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Research Paper

Bending of a Seven Layer Beam with Foam Cores

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The subject of the paper is a seven layer beam with foam cores. The structure of the beam is symmetrical. The beam is composed of the main core, two inner sheets, two second cores and two outer sheets. The main core and two face cores are metal and polyurethane foams, while the sheets are metal. The analytical model of the beam is developed. The displacement and strain fields are formulated with consideration of the Zig-Zag hypothesis of deformation of a flat cross-section of the beam. The governing differential equations for the seven layer beam are obtained based on the stationary total potential energy. The detailed studies are devoted to deflections and stresses of the beams under a uniformly distributed load. The influence of the foam type of cores on the deflections and stresses of the beam is analysed. Moreover, the numerical FEM-model of the beam is developed. The analytical solution is compared to numerical calculations – FEM studies (ABAQUS System and SolidWorks Simulation). The results of the analysis are presented in Tables and Figures.

Key words: composite beam; shear effect; Zig-Zag hypothesis; deflections and stresses.

1. Introduction

A generalization of classical sandwich structures involves, for example, multilayer structures. Allen [1] described analytical modeling and analysis of sandwich structures. Vinson [2] presented comprehensive discussion of the structural mechanics involved in the field of sandwich structures. Carrera [3] described a historical review of the theories that have been developed for the analysis of multilayered structures, especially of Zig-Zag theories. Magnucka-Blandzi and Magnucki [4] analyzed a simply supported sandwich beam with a metal foam core and varying mechanical properties of the core using nonlinear hypothesis of deformation of a plane cross section of the beam. Chakrabarti *et al.* [5]

described higher order zigzag theory for the static analysis of laminated sandwich beam with soft core where in-plane displacement variation is considered to be cubic for both the face sheets and the core. Magnucka-Blandzi [6] presented vibrations of simply supported sandwich beams with a metal foam core with three different hypotheses of the fields of displacement for the flat cross section of the beam. Magnucka-Blandzi [7] presented buckling of a simply supported three layer rectangular sandwich plate with isotropic facings and a core. Osei-Antwi *et al.* [8] analyzed models for predicting axial and shear stresses in multilayer sandwich structures composed of stiff core layers and intermediate laminates. Zhang *et al.* [9] described dynamic response of fully clamped slender metal foam core multilayer sandwich beams struck by a low-velocity heavy mass. Chen *et al.* [10] presented nonlinear free vibration characteristics of sandwich porous beams with non-uniform and uniform porosity distributions. Caliri Jr. *et al.* [11] presented a review of theories and solution methods for laminated and sandwich structures. Malinowski *et al.* [12] numerically studied buckling and post-buckling problems of an elastic seven-layered cylindrical shell under uniformly distributed pressure. Magnucki *et al.* [13] described the seven-layer beam with transverse sinusoidal corrugated main core. Paczos *et al.* [14] studied orthotropic sandwich beams that consist of five layers and presented experimental and numerical results and sensitivity analysis of the beam. Magnucka-Blandzi *et al.* [15, 16] presented bending, buckling and vibrations of sandwich beams with corrugated main core. Magnucka-Blandzi and Rodak [17] determined deflection and critical axial force for seven layer beams with a lengthwise trapezoidal corrugated main core and two crosswise trapezoidal corrugated cores of faces. Smyczyński and Magnucka-Blandzi [18] presented stability analysis of a simply supported three layer beam with nonlinear hypothesis of deformation of the cross section of the beam. Magnucka-Blandzi *et al.* [19, 20] studied strength, stability and vibrations of a metal seven-layer rectangular plate with trapezoidal corrugated cores. Sayyad and Ghugal [21] delivered a critical review of literature on bending, buckling and free vibration analysis of shear deformable isotropic, laminated composite and sandwich beams based on equivalent single layer theories, layerwise theories, Zig-Zag theories and exact elasticity solution. Magnucka-Blandzi [22] analyzed bending and buckling of a metal seven-layer beam with crosswise corrugated main core and compared with analysis of sandwich beam. Smyczynski and Magnucka-Blandzi [23] presented strength analysis of a simply supported three layer beam under three point bending. Abrate and Di Scoiva [24] presented a review of equivalent single layer theories for composite and sandwich structures. Smith *et al.* [25], Szyniszewski *e al.* [26] and Klasztorny *et al.* [27] described manufacturing, applications and mechanical properties of metal foams. Particular group of the multi-layer structures includes seven-layer plates with corrugated cores.

