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Research Paper

Approximate Estimation of Stability of Homogeneous Beam on Elastic Foundation

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The paper deals with a proposition of obtaining an analytical solution for a beam on elastic foundation. The main objective of presented work was stability analysis of the axially compressed beam. The analytical model was proposed. Shape function for inhomogeneous properties of the foundation was assumed. The Galerkin method was used to calculate the values of critical forces. Main conditions have been defined. The critical loads as a function of geometric and mechanical properties of the beam as well as inhomogeneous properties of the elastic foundation have been calculated.

Key words: analytical model; homogeneous beam; elastic foundation; critical force.

1. Introduction

The construction-foundation interaction is a major problem in the design of beams and other structures. The researchers describe several analytical models of beams resting on elastic foundation. Most of them are so complex that it reduces their practical application.

The problem of beams resting on elastic foundation is often found in the design of variety of constructions, e.g. buildings (frames and constructions), railroads, airports, highways, sports fields, parking lots, storage capacities as well as dams and embankments. In addition, they have also found the application in geotechnical engineering, marine engineering, bio-mechanics, harbour works, buried gas pipeline systems as well as constructions of machine foundations. The assumption of beams on an elastic foundation was first introduced and described by Winkler in 1867. The beam lies on elastic foundation when, under applied external loads, the reaction forces of the foundation are proportional at every point to the deflection of the beam at this point.

Beams supported on elastic foundation appear in a variety of technical problems. The derivation of the differential equation has been derived from the assumption that the supporting medium obeys the Hooke's law. In this theory, it is assumed that the effect of the foundation is the same as that of a great number of small springs and therefore that the reaction of the foundation is proportional to the local deflection. If such a foundation is subjected to a partially distributed surface loading, q, the springs will not be affected beyond the loaded region. A number of studies in the area of foundation-structure interaction have been conducted on the basis of Winkler hypothesis for its simplicity. The Winkler model is very simple but it does not accurately represents the characteristics of many practical foundations. One of the most important deficiencies of the Winkler model is that a displacement discontinuity appears between the loaded and the unloaded part of the foundation surface. In practise, the foundation surface does not show any discontinuity.

Buckling is characterized by a sudden sideways failure of a structural member subjected to loads, where the compressive stress at the point of failure is less than the ultimate compressive stress that the material is capable of withstanding. Buckling analysis has been conducted by various amount of researchers. The analysis included local and global buckling as well as pre- and post-buckling state. Thermal post-buckling analysis of anisotropic beams on an elastic foundation was described by Li and Qiao [1]. Deformation and buckling of a hinged buckled beam resting on an elastic foundation and subjected to a midpoint force was investigated by Hung and Chen [2]. In this work, all the initial stable configurations under different elastic foundation stiffness were determined. The analysis revealed that the initial stable configuration of the elastically supported buckled beam may be symmetric or asymmetric with respect to the centre line. The problems of buckling loads and natural frequencies of homogeneous beam on an elastic foundation were studied by Zhang *et al.* [3] where exact solutions were made. The comparison of analytical and numerical results related to buckling of beams on an elastic foundation was presented by Griffiths and Bee [4]. Static and dynamic analysis of buckling of functionally graded beam on an elastic foundation, subjected to uniform temperature rise loading and uniform compression, was conducted by Ghiasian *et al.* [5]. Post-buckling analysis of an axially loaded elastic beam resting on a linearly elastic medium was considered by Challamel [6]. The post-buckling analysis of thin-walled beams subjected to an axial compressive load and resting on Winkler-type continuous elastic foundation was formulated by Kameswara Rao and Mirza [7]. Post-buckling analysis of a column on an elastic foundation was discussed by

