\ &\|\boldsymbol{\varepsilon}(\mathbf{v}(x))\| \leqslant 1, \ \text{a.e. in } \Omega\} \ (\mathrm{P^*_{FMD}}). \end{aligned} \tag{2.8}$$

Upon putting the above problems in a rigorous form, see [2], one can prove their well-posedness. The solution $\breve{\boldsymbol{\tau}}$ to problem ($\mathrm{P_{FMD}}$) determines the optimal moduli of elasticity:

$$\breve{C}_{ijkl} = \lambda_1 \omega_{ij} \omega_{kl}, \qquad \omega_{ij}(x) = \frac{\breve{\tau}_{ij}(x)}{\|\breve{\boldsymbol{\tau}}(x)\|}, \qquad \lambda_1(x) = \Lambda \frac{\|\breve{\boldsymbol{\tau}}(x)\|}{\int\limits_\Omega \|\breve{\boldsymbol{\tau}}\|}. \tag{2.9}$$

Thus λ_1 is the only non-zero eigenvalue of tensor $\breve{\mathbf{C}}$. Despite this degeneracy, the elasticity problem of the body with optimal elastic moduli (2.9) is well-posed: the stress field $\breve{\boldsymbol{\sigma}}$ transmitting the given tractions to the given support exists, its uniqueness being still not proved. Moreover, this field is one of minimizers of problem ($\mathrm{P_{FMD}}$), or one can write $\breve{\boldsymbol{\sigma}} = \breve{\boldsymbol{\tau}}$.

3. Isotropic material design (IMD)

Consider now the isotropic designs. The unknown tensor $\mathbf{C}$ is assumed in the form

$$\mathbf{C} = nk\boldsymbol{\Lambda}_1 + 2\mu\boldsymbol{\Lambda}_2, \qquad \boldsymbol{\Lambda}_1 = \frac{1}{n}\delta_{ij}\delta_{kl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \qquad \boldsymbol{\Lambda}_2 = \mathbf{I} - \boldsymbol{\Lambda}_1. \tag{3.1}$$

n being the dimension of the problem. The bulk modulus $k(x) \geqslant 0$ and the shear modulus $\mu(x) \geqslant 0$ are independent design variables. The minimum compliance problem has the form (2.6), where $\mathbf{C}$ is given by (3.1) and minimization takes over both the moduli subject to the cost condition (2.4) with $\operatorname{tr}\mathbf{C} = 3k + 10\mu$ for $n = 3$ and $\operatorname{tr}\mathbf{C} = 2k + 4\mu$ for $n = 2$. The formula $(2.7)_1$ still holds, see [3, 7]. The counterpart of $(2.7)_2$ assumes the form

$$Z = \inf\left\{ \int\limits_\Omega \|\boldsymbol{\tau}\|_{\left(\frac{1}{\sqrt{n}}, \sqrt{3n-4}\right)} \,|\, \boldsymbol{\tau} \in \Sigma_T(\Omega) \right\} \quad (\mathrm{P_{IMD}}), \tag{3.2}$$

where

$$\|\boldsymbol{\tau}\|_{(\alpha,\beta)} = \alpha\,|\operatorname{tr}\boldsymbol{\tau}| + \beta\,\|\operatorname{dev}\boldsymbol{\tau}\|. \tag{3.3}$$

Assume that $n = 3$. Having solved problem (3.2), one can find the optimal moduli by the rules:

$$3\breve{k}(x) = \frac{\Lambda}{\int\limits_\Omega \|\breve{\boldsymbol{\tau}}\|_{\left(\frac{1}{\sqrt{3}}, \sqrt{5}\right)}} \frac{|\operatorname{tr}\breve{\boldsymbol{\tau}}(x)|}{\sqrt{3}}, \qquad 10\breve{\mu}(x) = \frac{\Lambda\sqrt{5}}{\int\limits_\Omega \|\breve{\boldsymbol{\tau}}\|_{\left(\frac{1}{\sqrt{3}}, \sqrt{5}\right)}} \|\det\breve{\boldsymbol{\tau}}(x)\|. \tag{3.4}$$

The material is necessary in the subdomains where both the moduli presented above vanish.

4. Young's modulus design (YMD)

Assume that the distribution of Poisson's ratio is given. Consider the problem of optimum design of Young's modulus to minimize the compliance with the cost condition (2.4), in which $\operatorname{tr}\mathbf{C} = aE$ for $n = 3$, here $a = (6-9\nu)(1+\nu)^{-1}(1-2\nu)^{-1}$. The formula $(2.7)_1$ holds, with a new problem ($\mathrm{P}_{\mathrm{YMD}}$) expressed by $(2.7)_2$, where the norm involved has now the following form:

$$\|\boldsymbol{\tau}\|_{\mathrm{YMD}} = \left(\frac{6-9\nu}{3\,(1+\nu)}\,(\operatorname{tr}\boldsymbol{\tau})^2 + \frac{6-9\nu}{1-2\nu}\,\|\operatorname{dev}\boldsymbol{\tau}\|^2\right)^{1/2}. \tag{4.1}$$

Let $\breve{\boldsymbol{\tau}}$ be a solution to problem $(2.7)_2$ with the norm (4.1). The optimal Young's modulus is given by

$$\breve{E}(x) = \frac{1}{a}\,\frac{\Lambda}{\displaystyle\int_{\Omega}\|\breve{\boldsymbol{\tau}}\|_{\mathrm{YMD}}}\,\|\breve{\boldsymbol{\tau}}(x)\|_{\mathrm{YMD}}\,. \tag{4.2}$$

The stress field in the optimal body coincides with one of the stress fields being solutions to $(2.7)_2$.