The subject of the study is a simply supported symmetrical seven layer beam of length L and width b. The beam is composed of the main core of thickness t_{c1} and Young's modulus E_{c1}, two inner sheets of thicknesses t_s and Young's modules E_{si}, two second cores of thicknesses t_{c2} and Young's modules E_{c2}, and two outer sheets of thicknesses t_s and Young's modules E_{so}. The main core and two face cores are metal or polyurethane foams, while the sheets are metal. The beam is under uniformly distributed transverse load of intensity q (Fig. 1).

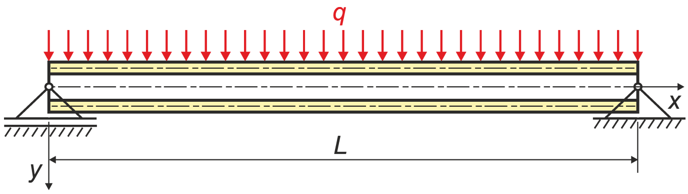

FIG. 1. Scheme of the symmetrical seven layer beam under transverse load.

The objectives of the study are:

- development of analytical model of the seven-layer beam with consideration of the shear effect and calculation of the deflection values for some selected beams;
- deflection calculation of the same beams in accordance with the classical Euler-Bernoulli beam theory;
- numerical calculation of these deflections with the use of FEM systems (SolidWorks and ABAQUS);
- comparison of the analytical and numerical results.

2. Analytical model of the beam

The Zig-Zag hypothesis is assumed for modelling of the beam. The straight line before bending transforms into a broken line (Fig. 2).

Displacements in subsequent layers of the beam with consideration of the hypothesis are:

- the main core – first core $\{-t_{c1}/2 \leq y \leq t_{c1}/2\}$

$$u(x,y) = -y\left(\frac{\mathrm{d}v}{\mathrm{d}x} - 2\frac{u_1(x)}{t_{c1}}\right), \tag{2.1}$$

- the inner sheets
  - the upper sheet $\{-(t_s + t_{c1}/2) \leq y \leq -t_{c1}/2\}$

$$u(x,y) = -\left(y\frac{\mathrm{d}v}{\mathrm{d}x} + u_1(x)\right), \tag{2.2}$$

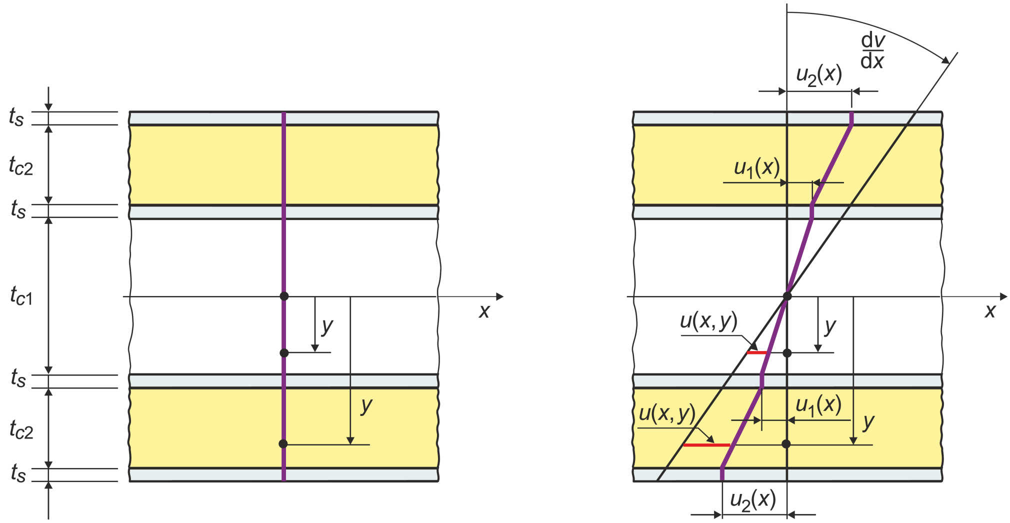

FIG. 2. The scheme of the deformation of straight line – the Zig-Zag hypothesis.