Kounadis *et al.* [8]. The investigation revealed that the critical state of perfect columns was a symmetric bifurcation point. Buckling analysis of elastic beams embedded in granular media was carried out by Mojdehi *et al.* [9]. Buckling loads of beams with different flexural rigidity, length, and boundary conditions within granular media of different depths were determined. The energy method was used to predict the buckling load with the use of a series of springs along the length of the beam based on a beam on an elastic foundation. Good agreement between the experimental results and the theoretical model was obtained. Torsional post-buckling analysis of thin-walled open section clamped beam supported on Winkler-Pasternak foundation was performed by Kameswara Rao and Bhaskara Rao [10]. The beam was subjected to an axial compressive load. The point of bifurcation for a clamped beam was calculated. In addition, the influence of continuous Winkler-Pasternak elastic foundation on the torsional post-buckling behaviour of the beam was performed. The analysis of bending, buckling, and vibration of a carbon nanotube-reinforced composite beams was described by Wattanasakulpong and Ungbhakorn [11]. The beams resting on Pasternak elastic foundation, including a shear layer and Winkler springs, were considered. It was found for buckling that the critical buckling loads increase with the increase of the stiffness of the springs. Buckling analysis of nanocomposite Timoshenko beams reinforced by single-walled carbon nanotubes and resting on an elastic foundation was investigated by Yas and Samadi [12]. Critical buckling loads were obtained for the beams with different boundary conditions. Post-buckling and nonlinear free vibration analysis of geometrically imperfect functionally graded beams resting on a nonlinear elastic foundation was presented by Yaghoobi and Torabi [13]. The material properties of functionally graded beams were assumed to be graded in the thickness direction. The assumptions of small strain and deformation have been used. The analysis of a steady-state response of an uniform infinite Euler-Bernoulli elastic beam resting on Pasternak elastic foundation was conducted by Froio *et al.* [14]. The beam was subjected to a concentrated load moving at constant velocity. The Fourier transform technique has been applied to derive an universal fully parametric analytical solution. In addition, analysis of the influence of the moving load velocity, the Pasternak modulus, and the damping ratio on the beam-foundation response has been made. The recursive differentiation method (RDM) has been introduced and employed by Hassan and Doha [15] to obtain analytical solutions for static and dynamic stability parameters of beams resting on two-parameter foundations in various different end conditions. The analysis revealed that the critical load of the first buckling mode was not always the smallest critical load in contrast to that common fact in the case of beams without foundation. The critical load of a higher mode may be smaller than the critical load of a lower buckling mode. Post-buckling analysis of beams made of functionally graded materials, resting on a non-linear elastic foundation, and subjected to an axial force was investigated by YAGHOOBI and TORABI [16]. The influence of foundation parameters, axial force, end supports, and material inhomogeneity on the post-buckling behaviour of beams has been taken under consideration. The analysis revealed that an increase in the values of the shearing layer stiffness results in decreasing the hardening characteristic of the beam.

The analysis of buckling of beams resting on an elastic foundation was presented by various amount of researchers. Most of them assume that the physical model of the foundation is presented as a great number of small springs. This assumption may cause some substantial difficulties in analytical solutions. In all cases, the calculations are dependent on the characteristics of springs. The most important factors that affect the properties of beams on an elastic foundation are stiffness of springs, their position as well as their distance from the supports. In addition, most problems related to beams on an elastic foundation are solved with the use of Winkler model. The modelling of foundation with the use of Winkler theory was considered insufficient in various technical problems. The main disadvantage is the fact that it overlooks the shear interaction between the spring elements. The authors did not find the articles which present the elastic foundation in a different form than the system of springs. In presented work, the authors proposed new analytical solution for homogeneous beams on an elastic foundation. The elastic foundation has been described by a mathematical function. The shape of elastic foundation is conditioned by the form of the graph of the assumed function. In this case, the elastic foundation can be described by any function which is in accordance with boundary conditions. Different shapes of the foundations can be considered.