5. Example and final remarks

The example concerns the YMD optimal in-plane design. We consider L-shaped plate, see Fig. 1a ($h_1 = h_2 = 2l$). The plate is fixed along its upper horizontal boundary. The right vertical segment is subjected to a constant tangent traction of intensity q. Poisson's ratio ν is constant and equal to 0.3. The problem ($\mathrm{P}_{\mathrm{YMD}}$) is solved numerically by the method elaborated

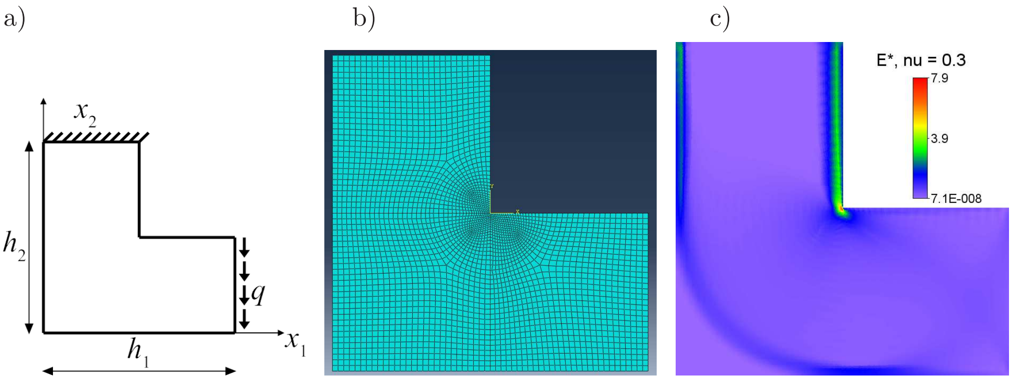

Fig. 1. a) L-shaped domain problem: geometry, load and boundary conditions, b) FEM mesh, c) the YMD prediction – scatter plot visualization of optimal Young's modulus $\breve{E}$.

in [4, 6] for the FMD approach; a non-uniform finite element mesh is shown in Fig. 1b. The optimal Young modulus $\breve{E}$ assumes the extreme values at the re-entrant corner and the smallest values around the left lower corner and between vertical strips close to the support, see Fig. 1c. Optimal compliance calculated by formula (2.7), with Z corresponding to the 2D counterpart of the norm (4.1), is equal to $Y = 19.55\frac{q^2 l}{\Lambda}|\Omega|$. The YMD prediction $\breve{E}$ compares favorably with those available in the literature, see [9] concerning the FMD and thickness optimization problems, for the case where no local stress constraints are imposed. The material domain is in fact cut out from the design domain.

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References

1. Bendsøe M.P., Guedes J.M., Haber R.B., Pedersen P., Taylor J.E., *An analytical model to predict optimal material properties in the context of optimal structural design*, Journal of Applied Mechanics, Transactions of ASME, **61**(4): 930–937, 1994, doi: 10.1115/1.2901581.

2. Bouchitté G., Buttazzo G., *Characterization of optimal shapes and masses through Monge-Kantorovich equation*, Journal of the European Mathematical Society, **3**(2): 139–168, 2001, doi: 10.1007/s100970000027.

3. Czarnecki S., *Isotropic material design*, Computational Methods in Science and Technology, **21**(2): 49–64, 2015, doi: 10.12921/cmst.2015.21.02.001.

4. Czarnecki S., Lewiński T., *A stress-based formulation of the free material design problem with the trace constraint and single loading condition*, Bulletin of the Polish Academy of Sciences: Technical Sciences, **60**(2): 191–204, 2012.

5. Czarnecki S., Lewiński T., *On minimum compliance problems of thin elastic plates of varying thickness*, Structural and Multidisciplinary Optimization, **48**(1): 17–31, 2013, doi: 10.1007/s00158-013-0893-x.

6. Czarnecki S., Lewiński T., *A stress-based formulation of the free material design problem with the trace constraint and multiple load conditions*, Structural and Multidisciplinary Optimization, **49**(5): 707–731, 2014, doi: 10.1007/s00158-013-1023-5.

7. Czarnecki S., Wawruch P., *The emergence of auxetic material as a result of optimal isotropic design*, Physica Status Solidi B, **252**(7): 1620–1630, 2015, doi: 10.1002/pssb.201451733.

8. Czubacki R., Lewiński T., *Topology optimization of spatial continuum structures made of non-homogeneous material of cubic symmetry*, Journal of Mechanics of Materials and Structures, **10**(4): 519–535, 2015, doi: 10.2140/jomms.2015.10.519.

9. Kočvara M., Stingl M., *Free material optimization for stress constraints*, Structural and Multidisciplinary Optimization, **33**(4): 323–335, 2007, doi: 10.1007/s00158-007-0095-5.

10. Rychlewski J., *On Hooke's law*, Journal of Applied Mathematics and Mechanics, **48**(3): 303–314, 1984, doi: 10.1016/0021-8928(84)90137-0.

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