– the lower sheet $\{t_{c1}/2 \le y \le t_{c1}/2 + t_s\}$

$$u(x, y) = -\left(y\frac{\mathrm{d}v}{\mathrm{d}x} - u_1(x)\right), \tag{2.3}$$

- the second cores
  – the upper core $\{-(t_{c2} + t_s + t_{c1}/2) \le y \le -(t_s + t_{c1}/2)\}$

$$u(x, y) = -y\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{u_2(x) - u_1(x)}{t_{c2}}\right) - \left(\frac{t_{c2} + t_s + t_{c1}/2}{t_{c2}}u_1(x) - \frac{t_s + t_{c1}/2}{t_{c2}}u_2(x)\right), \tag{2.4}$$

  – the lower core $\{t_s + t_{c1}/2 \le y \le t_{c2} + t_s + t_{c1}/2\}$

$$u(x, y) = -y\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{u_2(x) - u_1(x)}{t_{c2}}\right) + \left(\frac{t_{c2} + t_s + t_{c1}/2}{t_{c2}}u_1(x) - \frac{t_s + t_{c1}/2}{t_{c2}}u_2(x)\right), \tag{2.5}$$

- the outer sheets
  – the upper sheet $\{-(t_{c2} + 2t_s + t_{c1}/2) \le y \le -(t_{c2} + t_s + t_{c1}/2)\}$

$$u(x,y) = -\left(y\frac{\mathrm{d}v}{\mathrm{d}x} + u_2(x)\right), \tag{2.6}$$

– the lower sheet $\{t_{c2} + t_s + t_{c1}/2 \le y \le t_{c2} + 2t_s + t_{c1}/2\}$

$$u(x,y) = -\left(y\frac{\mathrm{d}v}{\mathrm{d}x} - u_2(x)\right). \tag{2.7}$$

Therefore, the strains in the layers of the beam are as follows:

- the main core – first core $\{-t_{c1}/2 \le y \le t_{c1}/2\}$

$$\varepsilon_x = \frac{\partial u}{\partial x} = -y\left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - \frac{2}{t_{c1}}\frac{\mathrm{d}u_1}{\mathrm{d}x}\right), \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\mathrm{d}v}{\mathrm{d}x} = 2\frac{u_1(x)}{t_{c1}}, \tag{2.8}$$

- the inner sheets
  – the upper sheet $\{-(t_s + t_{c1}/2) \le y \le -t_{c1}/2\}$

$$\varepsilon_x = \frac{\partial u}{\partial x} = -\left(y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \frac{\mathrm{d}u_1}{\mathrm{d}x}\right), \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\mathrm{d}v}{\mathrm{d}x} = 0, \tag{2.9}$$

  – the lower sheet $\{t_{c1}/2 \le y \le t_{c1}/2 + t_s\}$

$$\varepsilon_x = \frac{\partial u}{\partial x} = -\left(y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - \frac{\mathrm{d}u_1}{\mathrm{d}x}\right), \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\mathrm{d}v}{\mathrm{d}x} = 0, \tag{2.10}$$

- the second cores
  – the upper core $\{-(t_{c2} + t_s + t_{c1}/2) \le y \le -(t_s + t_{c1}/2)\}$

$$\begin{aligned}\varepsilon_x = \frac{\partial u}{\partial x} = -y&\left[\frac{d^2 v}{dx^2} - \frac{1}{t_{c2}}\left(\frac{\mathrm{d}u_2}{\mathrm{d}x} - \frac{\mathrm{d}u_1}{\mathrm{d}x}\right)\right] \\ &- \left(\frac{t_{c2} + t_s + t_{c1}/2}{t_{c2}}\cdot\frac{\mathrm{d}u_1}{\mathrm{d}x} - \frac{t_s + t_{c1}/2}{t_{c2}}\cdot\frac{\mathrm{d}u_2}{\mathrm{d}x}\right),\end{aligned} \tag{2.11}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{u_2(x) - u_1(x)}{t_{c2}}, \tag{2.12}$$