The main objective of presented work is to elaborate the mathematical model of the homogeneous beam and to investigate the influence of geometric and mechanical parameters of the beam as well as inhomogeneous parameters of the elastic foundation on the behaviour of the structure. The main issue of this work is the analysis and the calculation of the values of critical loads of homogeneous beam with inhomogeneous properties of the foundation. Analytical model was studied. The scheme of considered beam, geometry, and load is shown in Fig. 1. The beam is subjected to compressive axial force F_0. The work presents original

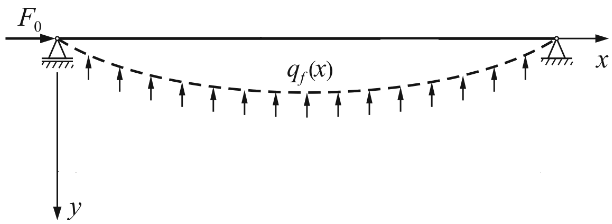

FIG. 1. The scheme of homogeneous beam.

approach to the problem of beams on an elastic foundation. Original function of deflection has been proposed.

2. ANALYTICAL MODEL OF THE BEAM

The model is derived with linear mechanical properties of the homogeneous beam. The scheme of the beam is shown in Fig. 1.

Differential equation of the beam is as follows:

$$EJ_z\frac{d^4v}{dx^4} + F_0\frac{d^2v}{dx^2} = -q_f(x), \tag{2.1}$$

where $q_f(x)$ is the intensity of load – reaction of the elastic foundation $\left[\frac{\mathrm{N}}{\mathrm{mm}}\right]$ and $q_f(x) = c(x)\cdot v(x)$; $c(x)$ is the property – foundation constant $\left[\frac{\mathrm{N}}{\mathrm{mm}^2}\right]$, $v(x)$ – deflection of the beam [mm].

Therefore, the differential equation (2.1) can be written in the following form:

$$EJ_z\frac{d^4v}{dx^4} + F_0\frac{d^2v}{dx^2} + c(x)\cdot v(x) = 0. \tag{2.2}$$

Shape function (Fig. 2) of inhomogeneous properties of the foundation is assumed (property foundation function):

$$c(x) = c_0 - c_1\sin^k(\pi\xi), \tag{2.3}$$

where $\xi = \frac{x}{L}$ (L – length of the beam) and $0 \ll \xi \ll 1$; k is a natural exponent.

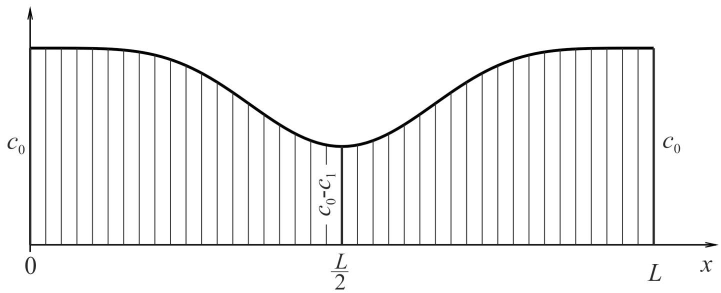

FIG. 2. Shape function of $c(x)$ parameter.

The higher value of parameter k, the narrower concavity the function has. The higher value of c_1 parameter, the deeper concavity the function has (Fig. 2).

The function of deflection of the homogeneous beam is assumed in the following form:

$$v(x) = v_a\cdot\sin(m\pi\xi)\cdot\sin^n(\pi\xi), \tag{2.4}$$

where m and n are natural numbers.

It was assumed that the function (2.3) is symmetrical to both ends of the beam (Fig. 2). Equation (2.4) can be written as follows:

$$\Phi(\xi) = \pi^4 E J_z \frac{v_a}{L^4} \cdot f_4(\xi) + F_0 \cdot \pi^2 \frac{v_a}{L^2} \cdot f_2(\xi) + [c_0 - c_1 \sin^k(\pi\xi)] \cdot v_a \sin(m\pi\xi) \sin^n(\pi\xi) = 0, \tag{2.5}$$

where f_2 and f_4 are derivatives of the Eq. (2.4) of second and fourth order respectively.