  – the lower core $\{t_s + t_{c1}/2 \le y \le t_{c2} + t_s + t_{c1}/2\}$

$$\begin{aligned}\varepsilon_x = \frac{\partial u}{\partial x} = -y&\left[\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - \frac{1}{t_{c2}}\left(\frac{\mathrm{d}u_2}{\mathrm{d}x} - \frac{\mathrm{d}u_1}{\mathrm{d}x}\right)\right] \\ &+ \left(\frac{t_{c2} + t_s + t_{c1}/2}{t_{c2}}\cdot\frac{\mathrm{d}u_1}{\mathrm{d}x} - \frac{t_s + t_{c1}/2}{t_{c2}}\cdot\frac{\mathrm{d}u_2}{\mathrm{d}x}\right),\end{aligned} \tag{2.13}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{u_2(x) - u_1(x)}{t_{c2}}, \tag{2.14}$$

- the outer sheets
  - the upper sheet $\{-(t_{c2}+2t_s+t_{c1}/2) \le y \le -(t_{c2}+t_s+t_{c1}/2)\}$

$$\varepsilon_x = \frac{\partial u}{\partial x} = -\left(y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \frac{\mathrm{d}u_2}{\mathrm{d}x}\right), \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\mathrm{d}v}{dx} = 0, \tag{2.15}$$

  - the lower sheet $\{t_{c2}+t_s+t_{c1}/2 \le y \le t_{c2}+2t_s+t_{c1}/2\}$

$$\varepsilon_x = \frac{\partial u}{\partial x} = -\left(y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - \frac{\mathrm{d}u_2}{\mathrm{d}x}\right). \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\mathrm{d}v}{\mathrm{d}x} = 0. \tag{2.16}$$

Stresses in the layers of the beam are formulated in accordance with the Hooke's law. Therefore, the elastic strain energy of the beam

$$U_\varepsilon = U_\varepsilon^{(c1)} + U_\varepsilon^{(is)} + U_\varepsilon^{(c2)} + U_\varepsilon^{(os)}, \tag{2.17}$$

where

- the elastic strain energy of the main core after integration with regard to thickness of this core

$$U_\varepsilon^{(c1)} = \frac{1}{2}E_{c1}bt_{c1}\int_0^L \left[\frac{1}{12}\left(t_{c1}\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - 2\frac{\mathrm{d}u_1}{\mathrm{d}x}\right)^2 + \frac{2}{1+\nu_{c1}}\frac{u_1^2(x)}{t_{c1}^2}\right]\mathrm{d}x, \tag{2.18}$$

- the sum of the elastic strain energy of two inner sheets after integration with regard to thickness of these sheets

$$U_\varepsilon^{(is)} = E_{si}bt_s\int_0^L \left[\frac{1}{12}C_{s1}t_s^2\left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right)^2 - C_{s2}t_s\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\frac{\mathrm{d}u_1}{\mathrm{d}x} + \left(\frac{\mathrm{d}u_1}{\mathrm{d}x}\right)^2\right]\mathrm{d}x, \tag{2.19}$$

  where dimensionless parameters

$$C_{s1} = \frac{1}{t_s^2}\left(3t_{c1}^2 + 6t_{c1}t_s + 4t_s^2\right), \qquad C_{s2} = \frac{1}{t_s}\left(t_{c1}+t_s\right),$$