The critical value of load (2.13) will be calculated using the Galerkin method. The main condition of this method is as follows:

$$\int_0^1 \Phi(\xi) \cdot \sin(m\pi\xi) \sin^n(\pi\xi) \, \mathrm{d}\xi = 0. \tag{2.6}$$

General solution can be defined in the following form:

$$\left(\frac{\pi}{L}\right)^2 E J_z \cdot J_4 - F_0 \cdot J_2 + \left(\frac{L}{\pi}\right)^2 \cdot J_0 = 0, \tag{2.7}$$

from which

$$F_0 = \frac{1}{J_2}\left[J_4 \cdot F_{\text{EULER}} + \left(\frac{L}{\pi}\right)^2 \cdot J_0\right], \tag{2.8}$$

where

$$F_{\text{EULER}} = \frac{\pi^2 E J_z}{L^2}, \tag{2.9}$$

$$J_4 = \left(\frac{1}{\pi}\right)^4 \int_0^1 \frac{d^4\widetilde{v}}{d\xi^4} \cdot \sin(m\pi\xi) \cdot \sin^n(\pi\xi) \, \mathrm{d}\xi, \tag{2.10}$$

$$J_2 = -\left(\frac{1}{\pi}\right)^2 \int_0^1 \frac{d^2\widetilde{v}}{d\xi^2} \cdot \sin(m\pi\xi) \cdot \sin^n(\pi\xi) \, \mathrm{d}\xi, \tag{2.11}$$

$$J_0 = \int_0^1 \left[c_0 - c_1 \sin^k(\pi\xi)\right] [\sin(m\pi\xi) \sin^n(\pi\xi)]^2 \, \mathrm{d}\xi. \tag{2.12}$$

The critical load $F_{0,\,\mathrm{cr}}$ (2.13) is a function of the geometrical and mechanical properties of the homogeneous beam as well as inhomogeneous parameters of the elastic foundation. The function is in the following form:

$$F_{0,\,\mathrm{cr}} = \min_{m,n} \left\{ \frac{J_4 \cdot F_{\mathrm{EULER}} + \left(\frac{L}{\pi}^2\right) \cdot J_0}{J_2} \right\}. \tag{2.13}$$

Sample analytical values of critical loads (2.13) and stresses in a function of variable m and n parameters were performed for the following data: $c_0 = 10$ MPa, $E = 200\,000$ MPa, $L = 1200$ mm, $J_z = 240$ mm^4, and $A = 180$ mm^2. The results for different proportions of c_1/c_0 (amplitudes of shape function) and k (shape parameter) are presented in Tables 1–9.

The analytical calculations presented in Tables 1–9 refer to elastic foundation with inhomogeneous mechanical properties. Comparing the elastic foundation with constant mechanical properties ($c_1 = 0$, then $n = 0$) and substitute to the Eq. (2.13) the following data: $J_0 = 5$, $J_2 = 32$, $J_4 = 2048$, and $m = 8$, critical

Table 1. Critical loads values for homogeneous beam with inhomogeneous properties of elastic foundation for $k = 1$.

c_1/c_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$F_{0,\,\mathrm{cr}}$ [kN]	43.221	41.316	39.404	36.779	34.154	31.530	28.179	24.585
σ_{cr} [MPa]	240.1	229.5	218.9	204.3	189.7	175.2	156.6	136.6
m	8	8	7	7	7	7	6	6
n	1	1	2	2	2	2	3	3

Table 2. Critical loads values for homogeneous beam with inhomogeneous properties of elastic foundation for $k = 3$.

c_1/c_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$F_{0,\,\mathrm{cr}}$ [kN]	43.602	42.078	40.364	38.279	35.868	33.490	30.972	27.402
σ_{cr} [MPa]	242.2	233.8	224.2	212.7	199.3	186.1	172.1	152.2
m	8	8	8	7	7	7	6	6
n	1	1	2	2	3	3	4	4

Table 3. Critical loads values for homogeneous beam with inhomogeneous properties of elastic foundation for $k = 5$.

c_1/c_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$F_{0,\,\mathrm{cr}}$ [kN]	43.820	42.481	40.941	39.110	36.949	34.703	32.444	29.466
σ_{cr} [MPa]	243.4	236	227.5	217.3	205.3	192.8	180.2	163.7
m	8	8	8	7	7	7	7	6
n	1	2	2	3	3	4	4	4

Table 4. Critical loads values for homogeneous beam with inhomogeneous properties of elastic foundation for $k = 10$.

c_1/c_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$F_{0,\,\mathrm{cr}}$ [kN]	44.114	43.068	41.820	40.513	38.577	36.641	34.705	32.770
σ_{cr} [MPa]	245.1	239.3	232.3	225.1	214.3	203.6	192.8	182.1
m	8	8	8	7	7	7	7	7
n	1	2	2	4	4	4	4	4

Table 5. Critical loads values for homogeneous beam with inhomogeneous properties of elastic foundation for $k = 15$.

c_1/c_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$F_{0,\,\mathrm{cr}}$ [kN]	44.271	43.412	42.337	41.262	39.584	37.849	36.115	34.381
σ_{cr} [MPa]	246	241.2	235.2	229.2	219.9	210.3	200.6	191
m	8	8	8	8	7	7	7	7
n	1	2	2	2	4	4	4	4

Table 6. Critical loads values for homogeneous beam with inhomogeneous properties of elastic foundation for $k = 20$.

c_1/c_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$F_{0,\,\mathrm{cr}}$ [kN]	44.373	43.620	42.694	41.738	40.252	38.651	37.050	35.449
σ_{cr} [MPa]	246.5	242.3	237.2	231.9	223.6	214.7	205.8	196.9
m	8	8	8	8	7	7	7	7
n	1	1	2	2	4	4	4	4

Table 7. Critical loads values for homogeneous beam with inhomogeneous properties of elastic foundation for $k = 30$.

c_1/c_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$F_{0,\,\mathrm{cr}}$ [kN]	44.509	43.892	43.190	42.400	41.054	39.614	38.173	36.733
σ_{cr} [MPa]	247.3	243.8	240	235.6	228.1	220.1	212.1	204.1
m	8	8	8	8	7	7	7	7
n	1	1	2	2	4	4	4	4

Table 8. Critical loads values for homogeneous beam with inhomogeneous properties of elastic foundation for $k = 40$.

c_1/c_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$F_{0,\,\mathrm{cr}}$ [kN]	44.603	44.080	43.543	42.860	41.511	40.162	38.813	37.465
σ_{cr} [MPa]	247.8	244.9	242	238.1	230.6	223.1	215.6	208.1
m	8	8	8	7	7	7	7	7
n	1	1	2	4	4	4	4	4

Table 9. Critical loads values for homogeneous beam with inhomogeneous properties of elastic foundation for $k = 50$.

c_1/c_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$F_{0,\,\mathrm{cr}}$ [kN]	44.675	44.223	43.771	43.102	41.814	40.526	39.238	37.949
σ_{cr} [MPa]	248.2	245.7	243.2	239.5	232.3	225.1	218	210.8
m	8	8	8	7	7	7	7	7
n	1	1	1	4	4	4	4	4

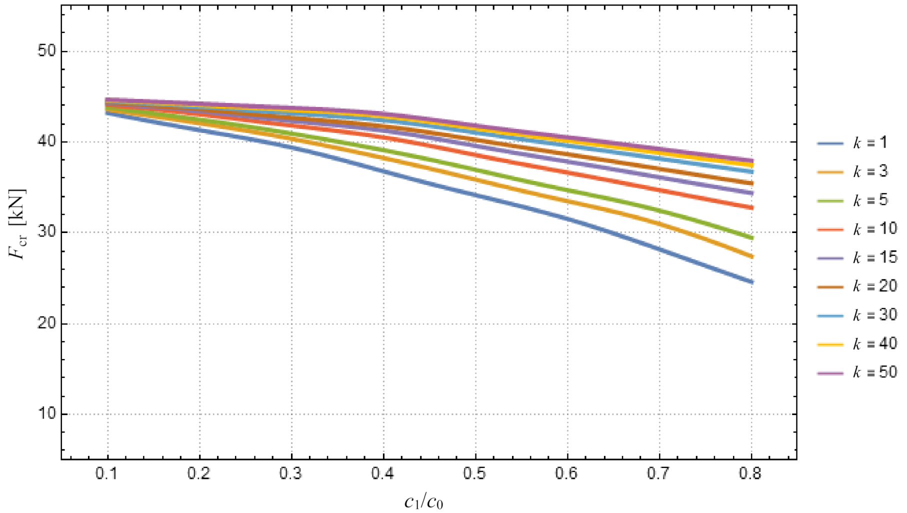

Fig. 3. The influence of k parameter and the c_1/c_0 ratio on critical loads values.