- the sum of the elastic strain energy of two second cores after integration with regard to thickness of these cores

$$U_\varepsilon^{(c2)} = E_{c2}bt_{c2}\int_0^L \{\Phi_1(x) - \Phi_2(x) + \Phi_3(x) + \Phi_4(x)\}\,\mathrm{d}x, \tag{2.20}$$

where functions

$$\Phi_1(x) = \frac{1}{12} C_{c1} \left[t_{c2} \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}u_2}{\mathrm{d}x} - \frac{\mathrm{d}u_1}{\mathrm{d}x} \right) \right]^2,$$

$$\Phi_3(x) = \frac{1}{4} \left[C_{c3} \frac{\mathrm{d}u_1}{\mathrm{d}x} - C_{c4} \frac{\mathrm{d}u_2}{\mathrm{d}x} \right]^2,$$

$$\Phi_2(x) = C_{c2} \left[t_{c2} \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}u_2}{\mathrm{d}x} - \frac{\mathrm{d}u_1}{\mathrm{d}x} \right) \right] \left[C_{c3} \frac{\mathrm{d}u_1}{\mathrm{d}x} - C_{c4} \frac{\mathrm{d}u_2}{\mathrm{d}x} \right],$$

$$\Phi_4(x) = \frac{1}{2\left(1 + \nu_{c2}\right)} \frac{\left(u_2(x) - u_1(x)\right)^2}{t_{c2}^2},$$

and dimensionless parameters

$$C_{c1} = \frac{1}{t_{c2}^2} \left[3t_{c1}^2 + 6t_{c1}\left(t_{c2} + 2t_s\right) + 4\left(t_{c2}^2 + 3t_{c2}t_s + 3t_s^2\right) \right],$$

$$C_{c2} = \frac{1}{2t_{c2}} \left(t_{c1} + t_{c2} + 2t_s\right),$$

$$C_{c3} = \frac{1}{t_{c2}} \left(t_{c1} + 2t_{c2} + 2t_s\right),$$

$$C_{c4} = \frac{1}{t_{c2}} \left(t_{c1} + 2t_s\right);$$

the sum of the elastic strain energy of two outer sheets after integration with regard to thickness of these sheets

$$U_\varepsilon^{(os)} = E_{so} b t_s \int_0^L \left[\frac{1}{12} C_{s3} t_s^2 \left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \right)^2 - C_{s4} t_s \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \frac{\mathrm{d}u_2}{\mathrm{d}x} + \left(\frac{\mathrm{d}u_2}{\mathrm{d}x} \right)^2 \right] \mathrm{d}x, \tag{2.21}$$

where dimensionless parameters

$$C_{s3} = \frac{1}{t_s^2} \left[3t_{c1}^2 + 6t_{c1}\left(2t_{c2} + 3t_s\right) + 4\left(3t_{c2}^2 + 9t_{c2}t_s + 7t_s^2\right) \right],$$

$$C_{s4} = \frac{1}{t_s} \left(t_{c1} + 2t_{c2} + 3t_s\right).$$

The work of the load

$$W = \int_0^L q v(x) \, \mathrm{d}x. \tag{2.22}$$

Based on the principle of stationary total potential energy $[\delta(U_{\varepsilon}-W)=0]$ with consideration of the expressions (2.17) and (2.22) three differential equations of equilibrium are obtained in the following form

$$C_{vv}\frac{\mathrm{d}^4 v}{\mathrm{d}x^4}-C_{vu1}\frac{\mathrm{d}^3 u_1}{\mathrm{d}x^3}-C_{vu2}\frac{\mathrm{d}^3 u_2}{\mathrm{d}x^3}=p, \tag{2.23}$$

$$C_{vu1}\frac{\mathrm{d}^3 v}{\mathrm{d}x^3}-C_{u1u1}\frac{\mathrm{d}^2 u_1}{\mathrm{d}x^2}+C_{u1u2}\frac{\mathrm{d}^2 u_2}{\mathrm{d}x^2}+C_{u1}u_1(x)-C_{u2}u_2(x)=0, \tag{2.24}$$