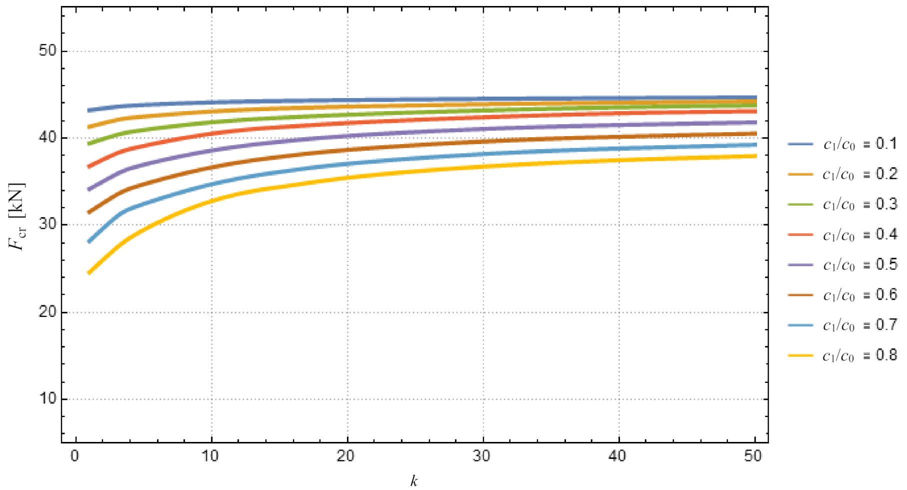

Fig. 4. The influence of c_1/c_0 ratio and k parameter on critical loads values.

loads values were calculated. The values are the same as calculated on the basis of Eq. (2.13) and referred to the inhomogeneous mechanical properties. The critical load value for constant and inhomogeneous properties of the beam is equal $F^{\circ}_{0,\,\mathrm{cr}} = 43.852$ kN.

The results presented in the tables above are summarized in Figs 3 and 4. It can be concluded that critical loads values are dependent on the values of k parameter (shape parameter) and c_1/c_0 ratio (amplitudes of shape function). The highest values of critical loads (2.13) can be obtained for the highest values of k parameter and the lowest values of c_1/c_0 ratio. The highest value is equal $F^{\circ}_{0,\,\mathrm{cr}} = 44.675$ kN and was obtained for $k = 50$ and $c_1/c_0 = 0.1$.

3. Conclusions

The problem of beams resting on elastic foundation was presented by various amount of researchers. They assume that the physical model of the foundation is presented as a great number of small springs. In this case, the results of calculations are dependent on the characteristics of springs. The article presents new approach to the problem of beams resting on an elastic foundation. The model of the foundation has been presented as a mathematical function which is on accordance with boundary conditions. The elastic foundation can be freely modeled. The only required condition is to apply the appropriate mathematical equation in order to determine the shape of the foundation.

The main issue of this work is the analysis and estimation the critical loads values of homogeneous beam on elastic foundation with inhomogeneous mechanical properties. Differential equation of the beam, according to Winkler model, was presented. Shape (of elastic foundation) and deflection functions were assumed. The work presents original approach to the problem of beams on elastic foundation. Original function of deflection (2.4) has been proposed. The critical loads values were calculated. The examples of calculations were shown. The values were calculated numerically and they were the same as those calculated on the basis of Eq. (2.13) and referred to the inhomogeneous mechanical properties. In order to increase the critical load value of the beam, k parameter has to be increased. It must be noticed that analytical values of critical loads are calculated in a function of variable m and n parameters. Therefore, appropriate values must be chosen so that the results are consistent with Eq. (2.13).

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