$$C_{vu2}\frac{\mathrm{d}^3 v}{\mathrm{d}x^3}+C_{u1u2}\frac{\mathrm{d}^2 u_1}{\mathrm{d}x^2}-C_{u2u2}\frac{\mathrm{d}^2 u_2}{\mathrm{d}x^2}-C_{u2}u_1(x)+C_{u2}u_2(x)=0, \tag{2.25}$$

where

$$C_{vv}=\frac{1}{12}\left[E_{c1}t_{c1}^3+2\left(E_{si}C_{s1}+E_{so}C_{s3}\right)t_s^3+2E_{c2}C_{c1}t_{c2}^3\right],$$

$$C_{vu2}=E_{so}C_{s4}t_s^2-E_{c2}t_{c2}^2\left(C_{c2}C_{c4}-\frac{1}{6}C_{c1}\right),$$

$$C_{vu1}=\frac{1}{6}\left[E_{c1}t_{c1}^2+6E_{si}C_{s2}t_s^2+E_{c2}t_{c2}^2\left(6C_{c2}C_{c3}-C_{c1}\right)\right],$$

$$C_{u1}=2\frac{E_{c1}}{1+\nu_{c1}}\frac{1}{t_{c1}}+\frac{E_{c2}}{1+\nu_{c2}}\frac{1}{t_{c2}},$$

$$C_{u1u1}=2\left[E_{c1}t_{c1}+E_{si}t_s+E_{c2}t_{c2}\left(\frac{1}{12}C_{c1}-C_{c2}C_{c3}+\frac{1}{4}C_{c3}^2\right)\right],$$

$$C_{u2}=\frac{E_{c2}}{1+\nu_{c2}}\frac{1}{t_{c2}},$$

$$C_{u1u2}=\frac{1}{6}E_{c2}t_{c2}\left[C_{c1}-6C_{c2}\left(C_{c3}+C_{c4}\right)+3C_{c3}C_{c4}\right],$$

$p=\frac{q}{b}$ – uniformly distributed pressure at the outer upper sheet of the beam, $C_{u2u2}=\frac{1}{6}E_{c2}t_{c2}\left[C_{c1}-12C_{c2}C_{c4}+3C_{c4}^2\right]+2E_{so}t_s$.

The bending moment $M_b(x)=b\int\limits_h y\sigma(x)\,\mathrm{d}y$ with consideration of the Hooke's law and after integration with regard to depth h of the beam, leads to the following equation

$$C_{vv}\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}-C_{vu1}\frac{\mathrm{d}u_1}{\mathrm{d}x}-C_{vu2}\frac{\mathrm{d}u_2}{\mathrm{d}x}=-\frac{M_b(x)}{b}. \tag{2.26}$$

It may be noticed that this equation is equivalent to the equation (2.23). Therefore, the system of equations for the analyzed problem comprises the equations (2.26), (2.24) and (2.25).

3. Bending of the beam – analytical study

The bending moment at the simply supported beam loaded with uniformly distributed transverse load of intensity q (Fig. 1)

$$M_b = \frac{1}{2}\left(1 - \frac{x}{L}\right)\frac{x}{L}qL^2. \tag{3.1}$$

The system of differential equations (2.26), (2.24) and (2.25) with consideration of the expression (3.1) is approximately solved with the use of three assumed functions

$$v(x) = v_a \sin\left(\pi\frac{x}{L}\right), \quad u_1(x) = u_{1a}\cos\left(\pi\frac{x}{L}\right), \quad u_2(x) = u_{2a}\cos\left(\pi\frac{x}{L}\right), \tag{3.2}$$

where v_a – maximal deflection, u_{1a}, u_{2a} – maximal displacements of the sheets.

Therefore, after application of the Galerkin's method, the system of three linear algebraic equations is obtained

$$\pi C_{vv}\frac{v_a}{L} - C_{vu1}u_{1a} - C_{vu2}u_{2a} = \frac{4}{\pi^4}pL^3, \tag{3.3}$$

$$\pi C_{vu1}\frac{v_a}{L} - \left[C_{u1u1} + \left(\frac{L}{\pi}\right)^2 C_{u1}\right]u_{1a} + \left[C_{u1u2} + \left(\frac{L}{\pi}\right)^2 C_{u2}\right]u_{2a} = 0, \tag{3.4}$$

$$\pi C_{vu2}\frac{v_a}{L} + \left[C_{u1u2} + \left(\frac{L}{\pi}\right)^2 C_{u2}\right]u_{1a} - \left[C_{u2u2} + \left(\frac{L}{\pi}\right)^2 C_{u2}\right]u_{2a} = 0. \tag{3.5}$$

From which, the maximal deflection of the beam

$$v_{\max} = v_a = \frac{4}{\pi^5}\left\{\left[C_{u1u1} + \left(\frac{L}{\pi}\right)^2 C_{u1}\right]\left[C_{u2u2} + \left(\frac{L}{\pi}\right)^2 C_{u2}\right] - \left[C_{u1u2} + \left(\frac{L}{\pi}\right)^2 C_{u2}\right]^2\right\}\frac{pL^4}{\det_0}, \tag{3.6}$$

where $\det_0$ – the eliminant – the matrix determinant.

Taking into account the classical Euler-Bernoulli beam theory (the linear hypothesis in which plane cross sections before bending remains plane after bending), the bending problem of the beam with regard to the expression (3.1) is expressed by one equation

$$C_{vv}\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = -\frac{qL^2}{2b}\left(1 - \frac{x}{L}\right)\frac{x}{L}. \tag{3.7}$$

From which, after double integration and with consideration of the boundary conditions $v(0) = v(L) = 0$, one obtains the maximal deflection of the beam

$$v_{\max}^{(E-B)} = \frac{5}{384}\frac{pL^4}{C_{vv}}. \tag{3.8}$$

The detailed studies are carried out for example beams of length $L = 800$ mm, width $b = 50$ mm, thicknesses $t_{c1} = 15$ mm, $t_{c2} = 10$ mm, $t_s = 0.8$ mm, and pressure $p = 0.05$ MPa. The values of material constants of beam parts are specified in Table 1.

Table 1. The values of Young's modules E [MPa] and Poisson ratios ν.

Material	Steel	Steel foam	Aluminium	Al foam	Polyurethane foam
E [MPa]	200 000	3150	79 000	200	4.5
ν	0.3	0.05	0.33	0.3	0.1

The seven type beams are selected to detailed studies. These types are characterized by various materials of the layers (Table 2).

Table 2. The seven type beams with various materials of the layers.

Beam	Main core – t_{c1}	Inner sheets – t_s	Second cores – t_{c2}	Outer sheets – t_s
B-1	Steel foam	Steel	Steel foam	Steel
B-2	Al foam	Aluminium	Steel foam	Steel
B-3	Al foam	Aluminium	Al foam	Aluminium
B-4	Polyurethane foam	Aluminium	Polyurethane foam	Aluminium
B-5	Polyurethane foam	Steel	Polyurethane foam	Steel
B-6	Polyurethane foam	Steel	Steel foam	Steel
B-7	Polyurethane foam	Aluminium	Al foam	Aluminium

The values of maximal deflections of the beams calculated based on the expressions (3.6) and (3.8) are specified in Table 3.

Table 3. The values of maximal deflections of the beams – analytical study.

Beam	B-1	B-2	B-3	B-4	B-5	B-6	B-7
$v_{\max}^{(\text{Analyt})}$ [mm]	1.92	2.76	6.45	56.06	52.43	10.71	21.87
$v_{\max}^{(\text{E-B})}$ [mm]	1.85	2.03	5.04	5.12	2.02	1.86	5.04

Comparison of these values of maximal deflections of the beam types calculated with the use of two methods (Zig-Zag and Euler-Bernoulli theories) allows to notice significant differences between the beams composed mainly of metal layers and the ones including polyurethane cores.

4. Bending of the beam – FEM study – ABAQUS system

4.1. Numerical FEM model

The numerical analysis of the seven layer beam is executed in two different FEM systems: ABACUS and SolidWorks Simulation. The models of the beam in these programs are the same to compare obtained results with each other and to check out differences between numerical and analytical solutions. Taking into account two planes of symmetry the beam is calculated only in 1/4 part. On both symmetry planes, normal displacements are fixed for all the layers, while in the support vertical displacements are fixed (along y axis). This boundary condition is applied to whole supported planes. The scheme of the beam and boundary conditions and load are presented in Fig. 3.

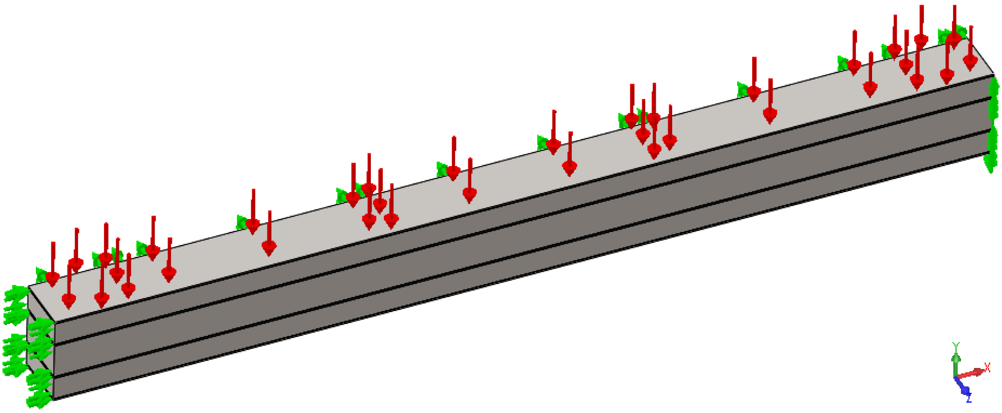

Fig. 3. The scheme of the beam with boundary conditions and load.

Total number of the nodes in SolidWorks Simulation system is 17 089, and total number of the elements of second order (tetrahedrons with 10 nodes) is 10966 for maximum element size not exceeding 8 mm. In ABACUS System the model includes 91 283 nodes and 19 200 elements. Two element types were used – 6400 of them were quadrilateral shell elements and 12 800 were solid quadratic hexahedral ones.

4.2. Bending – maximal deflections

The maximal deflection of the beam is in the plane of symmetry. The results of analytical and FEM analysis are specified in Table 4 for two systems, where $v_{\max}^{(\mathrm{FEM\_A})}$ is deflection calculated in ABACUS System and $v_{\max}^{(\mathrm{FEM\_S})}$ is deflection obtained in SolidWorks Simulation program.

Table 4. The values of maximal deflections of beams – analytical and FEM study.

Beam	B-1	B-2	B-3	B-4	B-5	B-6	B-7
$v_{\max}^{(\text{Analyt})}$ [mm]	1.92	2.76	6.45	56.06	52.43	10.71	21.87
$v_{\max}^{(\text{FEM\_A})}$ [mm]	1.91	2.74	6.41	55.40	51.77	10.74	21.99
$v_{\max}^{(\text{FEM\_S})}$ [mm]	1.91	2.73	6.39	54.72	51.15	10.70	21.86
$v_{\max}^{(\text{E-B})}$ [mm]	1.85	2.03	5.04	5.12	2.02	1.86	5.04

The results obtained in ABACUS System and SolidWorks Simulation program are similar and differs no more than 2.5 per cent from analytical solution. Bigger differences are in SolidWorks Simulation System.

5. Conclusions

The proposed objectives of the study are achieved that allowed to make the comparative analysis of the results. The results obtained with analytical and numerical methods specified in Table 4 allow to conclude that:

- The differences between the deflection values computed with analytical and numerical methods do not exceed 2.5 per cent.
- The results calculated with Euler-Bernoulli beam theory differ from the analytical and numerical values obtained for the B-1, B-2 and B-3 beams with metal foam cores by less than 36 per cent. In case of the polyurethane foam cores these differences are significant. For the B-5 beam the difference is more than 25-fold.
- In order to model the multilayered structures the use of the Zig-Zag theory is necessary.

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